A Theory of Arbitrage Capital[1]

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Abstract

Fire sales that occur during crises beg the question of why *sufficient* outside capital does not move in quickly to take advantage of fire sales, or in other words, why outside capital is so "slow-moving". We propose an answer to this puzzle in the context of an equilibrium model of capital allocation. There are states of the world in which asset prices fall low enough that it is profitable to carry liquid capital to acquire assets in such states. Set against this, keeping capital in liquid form entails costs in terms of foregone profitable investments. We show that a robust consequence of this trade-off between making investments today and waiting for arbitrage opportunities in future is the combination of occasional fire sales and limited stand-by capital. When there are learning-by-doing effects, such stand-by capital moves in to acquire assets only if fire-sale discounts are sufficiently deep. An extension of our model to several types of investments gives rise to a novel channel for contagion where sufficiently adverse shocks to one type can induce fire sales in other types that are fundamentally unrelated, provided these investments are arbitraged by a common pool of capital. (*JEL* G21, G28, G38, E58, D62)

Our understanding of financial crises has been enhanced by a large and rapidly growing empirical literature that has documented the incidence and severity of fire sales by distressed parties in a wide range of asset classes.[1] Indeed, it would not be too much of an exaggeration to say that fire-sales have been a defining feature of most financial crises, including the most recent crisis of 2007-09.

The term "fire sale" carries the connotation that assets are being sold at prices that are below some benchmark, fair fundamental price that would prevail in the absence of a crisis. However, the notion that assets are being sold at prices below their fundamental value begs an important question. How can fire sales take place in a world where arbitrage capital waits on the sidelines to take advantage of artificially low prices? If there were such arbitrageurs who wait on the sidelines, would they not compete with each other as soon as the crisis erupts, providing a cushion for prices? As well as these "positive" questions on the nature of the equilibrium outcome, there are also important normative questions on the social value of arbitrage capital. Is the equilibrium provision of arbitrage capital at the efficient level? If not, is the efficient level higher or lower than the equilibrium level? Our paper tries to answer these questions in a setting that is simple and transparent enough that helps uncover

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[1]Fire sales have been shown to exist in distressed sales of aircrafts by Pulvino (1998), in cash auctions in bankruptcies by Stromberg (2000), in creditor recoveries during industry-wide distress especially for industries with high asset-specificity by Acharya, Bharath and Srinivasan (2007), in equity markets when mutual funds engage in sales of similar stocks by Coval and Stafford (2007), and in an international setting where foreign direct investment increases during emerging market crises to acquire assets at steep discounts in the evidence by Krugman (1998), Aguiar and Gopinath (2005), and Acharya, Shin and Yorulmazer (2012).

the underlying economic mechanisms at work.

We show that the answers to both the positive and normative sets of questions rely on the interplay between two underlying allocative mechanisms in the economy. One operates at the ex post stage, and has to do with the efficient allocation of assets to those economic agents that can generate most value from them. The second operates at the ex ante stage, and has to do with how much of the economy's resources are set aside in the form of "idle" arbitrage capital that waits on the sidelines.

In our baseline set-up, engaging in production entails greater expertise to insiders through learning-by-doing effects, and these insiders are the natural holders of the assets in the sense of being able to generate greater value from them compared to outsiders who take over distressed assets of illiquid insiders. The greater is the discount in the value realized by outsiders, the more severe must be the fire sale before outsiders enter to cushion the distress. Thus, for a finite pool of arbitrage capital at the ex post stage, there are states of the world with possibly steep price discounts, which then create the incentive to hold unproductive "idle" arbitrage capital at the ex ante stage. In equilibrium, the two choices – whether to invest in profitable activities or to set aside funds for arbitrage in the future – earns the same rate of return when viewed *ex ante.* As a consequence, limited provision of arbitrage capital and fire sales emerge as robust features of the equilibrium. However, there are two important normative consequences of fire sales. First, there is ex post inefficiency due to assets being held by outsiders in some states of the world. Second, there is also ex ante inefficiency due to the ex ante allocation of resources to arbitrage capital.

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There are also implications for the depth of fire sales following long periods of low liquidity risk (good times). During periods when liquidity risk is low, illiquid projects are more attractive relative to holding cash balances. Hence, a higher fraction of agents choose to become insiders and there is less liquid capital put aside. As a result, when liquidity shocks do materialize fire-sale effects in asset prices are more severe as there was less liquid capital put aside for arbitrage. This can potentially explain why crises that erupt after good times are associated with sharper, more severe fire sales.[2]

The picture that emerges from our analysis is that private incentives lead to the over-provision of arbitrage capital relative to its socially optimal level. Intuitively, in states where there are few liquid insiders, asset prices must fall so that the market clears. Nevertheless, there is allocative efficiency as assets remain in the hands of insiders. The presence of arbitrage capital interferes with the efficient allocation. Ex post, there is inefficiency if arbitrageurs are less efficient than insiders in deploying assets. Importantly, even if arbitrageurs are as efficient as insiders, setting aside arbitrage capital ex ante implies passing over profitable investment opportunities. Somewhat counter-intuitively perhaps, relative to

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[2]Conversely, as economic times worsen, more capital is set aside for arbitrage. For instance, according to the article titled "Cashing in on the crash" in the *Economist* on August 23, 2007, vulture funds raised $15.1 billion in the first seven months of 2007, more than the $13.9 in all of 2006, to take advantage of fire sales due to expected distress in financial markets. The same article points out that while some hedge funds suffered, the others, such as Citadel, Ellington, and Marathon Asset Management had the ready cash. The article highlights the strategy of Citadel to keep more than a third of its assets in cash or liquid securities, allowing it to take advantage of fire sales when opportunities arise.

the competitive outcome, the social optimum features *lower* asset prices ex post and *greater* profitable investments ex ante.

Since our welfare results arise from the interplay between the ex post asset allocation and the ex ante provision of arbitrage capital, one important message from our paper is that the opening of capital markets in the ex post period where liquid insiders can raise new funding from outsiders does not eliminate the inefficiency, although there is some mitigation of the inefficiency. We demonstrate this feature in an extension of our framework where we allow insiders to raise additional funding from outsiders by selling financial claims. Since outsiders have the choice between acquiring physical assets or buying financial claims sold by insiders, arbitrage capital is allocated in such a way that the returns are equalized. The result is that fire-sale discounts in prices for acquisition of assets must equal that for provision of external finance, giving rise to a spillover or contagion from illiquidity in the market for real assets to that for financing of these assets. From the welfare standpoint, ex post efficiency of allocation is restored as arbitrageurs simply fund asset purchases of insiders, but they make same profits ex ante due to discounts they charge in funding insiders. As a result, it remains profitable for there to be some arbitrage capital in equilibrium and there continues to be some ex ante inefficiency in investment decisions.

In another extension of our benchmark model, we show that contagion can result across different asset classes whenever the provision of arbitrage capital in these asset markets is from a common pool; returns on different investments in the portfolio of an arbitrageur must be the same. The fact that the quantity of arbitrage capital is limited implies that these returns

are positive in all markets where arbitrageurs allocate capital.[3] This channel of contagion operates even though the fundamentals of two assets are independent and the existence of fire sales in one asset can give rise to fire sales in the other.

Related literature: Fire sales are, of course, not new to our paper. The idea that asset prices may contain liquidity discounts when potential buyers are financially constrained and assets are not easily redeployable were discussed by Williamson (1988) and Shleifer and Vishny (1992). This early literature suggests that firms, whose assets tend to be *specific* (that is, whose assets cannot be readily redeployed by firms outside of the industry) are likely to experience lower liquidation values because they may suffer from fire-sale discounts in cash auctions for asset sales, especially when firms within an industry get simultaneously into financial or economic distress. Since then, fire sales have often figured in models of crises (Allen and Gale, 1994, 1998, among others). Intimately tied to the notion of fire sales is the idea that arbitrageurs wanting to buy assets at steep discounts may also face financing frictions due to principal-agent problems. The resulting "limits of arbitrage" (Shleifer and Vishny, 1997) can entrench fire-sale prices for a period of time once they materialize.[4]

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[3]For (apparent) "dislocations" between different capital markets and the effect of liquidations in one market on prices in another, see an excellent discussion of the large body of extant empirical evidence in Duffie and Struvolici (2008). For similar evidence in an international setting, see Rigobon (2002) and Kaminsky and Schmukler (2002), and the discussion in Pavlova and Rigobon (2007). The literature by and large attributes such dislocations to investment-style restrictions or limited arbitrage capital.

[4]Mitchell, Pedersen and Pulvino (2007) provide compelling episodic evidence for the fact that capital appears to be "slow moving" when it enters markets affected by fire-sale discounts in prices.

Our contribution relative to this earlier literature is to focus on the ex ante decisions of investors, which much of the literature takes as given, and thereby to explain the origins of the limited nature of arbitrage capital as an equilibrium phenomenon. In this sense, our work is closest to the analysis by Allen and Gale (2004) of the portfolio choice of banks between holding safe versus risky assets. Gorton and Huang (2004), another closely related paper, also considers the equilibrium portfolio choice of firms, deriving that it is socially inefficient to hold large quantities of safe assets required to avoid fire sales, and studying in this context the role of government bailouts during crises. Our framework is tractable and facilitates crisp conclusions on key comparative statics and welfare questions. In particular, our result that arbitrage capital is endogenously lower in good times, and therefore, that crises arising right after good times feature deeper fire-sale discounts, is a noteworthy result, which (to our knowledge) has not been discussed so far in the literature.

