Chapter 9 Graphs

9.2 Graph Terminology

1. Introduction

- Types of Graphs
  - See page 597

2. Basic Terminology

- Definition 1 (page 598)
  - Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if {u,v} is an edge of G. If e={u,v}, the edge e is called incident with the vertices u and v. The edge e is also said to connect u and v. The vertices are called endpoints of the edge {u, v}.

2. Basic Terminology

- Definition 2 (page 598)
  - The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of the vertex. The degree of the vertex v is denoted by deg(v).
  - Example: See Figure 1 (page 598)
    - For Graph G, deg(e)=3, deg(g)=0.
    - For Graph H, deg(e)=6, deg(b)=6

2. Basic Terminology

- Definition 2 (page 598)
  - Remark:
    - A vertex of degree zero is called isolated. It follows that an isolated vertex is not adjacent to any vertex.
    - A vertex is pending if and only if it has degree one. Consequently, a pending vertex is adjacent to exactly one other vertex.

2. Basic Terminology

- The Handshaking Theorem
  - Let G=(V, E) be an undirected graph with e edges. Then

    $$2e = \sum_{v \in V} deg(v)$$

    (Note that this applies even if multiple edges and loops are present.)
  - Example 2 (page 598)
    - How many edges are there in a graph with ten vertices each of degree six?
    - Solution: e=30

2. Basic Terminology

- Theorem 2 (page 599)
  - An undirected graph has an even (偶数) number of vertices of odd (奇数) degree.
  - Proof:
    - V1--- the set of vertices of an even degree
    - V2--- the set of vertices of an odd degree
    - $2e = \sum_{v \in V} \deg(v)$

      $= \sum_{v \in V1} \deg(v) + \sum_{v \in V2} \deg(v)$

2. Basic Terminology

- Definition 3 (page 600)
  - When (u, v) is an edge of the graph G with directed edge, u is said to be adjacent to v and v is said to be adjacent from u.
  - The vertex u is called the initial vertex of (u,v), and v is called the terminal or end vertex of (u,v). The initial vertex and terminal vertex of a loop are the same.

2. Basic Terminology

- Definition 4 (page 600)
  - In a graph with directed edges the *in-degree of a vertex v (入度)*, denoted by $deg^{-}(v)$, is the number of edges with v as their terminal vertex.
  - The *out-degree of v (出度)*, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex .
  - Note: a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

2. Basic Terminology

- Theorem 3 (page 600)
  - Let G=(V, E) be a graph with directed edges. Then

    $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$

3. Some Special Simple Graphs

- Example 5 (Complete Graph, 完全图page 601)
  - The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.
  - The graph of K_n, for n=1, 2, 3, 4, 5, 6 are displayed in Figure 6.

3. Some Special Simple Graphs

- Example 6 (Cycles, page 601)
  - The cycle C_n, $n \geqslant 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ... , $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.
  - The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4 (page 601).

3. Some Special Simple Graphs

- Example 7 (Wheels, 轮, page 601)
  - We obtain the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geqslant 3$, and connect this new vertex of the n vertices in C_n, by new edges.
  - The wheels W_3, W_4, W_5, and W_6 are displayed in Figure 5 (page 601).

3. Some Special Simple Graphs

- Example 8 (n-Cubes, n-立方体, page 602)
  - The n-dimensional cube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n.
  - Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.
  - The graphs Q_1, Q_2, and Q_3 are displayed in Figure 6 (page 602).

3. Some Special Simple Graphs

- Question:
  - How to construct the (n+1)-cube Qn+1 from the n-cube Qn (page 602)?
  - Way:
    - by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels different only in the first bit,

4. Bipartite Graphs (二分图)

- Definition 5 (page 602)
  - A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.
  - Note
    - no edge in G connects either two vertices in V_1 or two vertices in V_2.

4. Bipartite Graphs (二分图)

- Example 8 (page 602)
  - C_6 is bipartite, as shown in Figure 7 (page 603)
  - Analysis:
    - $V_1 = \{V_1, V_3, V_5\}$
    - $V_2 = \{V_2, V_4, V_6\}$

4. Bipartite Graphs (二分图)

- Example 8 (page 602)
  - C_6 is bipartite, as shown in Figure 7 (page 603)
  - Analysis:
    - $V_1 = \{V_1, V_3, V_5\}$
    - $V_2 = \{V_2, V_4, V_6\}$
- Example 9 (page 602)
  - K_3 is not a bipartite.
  - Why?
    - Please see book.

4. Bipartite Graphs (二分图)

- Example 10 (page 603)
  - Are the graphs G and H displayed in Figure 8 bipartite?
  - Solution
    - G ---- two disjoint sets {a, b, d} and {c, e, f, g}
    - H ---- not bipartite

4. Bipartite Graphs (二分图)

- Theorem 4 (page 603)
  - A simple graph is bipartite if and only if it is possible to assign one of two colors to each vertex of the graph so that no two adjacent vertices are assigned to the same color.
  - Example 12
    - Use Theorem to determine whether the graphs in Example 11 are bipartite.

4. Bipartite Graphs (二分图)

- Example 13 (Complete Bipartite Graphs, page 604)
  - The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex set is in the second subset.
  - The complete bipartite graph $K_{2,3}$, $K_{3,3}$, $k_{3,5}$, and $k_{2,6}$ are displayed in Figure 9.

5. New Graphs from Old

- Definition 6 (page 607)
  - A subgraph (子图) of a graph G=(V, E) is a graph H=(W, F) where $W \subseteq V$ and $F \subseteq E$.
- Example 17
  - The graph G shown in Figure 14 is a subgraph of K_5.

5. New Graphs from Old

- Definition 7 (page 608)
  - The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.
  - The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.
- Example 18 (page 608)
  - Find the union of the graphs G_1 and G_2 in Figure 16(a).
  - Solution:
    - See page 554.

Homework

- Page 608~611
  - 6, 20, 21~25, 26, 34, 36(b)(d)(f)(h),46