Chapter 9 Graphs

9.7 Planar Graph

1. Introduction

- Introduction
  - Example 0 (page 657~658)
    - Is it possible to join these houses and utilities so that none of the connections cross?
    - Solution:
      - The problem can be modeled using the complete bipartite graph $K_{3,3}$.
      - Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?

1. Introduction

- Definition 1 (page 658)
  - A graph is called planar if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common point). Such a drawing is called a planar representation of the graph.
- Example 1 (page 658)
  - Is K_4 (shown in Figure 2 with two edges crossing) planar?
  - Solution: K_4 is planar because it can be drawn without crossing, as shown in Figure 3.

1. Introduction

- Example 2 (page 658)
  - Is Q_3 shown in Figure 4, planar?
  - Solution
    - Q_3 is planar, because it can be drawn without any edges crossing, as shown in Figure 5.

1. Introduction

- Is $K_{3,3}$, shown in Figure 6, planar?
  - Solution
    - Any attempt to draw K3,3 in the plane with no edges crossing is doomed (失败的).
    - 现在来说明为什么。在$K_{3,3}$的任何平面表示里，顶点v_1和v_2都必须同时与v_4和v_5连接。这四条边所形成的封闭曲线把平面分割成两个区域R_1和R_2，如图7(a)所示。顶点v_3属于R_1或R_2。当v_3属于闭曲线的内部R_2时，在v_3和v_4之间以及在v_3和v_5之间的边，把R_2分割成两个区域R_{21}和R_{22}，如图7(b)所示。

1. Introduction

- Is $K_{3,3}$, shown in Figure 6, planar?
  - Solution(cont.)
    - 下一步。注意没有办法来放置最后一个顶点v_6而又不迫使发生交叉。因为若v_6属于R_1，则不能不带交叉地画出v_6和v_3之间的边。若v_6属于R_{21}，则不能不带交叉地画出v_2和v_6之间的边。若v_6属于R_{22}，则不能不带交叉地画出v_1和v_6之间的边

      当v_3属于R_1时，可以使用类似的论证。所以，$K_{3,3}$是非平面图。

2. Euler's Formula

- The Regions of a Planar Graph
  - See Figure 8 (page 660)
    - For this planar graph, we have:

      $r=6$ (区域数)

      $e=11$ (边数)

      $v=7$
    - They satisfy the equation $r=e-v+2$.

2. Euler's Formula

- Theorem 1(Euler's Formula, page 606)
  - Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then r=e-v+2.
  - 证明
    - look at the blackboard or book.

2. Euler's Formula

- Example 4 (page 661)
  - Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how may regions does a representation of this planar graph split the plane?
  - Solution:
    - $2e=3\cdot 20$, $e=30$
    - $r = e-v+2 = 30-20+2 = 12$

2. Euler's Formula

- Corollary 1 (page 661)
  - If G is a connected planar simple graph with e edges and v vertices where $v \geqslant 3$, then $e \leqslant 3v-6$.
  - Proof:
    - Look at the blackboard and book.
    - The detailed proof is in page 608 with the help of the concept of the degree of a region (区域的度)

2. Euler's Formula

- Corollary 2 (page 661)
  - If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.
- Proof
  - If G has one or two vertices, the result is true.
  - If G has at least three vertices, by Corollary 1 we know that $e \leqslant 3v-6$, so $2e \leqslant 6v-12$. If the degree of every vertex were at least six (用反证法), then we would have $2e \geqslant 6v$. But this contradicts the inequality $2e \leqslant 6v-12$. It follows that there must be a vertex with degree no greater than five.

2. Euler's Formula

- Example 5 (page 662)
  - Show that K_5 is nonplanar using Corollary 1.
  - Solution
    - The graph K_5 has five vertices and ten edges. However, the inequality $e \leqslant 3v-6$ is not satisfied for this graph since $e=10$ and $3v-6=9$.
    - Therefore, K_5 is not planar.

2. Euler's Formula

- Corollary 3 (page 662)
  - If a connected planar simple graph has e edges and v vertices with $v \geqslant 3$ and no circuits of length three, then $e \leqslant 2v-4$
- Example 6 (page 663)
  - Use Corollary 3 to show that K3,3 is nonplanar.
  - Solution:
    - Since $K_{3,3}$ has no circuits of length three, Corollary 3 can be used. $K_{3,3}$ has six vertices and nine edge.
    - Since $e=9$ and $2v-4=8$, Corollary 3 shows that $K_{3,3}$ is nonplanar.

3. Kuratowski's Theorem

- Introduction (page 663)
  - $K_{3,3}$ and K_5 are not planar.
  - All nonplanar graphs must contain a subgraph that can be obtained from $K_{3,3}$ or K_5 using certain permitted operations.

3. Kuratowski's Theorem

- Operation-----Elementary Subdivision (初等细分或剖分)
  - Elementary subdivision
    - If {u, v} is an edge, then remove this edge and add a new vertex with edges {u, w} and {w, v}.
  - Homeomorphism (同胚)
    - The graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.

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  - The three graphs in Figure 12 (page 610) are homeomorphic

3. Kuratowski's Theorem

- Theorem 2 (page 664)
  - A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.
- Example 8 (page 610)
  - Determine whether the graph G shown in Figure 13 is planar?
  - Solution:
- Example 9 (page 610)
  - Is the Petersen graph, shown in Figure 14(a), planar?

Homework

- Page 665~666
  - 2, 3(read), 4, 5(read), 6, 7(read), 8,
  - 9(read), 12, 14, 16, 18, 20, 24