Another advantage of our tractable framework is to open up for scrutiny the arbitrageurs' access to different real and financial markets, and thereby identify a channel of contagion that relies purely on the limited nature of arbitrage capital. Our results on this front are closest to Gromb and Vayanos (2007) and Duffie and Struvolici (2008). Gromb and Vayanos (2007) consider arbitrageurs exploiting fire-sale opportunities across markets and this equilibrates returns they can earn in different markets. Duffie and Struvolici (2008) study a similar setting taking the financing friction of arbitrageurs as given. In both of these papers, the quantity of equilibrium arbitrage capital is exogenous, whereas our central theoretical concern is to endogenize the quantity of arbitrage capital and illustrate that its limited quantity as well as

its limited expertise make fire sales a robust equilibrium phenomenon.[5]

Rampini and Viswanathan (2010) consider a dynamic contracting set-up for exploring which firms make investments and which preserve debt capacities for the future. This question is related to our analysis. While we model fire sales as an outcome from market clearing when buyers are financially constrained, Rampini and Viswanathan model asset prices as being temporarily low due to low cash flow realizations. Acharya, Shin and Yorulmazer (2011) also consider gains from acquiring assets at fire-sale prices which make it attractive for firms (in their model, banks) to hold liquid assets ex ante, but that when such fire-sale states are not too likely, banks hold too little liquidity due to the asset-substitution problem. Their focus is on analyzing how government or central bank interventions to resolve banking crises (that may be desirable ex post) affect ex ante liquidity in potentially adverse ways.[6] Bolton, Santos, and Scheinkman (2011) and Diamond and Rajan (2011) present models wherein once an adverse state of the world arises, decisions of individual banks affect when assets get sold - right away or with delay. Immediate sales can increase returns to holding cash and lead to potentially excessive cash hoarding ex ante. In contrast to these models, our paper is more in the spirit of Allen and Gale (1994, 1998) and Gorton and Huang (2004) in that once the adverse state arises, there is an immediate (fire) sale of assets.

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[5]Note that contagion has been derived in many other settings through portfolio flows (Kodres and Pritsker, 2002) or utility-based assumptions (Kyle and Xiong, 2001).

[6]Huang and Wang (2010) also show that competitive market forces fail to lead to an efficient supply of liquidity. The market provision of liquidity is generally too low when the probability of a liquidity event is small and is too high when the probability of a liquidity event is large.

Our study is also related to the seminal work of Kiyotaki and Moore (1997) on credit cycles. In Kiyotaki and Moore (1997) and Krishnamurthy (2003), the underlying asset cannot be pledged because of inalienable human capital. Krishnamurthy (2003) differs from Kiyotaki and Moore (1997) in that all contingent claims on aggregate variables are subject to collateral constraints. However, land can be pledged and has value both as a productive asset and as collateral. Caballero and Krishnamurthy (2001) employ a Holmstrom and Tirole (1998) approach with exogenous liquidity shocks and allow firms to post collateral in a manner similar to Kiyotaki and Moore.

Finally, there are models in which there are pecuniary externalities from fire sales of assets. Lorenzoni (2008), for example, considers a competitive model of intermediaries in which ignoring these externalities leads to excessive borrowing ex ante and excessive volatility ex post. In our model, there are no externalities from fire sales and the result is that the competitive equilibrium features too much setting aside of idle capital ex ante for the purpose of undertaking arbitrage ex post.

1 Model

The timeline of the model is provided in Figure 1. There are three dates indexed by $t \in \{0, 1, 2\}$. There is a unit measure of risk-neutral agents. Each agent is endowed with 1 unit of the consumption good at $t = 0$.

There are two types of assets in the economy. There is a storage technology, referred to

as *cash*, that allows any agent to transfer one unit of the consumption good from date t to date $t + 1$. In this sense, the risk-free interest rate is zero.

There is also an (partially) illiquid investment opportunity, referred to as the *asset*, that agents can undertake at date 0. The investment opportunity is indivisible and needs the full unit of the consumption good as the input. The investment pays a return of R_t at $t = 1, 2$. For simplicity we assume that $R_t = R$ for $t = 1, 2$, and $R > 1/2$.[7]

Agents decide whether or not to undertake the illiquid investment at date 0. Those who decide to invest are known as *insiders*. Those who choose to keep their wealth in the storage technology are known as *arbitrageurs*. Insiders can be hit by a liquidity shock at $t = 1$, in which case they need to consume at $t = 1$ and want to convert their return of R at date 2 into cash at date 1. We call the insiders that get hit by a liquidity shock at $t = 1$ as *illiquid insiders*. Insiders that are not hit by the liquidity, denoted as *liquid insiders*, and the arbitrageurs consume at $t = 2$.[8]

The probability that an insider is hit by the liquidity shock is given by k. However, we allow aggregate uncertainty in the economy. When viewed from date 0, the probability

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[7]Alternatively, we could assume that the illiquid project pays off only at $t = 2$, that is, $R_1 = 0$. This would not change our results qualitatively. However, in our setup, insiders that are not hit by the liquidity shock can use the return at $t = 1$ for acquiring the assets of insiders that get hit by a liquidity shock and this adds richness to our model.

[8]We could assume that the arbitrageurs can get hit by a liquidity shock at $t = 1$ as well. This would not change our results qualitatively as long as arbitrageurs are less likely to be hit by the liquidity shock compared to the insiders. However, it would make it less attractive to become an arbitrageur since the only source of profit for arbitrageurs is the fire-sale of the assets of illiquid insiders.

k itself is uncertain. Nature first draws k from a known density $f(k)$ that is continuous over $[0, 1]$, and then determines the realizations for each agent as independent and identically distributed (i.i.d.) draws from coin tosses where the probability of a liquidity shock is fixed at k. By the law of large numbers, the proportion of illiquid insiders is exactly k, but this proportion is uncertain at the time of the investment. This aggregate uncertainty plays a key role in our model.

1.1 Insiders and Arbitrageurs

We will denote the proportion of agents who choose become arbitrageurs by w.

We assume that, by the nature of the investment, insiders benefit from learning-by-doing, so that insiders become proficient in managing the asset. This is an important feature of our model, as the role of arbitrageurs is double-sided. Although they stand on the sidelines ready to purchase the assets of illiquid insiders, they are not natural holders of the asset, and there are social costs as a result of their ownership of the assets, as we will describe below.

Proportion k of insiders turn out to be illiquid, and by date 1, the identity of the liquid and illiquid insiders is known. Illiquid insiders want to consume at $t = 1$ and try to convert their return of R at $t = 2$ into cash at $t = 1$. They do this by putting up their future return into the asset market in return for cash at $t = 1$, where the potential buyers of these assets are liquid insiders and arbitrageurs.

However, learning by doing matters for the terminal return from the asset. Between date 1 and date 2 (i.e. after the realization of the liquidity shocks, but before the realization of the

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return at $t = 2$), the return from the asset depends on who holds the asset. If the asset is held by an insider, the asset can be managed well due to the expertise gained by the insider in the initial period of production. In the hands of the insider, the terminal value of the asset at date 2 is given by $\bar{p} = R$. However, if the asset is run by an arbitrageur between date 1 and 2, the upkeep of the asset is not as good, so that the terminal value of the asset is given by $\underline{p}$ where

$$\underline{p} = R - \Delta, \tag{1}$$

and $\Delta \in [0, R]$ is the assumed to be the loss in the asset value in the hands of the arbitrageurs.

Our assumption that $\underline{p} < \bar{p}$ is motivated by the empirical evidence that during major financial crises, the entry of outside investors takes place only when incumbents face severe financial distress and lack the resources to take over failing rivals. The resolution of distressed banks is perhaps the clearest illustration of the comparative advantage of insiders in managing the assets. When faced with the imminent failure of a bank, regulators turn to other (healthier) banks to take over the ailing rival. It is only when potential acquiring banks are themselves distressed that regulators turn to outside investors. For these reasons, we believe that the case where $\underline{p} < \bar{p}$ is the natural one to examine in our model.

Nevertheless, it should be emphasized that our framework is rich enough to accommodate the alternative scenario where $\underline{p} > \bar{p}$, so that the outsiders have greater expertise and are able to generate more value than the insiders. We examine this case and show that the welfare results are made more subtle. The important point is that even with greater expertise ex post, the overall welfare calculation must take account also of the ex ante inefficiency associated

with "idle" arbitrage capital, and the greater expertise of outsiders may not eliminate the inefficiency of arbitrage capital.

Returning to our benchmark model, we continue under the assumption that $\underline{p} < \bar{p}$. A natural regularity condition in this context is that the ex ante productivity of investment justifies the cost of investment, namely $R > 1/2$ as assumed earlier. This condition states that the investment is not unjustified even under the most optimistic scenario where all the assets of illiquid insiders end up in the hands of liquid insiders.

1.2   Welfare

The welfare function is defined as the unweighted sum of the ex ante payoffs of all agents. In our framework, the welfare function is the expectation at date 0 of the total consumption across all agents - both insiders and arbitrageurs. Note that welfare takes account only of the total consumption rather than the distribution of consumption across agents.

