Chapter 9 Graphs

9.5 Euler and Hamilton Paths

1. Introduction

- Introduction
  - Read page 577
- Euler Paths and Circuits
  - Introduction
    - The seven bridges of Konigsberg
    - Please read Figure 1 and Figure 2. (page 633)

2. Euler Paths and Circuits

- Definition 1 (page 633)
  - An Euler circuit in a Graph G is a simple circuit containing every edge of G.
  - An Euler path in G is a simple path containing every edge of G.

2. Euler Paths and Circuits

- Example 1 (page 634)
  - Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?
  - Solution
    - G_1---- has an Euler circuit, for example a, e, c, d, e, b, a.
    - G_2 (or G_3) ---- does not have an Euler Circuit.
    - G_3 has an Euler path, namely, a, c, d, e, b, d, a, b.
    - G_2 does not have an Euler path.

2. Euler Paths and Circuits

- Example 2 (page 634)
  - Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?
  - Solution
    - The graph H_2 has an Euler circuit, for example, a, g, c, b, g, e, d, f, a.
    - Neither H_1 nor H_3 has an Euler circuit.
    - H_3 has an Euler path, namely c, a, b, c, d, b, but H1 does not.

2. Euler Paths and Circuits

- Necessary and Sufficient Conditions for Euler Circuits and Paths (欧拉回路和欧拉路径的充分必要条件, page 634)
  - What can we say if a connected multigraph has an Euler circuit?

    ---- Every vertex must have even degree

2. Euler Paths and Circuits

- Necessary and Sufficient Conditions for Euler Circuits and Paths (欧拉回路和欧拉路径的充分必要条件, page 634)
  - What can we say if a connected multigraph has an Euler circuit?

    ---- Every vertex must have even degree
- Is this necessary condition for the existence of an Euler circuit also sufficient?
  - read page 635 for the explanation.
  - Please also see Figure 5 on page 636 for constructing an Euler Circuit

2. Euler Paths and Circuits

- Theorem 1 (page 636)
  - A connected multigraph has an Euler circuit if and only if each of its vertices has even edges.
  - For example, now we can solve the Konigsberg bridge problem. Using theorem 1, we find it does not have an Euler circuit.
- Algorithm for Constructing Euler Circuits (page 636)

2. Euler Paths and Circuits

- Theorem 2 (page 637)
  - A connected multigraph has an Euler path but not Euler circuit if and only if it has exactly two vertices of odd degree.

2. Euler Paths and Circuits

- Example 4 (page 637)
  - Which graph shown in Figure 7 have an Euler path?
  - Solution
    - G_2--- exact two vertices of odd degree, namely b and d.

      one such Euler path: d,a,b,c,d,b
    - G_2--- exact two vertices of odd degree, namely b, d.

      path: b,a,g,f,e,d,c,g,b,c,f,d
    - G_3--- has no Euler path

3. Hamilton Paths and Circuits

- Definition
  - A path $x_0, x_1, ..., x_{n-1}, x_n$ in the graph G=(V, E) is called a Hamilton path if $V=\{x_0, x_1, ..., x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leqslant i \leqslant j \leqslant n$.
  - A path $x_0, x_1, ..., x_{n-1}, x_n, x_0$ (with n>1) in a graph G=(V, E) is called a Hamilton circuit if $x_0, x_1, ..., x_{n-1}, x_n$, is a hamilton path.

3. Hamilton Paths and Circuits

- Example 5 (page 639)
  - Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?
  - Solution:
    - G1--- has a Hamilton circuit: a,b,c,d,e,a.
    - G2--- no Hamilton circuit

      --- has a Hamilton path: a, b, c, d
    - G3--- no Hamilton circuit (or path)

3. Hamilton Paths and Circuits

- Any simple way to determine whether a graph has a Hamilton circuit or path?
  - no known simple necessary and sufficient criteria for the existence of Hamilton circuit. Further,
    - If a graph with a vertex of degree one cannot have a Hamilton circuit.
    - If a vertex has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
    - A Hamilton circuit cannot contain a smaller circuit within it.

3. Hamilton Paths and Circuits

- Example 6 (page 640)
  - Show that neither graph displayed in Figure 11 has a Hamilton circuit?
  - Solution:
    - G --- no Hamilton circuit

      --- has a vertex of degree one
    - H-----Since the degrees of the vertices a, b, d, and e are all two, every edge incident with these vertices must be part of any Hmilton circuit.

      Four edges in any Hamilton circuit are incident to c. This is impossible.

3. Hamilton Paths and Circuits

- Example 7 (page 641)
  - Show that Kn has a Hamilton circuit whenever n⩾3.
  - Solution:
    - We can form a Hamiltion circuit Kn beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly one.

3. Hamilton Paths and Circuits

- Theorem 3 (page 641) Dirc's Theorem
  - If G is a simple graph with n vertices with $n \geqslant 3$ such that the degree of every vertex in G is at least n/2, the G has a Hamilton circuit.
- Theorem 4 (page 641) Ore's Theorem
  - If G is a simple graph with n vertices with $n \geqslant 3$ such that $deg(u)+deg(v) \geqslant n$ for every pair of nonadjacent vertices u and v, then G has a Hamilton circuit.
  - Note: Dirac's theorem can be considered as a corollary of Ore's theorem.

Homework

- Page 643~647
  - 1(read), 2, 3(read), 4, 5(read), 6, 7,
  - 8(read),18, 19(read), 28, 42, 43(read),
  - 44, 45(read), 47(read)