Chapter 9 Graphs

9.3 Representing Graphs and Graph Isomorphism

1. Introduction

- Introduction
  - See page 611

2. Representing Graphs

- Adjacency lists (邻接表)
- Example 1 (page 611)
  - Use adjacency lists to describe the simple graph given in Figure 1.
  - Solution:
    - See Figure 1 and Table 1 (page 612)

2. Representing Graphs

- Example 2 (page 612)
  - Represent the directed graph shown in Figure 2 by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.
  - Solution

Initial Vertex	Terminal Vertex
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

3. Adjacency Matrices (邻接矩阵)

- Definition (page 612)
  - Suppose that G=(V, E) is a simple graph where |V|=n. Suppose that the vertices of G are listed arbitrary $v_1, v_2, \ldots, v_n$. The adjacency matrix A of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i,j)th entry when v_i and v_j are adjacent, 0 as its (i,j)th entry when they are not adjacent.

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of G,} \\ 0 & \text{otherwise} \end{cases}$$

3. Adjacency Matrices (邻接矩阵)

- Example 3 (page 613)
  - Use an adjacency matrix to represent the graph shown in Figure 3.
  - Solution:

	a	b	c	d
a	0	1	1	1
b	1	0	1	0
c	1	1	0	0
d	1	0	0	0

3. Adjacency Matrices (邻接矩阵)

- Adjacency matrix for graphs with loops and undirected multiple edges (page 613)
  - A loop at the vertex ai is represented by a 1 at the (i,i)th position of the adjacency matrix.
  - When multiple edges are present, the (i,j)th entry is the number of edges that are associated to $\{a_i, a_j\}$.

3. Adjacency Matrices (邻接矩阵)

- Example 5 (page 613)
  - Use an adjacency matrix to represent the pseudograph shown in Figure 5.
  - Solution:

	a	b	c	d
a	**0**	**3**	0	2
b	3	0	1	1
c	0	1	1	2
d	2	1	2	0

3. Adjacency Matrices (邻接矩阵)

- Adjacency matrix for directed graphs (page 614)
  - The matrix for a directed graph G=(V,E) has 1 in its (i,j)th position if there is an edge from vi to vj, where v_1, v_2, ..., v_n is an arbitrary listing of the vertices of the directed graph.

    a_{ij} = 1 if (v_i, v_j) is an edge of G

    0 otherwise
  - How about directed multigraphs?

4. Incidence Matrices

- Definition (page 614)
  - Let $G=(V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G.
  - Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M=[m_{ij}]$, where

$m_{ij} = 1$ when edge e_j is incident with v_i

$= 0$ otherwise

4. Incidence Matrices

- Example 6 (page 614)
  - Represent the graph shown in Figure 6 with an incidence matrix.
  - Solution:

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

4. Incidence Matrices

- Example 7 (page 615)
  - Represent the pseudograph shown in Figure 7 using an incidence matrix.
  - Solution:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	0	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3								
v_4								
v_5								

5. Isomorphism (同构) of Graphs

- Example 8 (page 615)
  - Show that the graph G=(V, E) and H=(W, F), displayed in Figure 8, are isomorphic.
  - Solution
    - Construct a one to one correspondence f: V--->W such that:

$$u_1\text{------}>v_1$$

$$u_2\text{------}>v_4$$

$$u_3\text{------}>v_3$$

$$u_4\text{------}>v_2$$

5. Isomorphism (同构) of Graphs

- Example 8 (page 615)
  - Solution
    - Based on the function f, we can have:

$\{u_1,u_2\} \in E$ and $\{f(u_1), f(u_2)\} = \{v_1, v_4\} \in F$

$\{u_1,u_3\} \in E$ and $\{f(u_1), f(u_3)\} = \{v_1, v_3\} \in F$

$\{u_2,u_4\} \in E$ and $\{f(u_2), f(u_4)\} = \{v_4, v_2\} \in F$

$\{u_3,u_4\} \in E$ and $\{f(u_3), f(u_4)\} = \{v_3, v_2\} \in F$

Therefore, Graph G and H are isomorphic.

5. Isomorphism (同构) of Graphs

- Definition 1 (page 615)
  - The simple graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V_1.

  Such a function f is called an isomorphism.