Denote by y_a the mass of assets held by arbitrageurs from $t = 1$ to $t = 2$. Then, the total assets held by the insiders are $(1 - w) - y_a$. Thus, interim welfare can be written as

$$\Pi\left(w, k\right) = \left(1 - w\right)\left[\left(1 - k\right)2R + k(R + \bar{p})\right] + w - \Delta \cdot y_a = \left(1 - w\right)2R + w - \Delta \cdot y_a \quad (2)$$

We can write ex ante welfare as the expectation of (2) with respect to the realization of the aggregate shock k, that is,

$$\Pi = E_k\left[\left(1 - w\right)2R + w - \Delta \cdot y_a\right] \quad (3)$$

From (3) we see that ex ante welfare is decreasing in y_a for any fixed value of w. The key

question is how Π behaves as a function of w. Since $R > 1/2$, we know that the risky asset has a higher return than the safe asset. Hence, ex ante welfare is decreasing in w so that $w = 0$ is the unique socially optimal level of w. This gives us the following proposition.

Proposition 1 *For $R > 1/2$, $w = 0$ is the unique socially optimal level of arbitrage capital.*

As noted already, any outcome in which the asset ends date 2 in the hands of an arbitrageur entails a welfare cost of Δ, and hence is not socially optimal. Two conditions are necessary and sufficient for the social optimum - namely, that the interim output at date 1 is maximized, and no assets end up in the hands of arbitrageurs. When $w = 0$, both conditions are satisfied. There are no arbitrageurs at date 1, and so all the assets end up in the hands of insiders only. Meanwhile, for $R > 1/2$, the interim output is maximized when $w = 0$. Hence, the outcome with no arbitrageurs is the unique socially optimal level of arbitrage capital.

In the next section, we characterize the equilibrium and show that the social optimum is never an equilibrium outcome. Thus, there is a strict separation between the *socially optimal outcome* and the *equilibrium outcome*.

2   Benchmark Equilibrium

Having characterized the socially optimal outcome, we now consider the competitive equilibrium. In the benchmark model, we assume that the liquid insiders cannot raise additional funding, and must rely solely on the resources from the successful project. Thus, until further

notice, we operate under the following assumption.[9]

Assumption 1. The insiders cannot raise outside capital and each liquid insider only has R units of cash that can be used to purchase distressed assets.

The payoffs of the model are determined through a competitive auction of illiquid insiders' assets at the interim date. We will then solve the model backward, by first considering the sale of illiquid insiders' assets and the resulting asset prices, and next, analyzing the ex ante choice to become insiders or arbitrageurs.

2.1 Asset Sales and Liquidation Prices

We keep track of two key features in the purchase of illiquid insiders' assets. First, arbitrageurs, using arbitrage capital, and the liquid insiders, using their first-period return, compete to purchase these assets. Second, liquid insiders may not have enough resources to acquire all illiquid insiders' assets. To focus on the interplay between these two features, we model asset sales as follows.

(i) All illiquid insiders' assets are pooled and competitively auctioned to the liquid insiders and arbitrageurs as described below.

(ii) The liquid insiders and arbitrageurs submit a demand schedule $y_i(p)$ that specifies the quantity demanded for illiquid insiders' assets for each price p. The index i belongs in $[0, (1-w)(1-k)]$ if i is a liquid insider, while $i \in [1-w, 1]$ if i is an arbitrageur.

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[9]We relax this assumption in Section 3.

(iii) We assume that insiders cannot raise additional financing. Hence, the resources available to each liquid insider for purchasing illiquid insiders' assets is the payoff R at $t = 1$ from the asset.

(iv) The price p clears the market, where assets allocated to liquid insiders and arbitrageurs add up at most to the proportion of illiquid insiders:[10]

$$\int_0^{(1-w)(1-k)} y_i(p)di + \int_{1-w}^1 y_i(p)di \leq (1-w)k. \tag{4}$$

(v) We pin down the price p by focusing on the symmetric case where all liquid insiders submit the same schedule, that is, $y_i(p) = y(p)$ for all $i \in [0, (1-w)(1-k)]$, and all arbitrageurs submit identical schedules, that is, $y_i(p) = y_a(p)$ for all $i \in [1-w, 1]$.

To solve for the competitive allocation, we first derive the demand schedule for liquid insiders. The expected profit of a liquid insider from the asset purchase is $y(p)[\bar{p} - p]$. The liquid insider wishes to maximize this profit subject to the budget constraint:

$$y(p) \cdot p \leq R. \tag{5}$$

Hence, for $p < \bar{p}$, liquid insiders are willing to purchase the maximum amount of assets using their resources. Thus, the optimal demand schedule for liquid insiders is

$$y(p) = \frac{R}{p}. \tag{6}$$

For $p > \bar{p}$, the demand is $y(p) = 0$, and for $p = \bar{p}$, $y(p)$ is infinitely elastic. In words, as long as purchasing assets is profitable, a liquid insider wishes to use up all its resources to purchase assets.

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[10]Since no insider asset is scrapped, the equation holds with equality in equilibrium.

We can derive the demand schedule for arbitrageurs in a similar way. Note that, arbitrageurs value these assets at $\underline{p}$. For $p < \underline{p}$, arbitrageurs are willing to supply all their funds for the asset purchase. Thus, their demand schedule is

$$y_a(p) = \frac{1}{p}. \tag{7}$$

For $p > \underline{p}$, the demand is $y_a(p) = 0$, and for $p = \underline{p}$, $y_a(p)$ is infinitely elastic.

Next, we characterize how illiquid insiders' assets are allocated and the price function that results. The equilibrium price function is also illustrated in Figure 2.

Lemma 1 *The equilibrium price function for liquidated assets $p^*(k)$ is given as follows:*[11]

$$p^*(k) = \begin{cases} \bar{p} & for & k \leqslant \underline{k} \\[2ex] \frac{(1-k)R}{k} & for & k \in (\underline{k}, \bar{k}] \\[2ex] \underline{p} & for & k \in (\bar{k}, \bar{\bar{k}}] \\[2ex] \frac{(1-k)R}{k} + \frac{w}{(1-w)k} & for & k > \bar{\bar{k}} \end{cases}, \tag{8}$$

where $\underline{k}$, $\bar{k}$, and $\bar{\bar{k}}$, are given respectively by equations (27), (28), and (29), in Appendix I. Arbitrageurs acquire assets whenever $k > \bar{k}$. Finally, as arbitrage capital w increases, the price weakly increases, that is, $\frac{dp^*}{dw} \geqslant 0$.

We know that in the absence of financial constraints, the efficient outcome is to sell all assets to liquid insiders. However, liquid insiders may not be able to pay the threshold price

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[11]Note that p^* depends on w as well as k, that is, we have $p^*(k, w)$. To simplify notation we use $p^*(k)$.

of $\underline{p}$ for all assets. If price falls further, buying these assets becomes profitable for arbitrageurs and they participate in the auction, resulting in misallocation of assets whenever $\Delta > 0$.

Specifically, in the first region, that is, for $k \leqslant \underline{k}$, the number of failures is small and liquid insiders have enough liquidity to acquire assets at the full price $\bar{p}$.

For moderate proportion of failures, that is, for $k \in (\underline{k}, \bar{k}]$, however, liquid insiders can no longer pay the full price for all assets but can still pay at least the threshold value of $\underline{p}$, below which arbitrageurs have a positive demand. In this region, liquid insiders use all available funds and the price falls as the proportion of failures increases. This effect comes from cash-in-the-market pricing, as in Allen and Gale (1994, 1998), and is akin to the industry equilibrium hypothesis of Shleifer and Vishny (1992) who argue that when industry peers of a firm in distress are financially constrained, the peers may not be able to pay a price for assets of the distressed firm that equals the value of these assets to them.

However, as the proportion of illiquid insiders increases even further, liquid insiders cannot pay the threshold price of $\underline{p}$ for all assets and profitable options emerge for arbitrageurs. Hence, arbitrageurs are willing to supply their funds for the asset purchase. With the injection of arbitrageurs' funds, prices can be sustained at $\underline{p}$ until a critical proportion of failures $\bar{\bar{k}}$.

In the extreme, the number of failures may be so large that even the injection of arbitrageur capital is not enough to sustain the price at $\underline{p}$. This can be considered as an aggregate shortage of liquidity in that there is cash-in-the-market pricing even when all liquidity in the economy is channeled for asset purchases.

Note that the resulting price function is downward-sloping in the proportion of illiquid

insiders k in two separate regions. In the first downward-sloping region, arbitrageurs have not yet entered the market $(k \in (\underline{k}, \bar{k}])$ and there is cash-in-the-market pricing given the limited funds of liquid insiders. In the second downward-sloping region $(k > \bar{\bar{k}})$, even the funds of arbitrageurs are not enough to sustain the price at $\underline{p}$, their highest valuation of assets.

2.2 Inefficiency of Equilibrium

Insiders' expected profit, denoted by $E(\pi)$, consists of profit from their own investments, profit from asset purchases, and the amount they recover for their assets when they are hit by the liquidity shock at $t = 1$, which can be derived using the price in equation (8). In particular, we have that the profit of each insider is

$$E(\pi) = E\left[(1-k)R\frac{\bar{p}}{p^*} + kp^* + R - 1\right],\tag{9}$$

where E denotes expectation over k.

In contrast, the only source of profit for arbitrageurs is the asset purchase at fire-sale prices. Therefore, we have that the profit of each arbitrageur is

$$E(\pi_a) = E\left[\max\left\{0, \left(\frac{\underline{p}}{p^*} - 1\right)\right\}\right].\tag{10}$$

In the competitive equilibrium, the capital allocation (characterized by arbitrage capital w) must be such that the two payoffs are equalized at the ex ante stage, so that

$$E(\pi) = E(\pi_a),\tag{11}$$

as otherwise, there is an incentive for some insiders to become arbitrageurs instead or vice-versa. Then, the following proposition formally characterizes agents' choices. Under a tech-

nical condition that "the distribution $f(k)$ not converge to zero too rapidly as k goes to 1" (or in other words, that there is a sufficiently "thick tail" that there will be a large number of failures and arbitrageurs will make profits), we obtain the following proposition.

Proposition 2 *Let $f(k)$ be a continuous probability distribution over $[0, 1])$ with an order of k less than 1. In the competitive equilibrium, a proportion $w^* \in \left(0, \frac{p}{1+\underline{p}}\right)$ of agents choose to become arbitrageurs, where w^* satisfies the indifference equation in (11). Further, $\overline{\overline{k}} < 1$, so that there are states of the world where $p^* < \underline{p}$.*

Formally, for $w = 0$, we have $\underline{p}/p = \left(\underline{p}/R\right)\left(\frac{1}{1-k} - 1\right)$, which converges to $+\infty$ as k converges to 1. Even though $f(k)$ can converge to 0 as k converges to 1, as long as $f(k)$ has an order of k less than 1, we have that in the limit the expected arbitrage profits are "too large" when there are no other arbitrageurs in the market: $\lim_{k \to 1} \left(\underline{p}/p\right) f(k) = +\infty$. And this cannot be an equilibrium. For distribution f that has an order of k greater than 1 and $\lim_{k \to 1} f(k) = 0$, we have $\lim_{k \to 1} \left(\underline{p}/p\right) f(k) = 0$, so that, for $w = 0$, we have $E\left(\pi_a\right) < +\infty$. Under such probability distributions, it is possible to have no arbitrage capital ($w = 0$) in equilibrium. For example $f(k) = c\left(1 - k\right)^a$, for $k \in [0, 1]$ and $a > 1$, would give such a result. Nevertheless, the case of interest for our analysis is the one where there is some arbitrage capital in equilibrium, and the technical condition in the Proposition shows that this is robustly the case. Note also that there cannot be an equilibrium with just arbitrageurs ($w = 1$) as there would be no profits for arbitrageurs due to lack of any fire sales.

Hence, we focus for the rest of our analysis on the equilibrium when the fractions of agents who choose to become insiders and arbitrageurs are bounded away from 0. An important

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implication is that cash-in-the-market prices are robust to the endogenous choice of arbitrage capital. That is, there will always be states of nature where the price falls not only below the fundamental value of $\bar{p}$ but also below $\underline{p}$, the value arbitrageurs attach to these assets. In these states, there is an aggregate shortage of liquidity as all capital with insiders and arbitrageurs is not sufficient to keep the asset prices above or equal to $\underline{p}$ which is necessary for efficient ex-post allocation of assets. This is a robust feature of our model. In order for there to be arbitrage capital in equilibrium, there must be states of the world where arbitrageurs make profits. In these states prices are below the arbitrageurs' valuation of assets. And, this is indeed the case in equilibrium.

2.3 Comparative Statics

Next, we formalize some comparative statics features of our model in the benchmark case under Assumption 1.

We have the following relationships between the level of arbitrage capital and (i) liquidity risk proxied by the aggregate distribution of liquidity shocks, and (ii) asset specificity.

Proposition 3 *Equilibrium level of arbitrage capital w^* satisfies two features:*

(i) *Suppose f and g are two probability densities for k, where f dominates g in the sense of first-order stochastic dominance. Let w_f^* and w_g^* be the equilibrium level of arbitrage capital under densities f and g, respectively. Then, $w_f^* > w_g^*$.*

(ii) *Let $\widehat{w} = \frac{R\underline{p}\Delta}{R\underline{p}\Delta + R\bar{p} + \underline{p}^2}$. For $w^* < \widehat{w}$, as the difference of expertise between insiders and*

arbitrageurs widens the equilibrium proportion of arbitrageurs decreases, that is, $\frac{dw^*}{d\Delta} < 0$.

Consider (ii) first. As the difference between the expertise levels of insiders and arbitrageurs widens (i.e., as insiders' assets become more specific), the return arbitrageurs make from these assets decreases. In turn, the region over which arbitrageurs enter the market shrinks. Thus, asset specificity reinforces fire-sale discounts in prices further.

Next, consider (i). During good times, it is more likely that illiquid projects perform well. The decrease in liquidity risk has two effects on agents' choice that go in the same direction. First, the expected return from being an insider increases. Also, the proportion of illiquid insiders decreases, which limits the fire-sale opportunities for arbitrageurs. Hence, during good times, we would expect a higher fraction of agents to become insiders and take illiquid projects and a smaller fraction to set aside capital for arbitrage.

Furthermore, from the price function in equation (8), we know that as the fraction of arbitrageurs w^* decreases, we observe bigger deviations in the price of illiquid insiders' assets from the fundamental value of $\bar{p}$. Hence, a corollary of Proposition 3 is that when adverse liquidity shocks arise during good times, fire-sale effects in asset prices are more severe, resulting in lower asset prices and higher price volatility. This result is a novel contribution of our analysis and provides one explanation for why crises that follow good times are associated with greater asset price deterioration.[12]

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[12]Acharya and Viswanathan (2011) build an alternative explanation in a model where there is greater entry of poorly-capitalized institutions when fundamentals are stronger, but in their model insiders serve as

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Corollary 1 *Adverse liquidity shocks during good times measured by high values of k result in bigger deviations in the price of illiquid insiders' assets from the fundamental value of $\bar{p}$, that is, $(\bar{p} - p^*(k))$ increases.*

2.4   Discussion: Efficiency of arbitrage capital

We have the stark contrast between Proposition 2, which states that the equilibrium level of arbitrage capital is strictly positive, and Proposition 1 which states that the socially optimal level of arbitrage capital is zero. In fact, the result that zero arbitrage capital is socially optimal holds even when arbitrageurs are as efficient as insiders ($\Delta = 0$) in running the assets. This is because even though there is no ex-post allocation inefficiency in this case, profitable opportunities are passed ex ante as capital remains idle waiting for arbitrage opportunities that do not create any social welfare.

Thus, for arbitrage capital to have social value, there has to be appeal to other rationales. One candidate is risk aversion, which would introduce a motive to reduce the price fluctuations across states of the world (Allen and Gale, 2004, 2005). Nevertheless, even with risk aversion, the ex ante gains from risk-sharing have to be sufficiently large that it swamps the productive inefficiency. Any *presumption* that arbitrage capital has social value must thus be justified.

An alternative channel through which arbitrage capital may have value is to moderate amplifying effects of financial distress whenever some fragility exists in the economic system that triggers snowball effects (e.g., due to marking-to-market constraints as in Cifuentes, Fer-

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arbitrageurs and there is no arbitrage capital set aside in equilibrium.

ucci and Shin, 2005). In such a context, mitigating the initial shocks through the cushioning effect of arbitrage capital could have substantial welfare benefits. However, as with the case for risk-aversion, the final assessment should be based on a comparison of the magnitudes, and any presumption one way or the other would be unjustified.

Finally, another rationale for arbitrage capital is that the arbitrageurs are experts in managing distressed assets. In such a set-up, arbitrageurs are willing to pay a higher price for illiquid insiders' assets and they will be the first to acquire these assets. In this case, while investing in the liquid asset yields lower returns compared to the risky investment, it allows arbitrageurs – as take-over experts – to acquire illiquid insiders' assets and generate higher returns from these distressed assets compared to the insiders. We sketch this version below.

2.4.1 Arbitrageurs as take-over experts

One potential interpretation of arbitrageurs can be that they may be experts in taking over and managing distressed assets, in which case, they value illiquid insiders' assets higher than liquid insiders. Hence, arbitrageurs are willing to pay a higher price for illiquid insiders' assets and they will be the first to acquire these assets. However, when arbitrageur funds are limited, for sufficiently large proportion of failures, prices fall and even though insiders are inefficient in managing distressed assets, they will acquire some of these assets.

Formally, let arbitrageurs generate a return of $\widehat{p} = \bar{p} + \delta$, with $\delta > 0$, from illiquid insiders' assets. Hence, for $p < \widehat{p}$, arbitrageurs are willing to supply all their funds for the asset purchase and their demand schedule is $y_a(p) = \frac{1}{p}$. For $p > \widehat{p}$, their demand is $y_a(p) = 0$, and

for $p = \widehat{p}$, $y_a(p)$ is infinitely elastic.

In the absence of financial constraints, the efficient outcome is to sell the assets to arbitrageurs. However, when arbitrage capital is limited, arbitrageurs may not be able to pay the price of $\widehat{p}$ for all assets and some of the illiquid insiders' assets get acquired by liquid insiders.