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5. Isomorphism (同构) of Graphs

- Definition 1 (page 615)
  - Remark (备注)
    - It is often difficult to determine whether two simple graphs are isomorphic.
    - However, we can often show that two simple graphs are not isomorphic by showing that they do not share a property that isomorphic simple graphs must both have.

5. Isomorphism (同构) of Graphs

- Example 9 (page 616)
  - Show that the graphs displayed in Figure 9 are not isomorphic.
  - Solution:
    - Both G and H have five vertices and six edges.
    - However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

5. Isomorphism (同构) of Graphs

- Example 10 (page 616)
  - Determine whether the graphs shown in Figure 10 are isomorphic.
  - Solution
    - The graphs G and H both have eight vertices and ten edges. They also both have four vertices of degree two and four of degree three. Since these invariants all agree, it is still conceivable that these graphs are isomorphic.

5. Isomorphism (同构) of Graphs

- Example 10 (page 616)
  - Solution
    - However, G and H are not isomorphic. To see this, note that since deg(a)=2 in G, a must correspond to either t,u,x,or y in H, since these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.
    - 为什么G和H是不同构的?

5. Isomorphism (同构) of Graphs

- Example 10 (page 616)
  - Remark (备注):
    - Another way to see that G and H are not isomorphic is to note that the subgraphs of G and H made up of vertices of degree three and the edges connecting them must be isomorphic if these two graphs are isomorphic (the reader should verify this).

      However, these subgraphs, shown in Figure 11, are not isomorphic.

5. Isomorphism (同构) of Graphs

- Example 11 (page 617)
  - Determine whether the graphs G and H displayed in Figure 12 are isomorphic.
  - Solution
    - We now will define a function f and then determine whether it is an isomorphism. Since deg(u1)=2 and since u1 is not adjacent to any other vertex of degree two, the image of u1 must be either v4 or v6, the only vertices of degree two in H not adjacent to a vertex of degree two. We arbitrarily set f(u1)=v6. [If we found that this choice did not lead to isomorphism, we would then try.]

5. Isomorphism (同构) of Graphs

- Example 11 (page 617)
  - Solution
    - Since u2 is adjacent to u1, the possible images of u2 are v3 and v5. We arbitrarily set f(u2)=v3. Continuing in this way, using adjacency of vertices and degrees as a guide, we set f(u3)=v4, f(u4)=v5, f(u5)=v1, f(u6)=v2.
    - Now we have a one-to-one correspondence between the vertex set of G and the vertex set of H, namely:

      f(u1)=v6, f(u2)=v3, f(u3)=v4,

      f(u4)=v5, f(u5)=v1, f(u6)=v2.

5. Isomorphism (同构) of Graphs

- Example 11 (page 617)
  - Solution
    - To see whether f preserves edges, we examine the adjacency matrix of G, and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G,

$$
A_G = \begin{array}{c|cccccc}
 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
\hline
u_1 & 0 & 1 & 0 & 1 & 0 & 0 \\
u_2 & 1 & 0 & 1 & 0 & 0 & 1 \\
u_3 & 0 & 1 & 0 & 1 & 0 & 0 \\
u_4 & 1 & 0 & 1 & 0 & 1 & 0 \\
u_5 & 0 & 0 & 0 & 1 & 0 & 1 \\
u_6 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

5. Isomorphism (同构) of Graphs

- Example 11 (page 617)
  - Solution

$A_H =$

	v_6	v_3	v_4	v_5	v_1	v_2
v_6	0	1	0	1	0	0
v_3	1	0	1	0	0	1
v_4	0	1	0	1	0	0
v_5	1	0	1	0	1	0
v_1	0	0	0	1	0	1
v_2	0	1	0	0	1	0

5. Isomorphism (同构) of Graphs

- Example 11 (page 617)
  - Solution
    - Since $A_G = A_H$, it follows that f preserves edges. We conclude that f is an isomorphism, so that G and H are isomorphic.
    - Note that if f turned out not to be an isomorphism, we would not have established that G and H are not isomorphic, since another correspondence of the vertices in G and H may be an isomorphism.

Homework

- Page 618~621
  - 2, 4, 6, 8, 12, 14, 18, 20,26, 28, 30, 34, 36, 38, 40,42