If the proportion of illiquid insiders is sufficiently small, arbitrageurs have enough funds to pay the full price $\widehat{p}$ for all assets. More specifically, for $k \leq \underline{k}_a$, where

$$\underline{k}_a = \frac{w}{(1-w)\widehat{p}}, \tag{12}$$

the auction price is $p_a^* = \widehat{p}$ and each arbitrageur is allocated a share $y_a(\widehat{p}) = \frac{(1-w)k}{w}$.

For moderate values of k, arbitrageurs cannot pay the price $\widehat{p}$ for all assets but can still pay at least $\bar{p}$, below which insiders have a positive demand. Formally, for $k \in (\underline{k}_a, \bar{k}_a]$, where

$$\bar{k}_a = \frac{w}{(1-w)\bar{p}}, \tag{13}$$

the price is set at $p_a^* = \frac{w}{(1-w)k}$, and all assets are acquired by arbitrageurs.

For $k > \bar{k}_a$, arbitrageurs cannot pay $\bar{p}$ for all assets, and liquid insiders supply their funds for the asset purchase. With the injection of insiders' funds, prices can be sustained at $\bar{p}$ until some critical proportion of failures $\bar{\bar{k}} \geqslant \bar{k}_a$. We obtain the resulting price function (also see Figure 3):

$$
p_a^*(k) = \begin{cases} \widehat{p} & \text{for} \quad k \leqslant \underline{k} \\[2ex] \frac{w}{(1-w)k} & \text{for} \quad k \in (\underline{k}, \bar{k}] \\[2ex] \bar{p} & \text{for} \quad k \in (\bar{k}, \bar{\bar{k}}] \\[2ex] \frac{(1-k)R}{k} + \frac{w}{(1-w)k} & \text{for} \quad k > \bar{\bar{k}} \end{cases}. \tag{14}
$$

Note that as the proportion w of agents that choose to become arbitrageurs increases, the boundaries $\underline{k}_a, \bar{k}_a$ and $\bar{\bar{k}}$ increase, as well as the price p_a^* in the second and the fourth regions. Hence, as w increases, the price p^* weakly increases, that is, we have $\frac{dp_a^*}{dw} \geqslant 0$.[13]

The social planner maximizes the expected total output generated by the economy:

$$
\Pi = E\left[w + (1-w)(1-k)R + y_I \bar{p} + y_A \widehat{p} \right], \tag{15}
$$

where y_I and y_A represent the units of illiquid insiders' assets acquired by insiders and arbitrageurs, respectively, and $y_I + y_A = (1-w)k$.[14]

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[13] Note that the equilibrium level of arbitrage capital is determined by the same condition as in the benchmark model, with the difference that the expected profit for arbitrageurs' is $E(\pi_a) = E\left[\frac{1}{p}(\widehat{p} - p) \right]$, and in equilibrium, a proportion $w^* \in \left(0, \frac{\widehat{p}}{1+\widehat{p}} \right)$ of agents choose to become arbitrageurs, where w^* satisfies the indifference equation in (11).

[14] Furthermore, we obtain:

$$
y_A = \begin{cases} (1-w)k & \text{for} \quad k \leqslant \underline{k} \\[2ex] (1-w)k & \text{for} \quad k \in (\underline{k}, \bar{k}] \\[2ex] \frac{w}{\bar{p}} & \text{for} \quad k \in (\bar{k}, \bar{\bar{k}}] \\[2ex] \frac{w(1-w)k}{(1-w)(1-k)R+w} & \text{for} \quad k > \bar{\bar{k}} \end{cases} \quad \text{and} \quad y_I = \begin{cases} 0 & \text{for} \quad k \leqslant \underline{k} \\[2ex] 0 & \text{for} \quad k \in (\underline{k}, \bar{k}] \\[2ex] (1-w)k - \frac{w}{\bar{p}} & \text{for} \quad k \in (\bar{k}, \bar{\bar{k}}] \\[2ex] \frac{(1-w)^2 k(1-k)R}{(1-w)(1-k)R+w} & \text{for} \quad k > \bar{\bar{k}} \end{cases}.
$$

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Note that the condition $R > 1/2$ is sufficient for the risky investment to have a higher expected return than the investment in the safe asset. While investing in the liquid asset yields lower returns compared to the risky investment, it allows arbitrageurs – take-over experts – to acquire illiquid insiders' assets and generate higher returns from these distressed assets compared to the insiders. Hence, for sufficiently high values of δ, that is, when arbitrageurs are sufficiently more efficient in running distressed assets compared to insiders, the forgone expected output from investing in the liquid assets is compensated by the efficiency gain from the take-over expertise of arbitrageurs. In that case, it would be socially optimal to set aside some arbitrage capital to acquire distressed assets.

While it is plausible in some contexts that arbitrageurs are able to generate more value than the insiders as in the extension we just considered, we note that such an assumption runs counter to the intuition that insiders gain by "learning-by-doing" so that they are the natural holders of the assets. As mentioned already, Acharya, Shin and Yorulmazer (2012) exhibit evidence that during emerging market crises, foreign arbitrageurs re-sell, or "flip", the assets they acquired during the fire-sale to local insiders, suggesting they were temporary owners of assets due to aggregate shortage of liquidity rather than permanent owners due to expertise. Importantly, even with greater expertise for arbitrageurs, the overall welfare calculation must take account also of the ex-ante inefficiency associated with "idle" arbitrage capital, and the greater expertise of arbitrageurs may not eliminate the inefficiency of arbitrage capital.

We now return to our benchmark model where arbitrageurs are inefficient relative to insiders and generalize it to a setting with capital markets.

3 Introducing a Capital Market

The inefficiency identified in the benchmark equilibrium reflects Assumption 1, which stated that the liquid insiders cannot raise outside capital and each only has R units of the consumption good that can be used to purchase distressed assets. It might appear that the inefficiency is somehow fragile to the introduction of a capital market, where we relax Assumption 1 and allow liquid insiders to sell claims to the arbitrageurs. However, this is not the case. It turns out that even when insiders can pledge all of their cash flows fully to arbitrageurs and raise "external financing" at date 1, the inefficiency persists. On the other hand, the introduction of a capital market mitigates the inefficiency, in a way to be made more precise below.

To see this, we relax Assumption 1 and allow liquid insiders to generate funds from arbitrageurs against the assets they acquire:

Assumption 2. The liquid insiders can raise outside capital by issuing shares to arbitrageurs and deploy it along with R units of cash that each liquid insider has to purchase assets.

In particular, liquid insiders issue shares, which is a claim on the return at $t = 2$ from a unit of illiquid insiders' assets they acquire or their own return at $t = 2$, to generate funds per unit of share issued. In general, we can assume that due to various imperfections such as asymmetric information, moral hazard, etc., insiders may not be able to fully pledge their future cash flows (à la Holmstrom and Tirole (1998)). However, we consider here the case

with full pledgeability as this gives liquid insiders the best chance to purchase liquidated

assets and stacks the odds against arbitrage capital being attractive ex ante.[15]

We denote as $q(k)$ the price of equity share in liquid insiders, purchased by arbitrageurs.

Hence, when the proportion of illiquid insiders is k, the amount of funding available with

the liquid insiders for the purchase of assets, including funds that can be generated against

returns from purchased assets, is given as:

$$L(k) = (1 - w)(1 - k)\left[R + sq(k)\right],\tag{16}$$

where s is the units of shares issued by each liquid insider, which must be less than or equal

to $1 + m$, the sum of the units of assets m acquired by each liquid insider and 1 for their

own return at date 2. Clearly, this total liquidity available with the liquid insiders for asset

purchases is higher compared to the benchmark case. As a result, the region over which we

observe cash-in-the-market pricing is smaller, i.e., it starts at a larger proportion of failures,

compared to the benchmark case (Lemma 1).

Now, we have two markets: one for assets of illiquid insiders and one for shares of liquid

insiders. To find the equilibrium prices and allocations in these two markets, we formally

state the optimization problem that liquid insiders and arbitrageurs face.

If a liquid insider issues s units of shares at the price $q(k)$ and purchases m units of assets

at the price $p(k)$, it makes an expected profit of $m\left(\bar{p} - p(k)\right) - s\left(\bar{p} - q(k)\right).$

Note that in any equilibrium, $q(k)$ cannot exceed $\bar{p}$. Thus, we have $q(k) \leqslant \bar{p}$, and liquid

insiders issue equity just enough for the asset purchase, not more. Using this, we can state a

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[15]Details of the case with partial pledgeability are available upon request.

liquid insider's maximization problem as:

$$\max_{m,s} \quad m\left(\bar{p} - p(k)\right) - s\left(\bar{p} - q(k)\right) \tag{17}$$

$$\text{s.t.} \quad s \cdot q(k) + R \geqslant m \cdot p(k)$$

$$s \leqslant 1 + m.$$

For $q(k) \leqslant p(k)$, liquid insiders cannot make positive profits by issuing equity to purchase assets. Thus, when $q(k) \leqslant p(k)$, $s = 0$ and $m = \frac{R}{p(k)}$. And when $q(k) > p(k)$, liquid insiders make positive profits from asset purchase using the funds they generate by issuing equity. Hence, they would like to issue as much equity as possible, that is, $s = 1 + m$.

We can state each arbitrageur's maximization problem in a similar way:

$$\max_{x,z} \quad x\left(\underline{p} - p(k)\right) + z\left(\bar{p} - q(k)\right) \tag{18}$$

$$\text{s.t.} \quad x \cdot p(k) + z \cdot q(k) \leqslant 1$$

where x and z represent the units of assets and the proportion of shares in liquid insiders purchased by arbitrageurs, respectively.

When the share price of liquid insiders, $q(k)$, is low compared to the price of illiquid insiders' assets, $p(k)$, arbitrageurs prefer to purchase shares of liquid insiders. However, if $p(k)$ instead becomes low compared to $q(k)$, then arbitrageurs may prefer to acquire the assets directly.

When $p(k) > \underline{p}$, arbitrageurs do not want to purchase assets and $x(q, p) = 0$. When

$p(k) < \underline{p}$, arbitrageurs choose x to maximize:

$$
\begin{aligned}
& x\left(\underline{p} - p(k)\right) + \left(\frac{1 - xp(k)}{q(k)}\right)(\bar{p} - q(k)) \\
= \ & x\left(\underline{p} - \frac{p(k)\bar{p}}{q(k)}\right) + \left(\frac{\bar{p}}{q(k)} - 1\right).
\end{aligned}
\tag{19}
$$

Thus, if $p(k) < \underline{p}$ and $\underline{p}\ q(k) > \bar{p}\ p(k)$, then arbitrageurs use all their funds for the asset purchase, that is $x = \frac{1}{p(k)}$. When $p(k) < \underline{p}$ and $\underline{p}\ q(k) < \bar{p}\ p(k)$, arbitrageurs use all their funds for the equity purchase, that is $z = \frac{1}{q(k)}$, and when $\underline{p}\ q(k) = \bar{p}\ p(k)$, arbitrageurs are indifferent between the equity and the asset purchase.

In equilibrium, demand for shares of liquid insiders and assets of illiquid insiders should equal their supply. Hence, we have the market clearing conditions:

$$
(1-w)(1-k)s \ = \ wz \qquad \text{(equity market)} \tag{20}
$$

$$
(1-w)(1-k)m + wx \ = \ (1-w)k \qquad \text{(asset market)} \tag{21}
$$

We concentrate on the equilibrium where the participation of arbitrageurs in the equity market is maximum, which results in the maximum price for assets. However, even in this case, we show that for a large proportion of failures, the share price of liquid insiders falls below $\underline{p}$.

The price functions for illiquid insiders' assets and for shares of liquid insiders are given as follows (illustrated in Figure 4):

Lemma 2 *In equilibrium, prices for real assets and financial shares are respectively:*

$$p^*(k) = \begin{cases} \bar{p} & if \quad k \leqslant \hat{k} \\ \\ \frac{(1-w)(1-k)R+w}{(1-w)k} & if \quad k > \hat{k} \end{cases} \tag{22}$$

and

$$q^*(k) = \begin{cases} \bar{p} & if \quad k \leqslant \bar{\bar{k}} \\ \\ \mu p^*(k) & if \quad k > \bar{\bar{k}} \end{cases}, \tag{23}$$

where $\mu = \frac{\bar{p}}{\underline{p}}$, $\hat{k} = \frac{(1-w)R+w}{(1-w)(R+\bar{p})}$, *and* $\bar{\bar{k}} = \frac{(1-w)R+w}{(1-w)(R+\underline{p})}$.

As Lemma 2 shows, the price of shares of liquid insiders follows a pattern that reflects aggregate shortage of liquidity. When the proportion of failures is large $(k > \bar{\bar{k}})$, cash-in-the-market pricing results in the price of assets falling even below the threshold value of arbitrageurs, $\underline{p}$. Since purchasing assets at such prices becomes profitable for arbitrageurs, in equilibrium they need to be compensated for purchasing shares of liquid insiders. As a result, share price of liquid insiders falls below their fundamental value, $\bar{p}$. In other words, liquid insiders can raise equity financing only at discounts. Thus, limited funds within the whole system and the resulting cash-in-the-market pricing affects not only the price of real assets but also the price of shares of liquid insiders. Furthermore, the discount that liquid insiders need to suffer in issuing equity is higher when illiquidity is more severe (high k).

In our model all the return generated by the asset and cash is shared by arbitrageurs and insiders (liquid and illiquid). Hence, the sum of expected profits of all insiders and arbitrageurs is equal to the expected net output, that is,

$$(1 - w)E(\pi) + wE(\pi_a) = \Pi - 1 \tag{24}$$

In the benchmark case without the capital market arbitrageurs acquire assets in states where $k > \bar{k}$. In particular, the total amount of illiquid insiders' assets acquired by arbitrageurs is given as:

$$
y_a(k) = \begin{cases}
0 & \text{if} \quad k \leqslant \bar{\bar{k}} \\[2mm]
(1-w)\left[k - \frac{(1-k)R}{\underline{p}}\right] & \text{if} \quad k \in (\bar{k}, \bar{\bar{k}}] \\[2mm]
\frac{w}{p^*(k)} & \text{if} \quad k > \bar{\bar{k}}
\end{cases}
\tag{25}
$$

where $p^*(k)$ is given in equation (8). Note that with the introduction of capital markets, all arbitrageur funds can be obtained by liquid insiders through sale of shares, even though such sales can be at a discount. Hence, with the introduction of capital markets $y_a = 0$. Since, all the assets are acquired by insiders, who are the efficient users of these assets, expected output increases with the introduction of capital markets.

One important observation is that the introduction of capital markets do not affect arbitrageurs' expected profit. The reason for this is that even though arbitrageurs can acquire shares of liquid insiders, in equilibrium, arbitrageurs make the same profit from asset and share purchases. Further, as the region where there is aggregate shortage of liquidity remains the same $(k > \bar{\bar{k}})$, for the same level of arbitrageur capital w, $E(\pi_a)$ is the same as in the case with no capital markets. However, the introduction of capital markets increases the expected output so that insider profits, $E(\pi)$, are higher now. Since insider profits are greater but those of arbitrageurs are the same (for a given level of arbitrage capital), the indifference equation (11) implies that the equilibrium allocation to arbitrage capital falls with greater access of insiders to external financing.

These results are summarized in the proposition below.

Proposition 4 *In the competitive equilibrium with external financing market, a proportion $w^{**} > 0$ of agents choose to become arbitrageurs, where w^{**} is smaller than w^*, the equilibrium proportion of arbitrageurs when there is no access to external financing (as characterized in Proposition 2).*

In other words, the inefficiency of arbitrage capital allocation in the competitive equilibrium does not change qualitatively when capital markets are considered. Profitable opportunities are still bypassed ex ante as some capital remains on standby waiting for purchase of equity issues of insiders at fire-sale prices. However, there is full ex post allocation of liquidated assets to insiders and the equilibrium level of arbitrage capital is lower (and welfare is higher) compared to the case without market for external financing.

4   Contagion via Limited Arbitrage

In this section, we extend the benchmark model to examine how limited arbitrage capital generates contagion. We introduce another asset to our benchmark model which can also be interpreted as another country or sector. The objective is to analyze how illiquidity and the allocation of funds between the two assets can lead to contagion from one asset to the other, resulting in excessive co-movement across assets that have independent fundamentals. Even though the returns from the two assets are independent, contagion results from the fact that when arbitrage capital for different assets and markets comes from a common pool of

investors, their equilibrium capital allocation requires that they earn the same rate of return across different assets and markets.[16]

Suppose that there are two ex ante identical assets, denoted by $i \in \{1, 2\}$, each with a measure $(1 - w_i)$ of insiders invested in. Hence, the total arbitrageur capital is $w_1 + w_2$. To simplify notation, we assume that these two assets have identical features to the asset introduced in the benchmark model except that their shocks are independent. Therefore, we consider the symmetric case where equilibrium allocation of arbitrage capital in each asset is the same: $w_1 = w_2 = w$. We allow insiders in one asset to access markets in the other asset, where we assume that insiders invested in asset i are as efficient as insiders invested in asset j in running asset j and vice versa.

Insiders in asset i are willing to pay a maximum price of $\bar{p}_i = \bar{p}$, whereas arbitrageurs are willing to pay a maximum price of $\underline{p}_i = \underline{p}$, for distressed asset $i = 1, 2$. Both liquid insiders (invested in either asset) and arbitrageurs can acquire assets $i = 1, 2$ that are put up for sale. In essence, the market for asset sales is fully integrated across the two assets. The implication of this is the following. Suppose that a fraction k_i of insiders in asset i get hit by a liquidity shock at $t = 1$. Whether that is sufficient to induce cash-in-the-market pricing for asset i depends also upon the quantity of asset j being put up for sale, in other words, on k_j, the fraction of illiquid insiders in asset j at $t = 1$.

Formally, we obtain the following lemma that characterizes the contagion effects on the

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[16]Other types of contagion explored in the literature include contagion through inter-linkages and direct exposures (Allen and Gale, 2000), information contagion (Chen, 1999), to cite a few.

price of asset i from asset j (also illustrated in Figure 5).

Lemma 3 *The price of assets as a function of the proportion of illiquid insiders in both assets (i and j, $i \neq j$) is as follows:*

$$p_i^*(k_i, k_j) = \begin{cases} \bar{p} & for & k_i \leqslant \underline{k}_i(k_j) \\[2ex] \frac{(2-k_1-k_2)R}{k_1+k_2} & for & k_i \in (\underline{k}_i(k_j), \bar{k}_i(k_j)] \\[2ex] \underline{p} & for & k_i \in (\bar{k}_i(k_j), \bar{\bar{k}}_i(k_j)] \\[2ex] \frac{(2-k_1-k_2)R}{k_1+k_2} + \frac{2w}{(1-w)(k_1+k_2)} & for & k_i > \bar{\bar{k}}_i(k_j) \end{cases} \tag{26}$$

where $\underline{k}_i(k_j), \bar{k}_i(k_j)$ and $\bar{\bar{k}}_i(k_j)$ are given in equations (50), (51), and (54), respectively, in Appendix I.

The key aspects of the contagion are as follows: (i) The threshold $\underline{k}_i$ is decreasing in k_j, so that asset i enters the cash-in-the-market region relatively sooner when the insiders in asset j experiences higher liquidity shocks; and, (ii) For $k_i > \bar{\bar{k}}_i(k_j)$, total liquidity of insiders and the arbitrageurs is not sufficient to keep the asset prices above $\underline{p}$. In equilibrium, arbitrageurs allocate their funds in these two assets such that they make the same profit from the two assets, which implies that $p_i = p_j$. Now, the threshold $\bar{\bar{k}}_i$ is decreasing in k_j so that asset i enters the second cash-in-the-market region also relatively sooner when asset j experiences more severe illiquidity.

In this case when the asset markets are perfectly integrated in terms of mobility of insiders as well as arbitrage capital, what is the equilibrium level of arbitrage capital allocation at

$t = 0$ (denoted as $2w^I$, so w^I per asset)? And how does it relate to the equilibrium level of arbitrage capital in the "autarky" case (denoted as w^A per asset, characterized earlier in Proposition 2) when the two asset markets are completely segmented in terms of mobility of capital?

Suppose that the arbitrage capital in case of integrated asset markets is $w^I = w^A$. Then, it can be shown that the integrated case maps one-for-one into the autarky case with one important difference that the proportion of illiquid insiders, k, in each asset in the autarky case is replaced by the average proportion of illiquid insiders, $\frac{k_1 + k_2}{2}$, in the integrated case. In particular, the thresholds of failures at which the two cash-in-the-market regions obtain are also exactly identical. These are the thresholds $\underline{k}$, $\bar{k}$, and $\bar{\bar{k}}$, of Proposition 2. The key difference, however, is in the likelihood of these outcomes. Clearly, the distribution $f^A(k)$ of the proportion of illiquid insiders per asset is different from the distribution (denoted as) $f^I(k)$ of average proportion of illiquid insiders across assets.

This mapping leads to the following characterization of when arbitrage capital per asset falls in the case when the asset markets are integrated relative to the case when markets are segmented. We show that this is the case when the probability of events, wherein the average proportion of illiquid insiders is high enough that arbitrageurs enter asset markets, is (event by event) smaller than the probability of seeing the same proportion of illiquid insiders per asset. That is, if the diversification effect of independent outcomes across assets makes it less likely that the average proportion of illiquid insiders will exceed a critically high threshold, then equilibrium arbitrage capital in the integrated case is smaller per asset than in autarky.

Proposition 5 *If $f^A(k) \geq f^I(k)$, $\forall k > \bar{k} \equiv \frac{R}{R+\bar{p}}$ (and strictly greater for at least some k greater than $\bar{\bar{k}}$), then the arbitrage capital per asset in the integrated capital markets case, $\frac{w^I}{2}$, is greater than the arbitrage capital per asset in the autarky case, w^A.*

The above condition is always met, for example, in the case of uniform distribution for $f^A(k)$ and $\bar{k} > \frac{1}{2}$. By implication, when arbitrage capital per asset falls, we can also show that overall welfare is improved due to a lower expectation of the ex-post misallocation cost. We stress, however, that this is in general not always the case. Shaffer (1994) and Winton (1999) explain that when the critical threshold for failures is not too high "pooling intensifies joint failure risk". Intuitively, the contagion effect, namely that even when one asset has a small proportion of illiquid insiders that it can experience cash-in-the-market pricing due to a large proportion of illiquid insiders in the other asset, can now potentially dominate the diversification effect. If this is the case, equilibrium arbitrage capital per asset may be greater in the integrated case than in the autarky case, and, in turn, overall welfare may be lower upon market integration due to greater ex-post misallocation of assets.

5   Conclusion

Our framework sheds light on fire sales as an equilibrium phenomenon when investors can choose ex ante how much arbitrage capital to hold. The joint occurrence of fire sales and limited arbitrage capital that moves in "slowly" to acquire assets (that is, only when price discounts are sufficiently steep) is a robust feature arising from the fundamental trade-off

faced by investors. Arbitrage capital can take advantage of depressed prices in crisis states, but entails costs in the form of foregone profitable investments and not investing in expertise. Equalizing the ex ante return from the two activities leads to the interior nature of the equilibrium. Equilibrium arbitrage capital is limited and fire sales during crises become a robust phenomenon.

We also demonstrated how this equilibrium construction can be used to good effect in two applications. First, we showed that (perhaps surprisingly) setting aside of arbitrage capital can be inefficient from the standpoint of ex-ante investment. Although arbitrage capital cushions financial distress in crisis states, it leads to foregoing of ex-ante profitable investments. Our second application of the equilibrium construction was to examine a novel channel of contagion between asset classes that have independent fundamentals. The contagious link arises from the fact that arbitrageurs must earn the same rate of return on capital from different markets to which they supply liquidity. In particular, this contagion also carries over from real asset markets to markets for financing asset purchases.

It would be interesting in future research to examine a dynamic setting in which one can study how arbitrage capital allocation shifts over time, in particular, as crises approach, and calibrate the resulting prices and contagion across markets to empirically observed patterns.

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Appendix I: Proofs

Proof of Lemma 1: The price cannot be greater than $\bar{p}$ since in this case we have $y_i(p) = y_a(p) = 0$. If $p \leqslant \bar{p}$, and the proportion of illiquid insiders is sufficiently small, liquid insiders have enough funds to pay the full price $\bar{p}$ for all assets. More specifically, this is the case when liquid insiders' liquidity $(1-w)(1-k)R$ is adequate to purchase all illiquid insiders' assets $(1-w)k$ at the full price $\bar{p}$. Thus, for $k \leq \underline{k}$, where

$$\underline{k} = \frac{R}{R + \bar{p}} = \frac{1}{2},\tag{27}$$

the auction price is $p^* = \bar{p}$. At this price, liquid insiders are indifferent between any quantity of assets purchased. Hence, each liquid insider is allocated a share $y_i(\bar{p}) = k/(1-k)$.

For moderate values of k, liquid insiders cannot pay the full price for all assets but can still pay at least the threshold value of $\underline{p}$, below which arbitrageurs have a positive demand.

Formally, for $k \in (\underline{k}, \bar{k}]$, where

$$\bar{k} = \frac{R}{R + \underline{p}} = \frac{R}{2R - \Delta},$$ (28)

the price is set at $p^* = (1 - k) R/k$, and again, all assets are acquired by liquid insiders.

For $k > \bar{k}$, liquid insiders cannot pay the threshold price of $\underline{p}$ for all assets and profitable options emerge for arbitrageurs. Hence, for $k > \bar{k}$, arbitrageurs have a positive demand and are willing to supply their funds for the asset purchase. With the injection of arbitrageurs' funds, prices can be sustained at $\underline{p}$ until some critical proportion of failures $\bar{\bar{k}} \geqslant \bar{k}$. However, for $k > \bar{\bar{k}}$, even the injection of arbitrageur capital is not enough to sustain the price at $\underline{p}$.

Formally, for $k > \bar{k}$, the amount of arbitrage capital v needed to maintain the price at $\underline{p}$, is given by the market-clearing condition: $(1 - w)(1 - k)R + v = (1 - w)k\underline{p}$. This gives $v = (1 - w)[k\underline{p} - (1 - k)R]$. In turn, we obtain the threshold k above which the price cannot be sustained at $\underline{p}$ even with entry of all arbitrageur funds w. That is, for $k \in (\bar{k}, \bar{\bar{k}}]$, where

$$\bar{\bar{k}} = \min \left\{ 1, \frac{(1 - w)R + w}{(1 - w)(R + \underline{p})} \right\},$$ (29)

the price is set at $\underline{p}$. At this price, arbitrageurs are indifferent between any quantity of assets purchased. Hence, each liquid insider receives a share of $y_i(\underline{p}) = \frac{R}{\underline{p}}$, and the rest, $y_a(\underline{p}) = \frac{1-w}{w} \left(k - \frac{(1-k)R}{\underline{p}} \right)$, is allocated to the arbitrageurs.

For $k > \bar{\bar{k}}$, the price is again strictly decreasing in k and is given by

$$p^*(k) = \frac{(1 - k)R}{k} + \frac{w}{(1 - w)k},$$ (30)

and $y_i(p^*) = \frac{R}{p^*}$, and $y_a(p^*) = \frac{1}{p^*}$.

Note that the proportion w of agents that choose to become arbitrageurs affects the price p^* only in the fourth region where $k > \bar{\bar{k}}$, as well as the boundary $\bar{\bar{k}}$ of the fourth region itself. In particular, for higher values of w, p^* is higher in this region. Furthermore, the region itself shifts to the left as the fraction of entrepreneurs increases, that is,

$$\frac{d\bar{\bar{k}}}{dw} = \frac{1}{(1-w)^2 \, (R + \underline{p})} > 0. \; \diamond \tag{31}$$

Proof of Proposition 2: We prove the results that $w^* \in (0,1)$ and $\bar{\bar{k}} < 1$ jointly by analyzing all the four possible regions for k that are given in equation (8). While the price depends on w and k, for simplicity of notation we use p instead of $p(k, w)$.

From the indifference equation (11), we have

$$E\left[(1-k)\left(\frac{R\bar{p}}{p}\right) + kp + R - 1\right] = E\left[\frac{1}{p} \, (\underline{p} - p)^+\right], \tag{32}$$

which implicitly gives the equilibrium level of w^* as

$$E\left[h\left(k, w^*\right)\right] = 0, \; \text{where} \tag{33}$$

$$h\left(k, w\right) = \frac{1}{p} \, (\underline{p} - p)^+ - (1-k)\left(\frac{R\bar{p}}{p}\right) - kp - R + 1. \tag{34}$$

Next, we show that $E\left[h\left(k, w\right)\right]$ is weakly decreasing in w. Note that the price $p^*(k)$ given in equation (8) is continuous in w. Hence, $h\left(k, w\right)$ is continuous in w. Thus, using Leibnitz's rule, we can show that

$$\frac{\partial E\left(h\right)}{\partial w} = \int_{k=0}^{1} \left(\frac{\partial h}{\partial w}\right) f(k) dk. \tag{35}$$

Note that w affects $h(k,w)$ only through the price p. From equation (8), for $k \leqslant \bar{\bar{k}}$, price p is independent of w, so that $\frac{\partial h}{\partial w} = 0$.

For $k > \bar{\bar{k}}$, we have

$$h\left(k, w^*\right) = \frac{\underline{p} - (1-k)R\bar{p}}{p} - kp, \tag{36}$$

which gives us

$$\frac{\partial h}{\partial w} = -\left(\frac{\underline{p} - (1-k)R\bar{p}}{p^2} + k\right)\underbrace{\left(\frac{\partial p}{\partial w}\right)}_{>0}. \tag{37}$$

Hence, for $k > \bar{\bar{k}}$, $\frac{\partial h}{\partial w}$ has the opposite sign as the expression $\left[\underline{p} - (1-k)R\bar{p} + kp^2\right]$. Hence, in this region, for $\left[\underline{p} - (1-k)R\bar{p} + kp^2\right] > 0$, [17] we have $\frac{\partial h}{\partial w} < 0$, which means that there is a unique w^* that satisfies the indifference equation (11).

Next we show that $w^* \in \left(0, \frac{\underline{p}}{1+\underline{p}}\right)$.

First, we show that $w^* \geqslant \frac{\underline{p}}{1+\underline{p}}$ cannot be an equilibrium. In that case, price never falls below $\underline{p}$ and $E\left(\pi_a\right) = 0$, and $E\left(\pi\right) = E\left[(1-k)R + kp + R - 1\right] > 0$. Hence, $w^* \geqslant \frac{\underline{p}}{1+\underline{p}}$ cannot be an equilibrium as some arbitrageurs would deviate and become insiders.

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[17] A sufficient condition for this inequality to hold is $\bar{p} < \frac{R-\Delta}{R-1}$. For $k = 1$, we have $\left[\underline{p} - (1-k)R\bar{p} + kp^2\right] = \underline{p} + p^2 > 0$. Note that $\left[\underline{p} - (1-k)R\bar{p} + kp^2\right] > \left[\underline{p} - (1-k)R\bar{p}\right]$, which are both increasing in k, and the inequality $\left[\underline{p} - (1-\bar{\bar{k}})R\bar{p}\right] > 0$ is a sufficient condition. Note that the minimum value $\bar{\bar{k}}$ can take is $\frac{R}{R+\underline{p}}$, which is when $w = 0$. We have $\underline{p} - (1-\bar{\bar{k}})R\bar{p} \geqslant \underline{p} - \left(\frac{\underline{p}}{R+\underline{p}}\right)R\bar{p}$. And, we can show that $\underline{p} - \left(\frac{\underline{p}}{R+\underline{p}}\right)R\bar{p} > 0$, for $\bar{p} < \frac{R-\Delta}{R-1}$. Note that $\bar{p} = R$ and for $\Delta < R(2-R)$, we always have $\bar{p} < \frac{R-\Delta}{R-1}$.

For $w = 0$, we have

$$p^*(k) = \begin{cases} \bar{p} & \text{for} \quad k \leqslant \underline{k} \\ \\ \frac{(1-k)R}{k} & \text{for} \quad k > \underline{k} \end{cases} \quad , \text{ and} \tag{38}$$

$$E(\pi) = E\left[\frac{(1-k)R\bar{p}}{p} + kp + R - 1\right] = E\left[(1-k)R + k\bar{p} + R - 1\right], \tag{39}$$

where $\pi = (1-k)R + k\bar{p} + R - 1$ for $k \in [0,1]$. Note that $\frac{d\pi_i}{dk} > 0$ since $R > \bar{p}$. Furthermore,

$E(\pi) < R$, and hence bounded.

For $w = 0$, we have

$$E(\pi_a) = E\left[\frac{1}{p}\left(\underline{p} - p\right)^+\right]. \tag{40}$$

Note that $\lim_{k \to 1} p = 0$ so that $\lim_{k \to 1} \left(\underline{p}/p\right) = +\infty$. Hence, if the probability distribution $f(k)$ is

such that it does not converge to 0 "too fast" as k converges to 1, then $\lim_{k \to 1} \left(\underline{p}/p\right) f(k) = +\infty$

so that $E(\pi_a) = +\infty$. For example, for all continuous $f(k)$ that converge to a positive value

as k converges to 1, we have $\lim_{k \to 1} \left(\underline{p}/p\right) f(k) = +\infty$ so that $E(\pi_a) = +\infty$.

Hence $w = 0$ cannot be an equilibrium as someone would deviate and take advantage of

the potential profits from fire sales. Hence, in equilibrium, we have a unique in equilibrium

$w^* \in (0,1)$. $\Diamond$

Proof of Proposition 3: Recall that the equilibrium level of arbitrage capital w^* is implicitly

given by the equation $E\left[h\left(k, w^*\right)\right] = 0$, where $h(k,w)$ is as defined in equation (34). Using

this implicit condition, we prove the two parts of the proposition as follows.

Part (i): Note that $\frac{\partial h(k,w)}{\partial k} > 0$ is a sufficient condition for $E_f\left[h\left(k,w\right)\right] > E_g\left[h\left(k,w\right)\right]$

when f FOSD g, where E_f and E_g represent expectations over probability distributions f

and g, respectively. We already showed that $\frac{\partial E[h(k,w)]}{\partial w} < 0$, so that it is sufficient to show

$\frac{\partial h(k,w)}{\partial k} > 0$ to prove the result. To do that, we look at the four possible regions for k.

(1) For $k \leqslant \underline{k}$, we have $p = \bar{p}$, which gives us $h(k,w) = -(1-k)R - k\bar{p} + R + 1$. Hence,

$\frac{\partial h}{\partial k} = R - \bar{p} > 0$.

(2) For $k \in (\underline{k}, \bar{k}]$, we have $p = \frac{(1-k)R}{k}$, which gives us

$$h(k,w) = -(1-k)\left(\frac{R\bar{p}}{p}\right) - kp - R + 1 = -k\bar{p} - (1-k)R - R + 1. \qquad (41)$$

Hence, $\frac{\partial h}{\partial k} = R - \bar{p} > 0$.

(3) For $k \in (\bar{k}, \bar{\bar{k}}]$, we have $p = \underline{p}$, and

$$h(k,w) = -(1-k)\left(\frac{R\bar{p}}{\underline{p}}\right) - k\underline{p} - R + 1, \text{ which gives us } \frac{\partial h}{\partial k} = \frac{R\bar{p}}{\underline{p}} - \underline{p} > R - \bar{p} > 0. \qquad (42)$$

(4) For $k > \bar{\bar{k}}$, we have $p = \frac{(1-w)(1-k)R+w}{(1-w)k}$, and

$$h(k,w) = \frac{(1-w)k\underline{p} - (1-w)k(1-k)R\bar{p} - (1-w)(1-k)R - w}{(1-w)(1-k)R + w} - (1-k)R - \frac{w}{(1-w)} - R + 1. \qquad (43)$$

Note that, in the above expression, the denominator of the first term is decreasing in k, whereas the second term is increasing in k. Hence, if the numerator of the first expression is increasing in k, then it is sufficient for $\frac{\partial h(k,w)}{\partial k} > 0$. The derivative of the numerator of the first expression with respect to k is given as:

$$(1-w)\left[\underline{p} - (1-k)R\bar{p} + kR\bar{p} + R\right] = (1-w)\left[\underline{p} - R\bar{p} + 2kR\bar{p} + R\right]. \qquad (44)$$

Next, we show that, for $R > \underline{p}$, we have $\left[\underline{p} - R\bar{p} + 2kR\bar{p} + R\right] > 0$.

Let $A = \underline{p} - R\bar{p} + 2kR\bar{p} + R$. We have $\frac{\partial A}{\partial k} = 2R\bar{p} > 0$. Hence, if $A > 0$ for $k = \bar{\bar{k}}$, we have $A > 0$ for all $k > \bar{\bar{k}}$.

We have $\frac{d\bar{\bar{k}}}{dw} = \frac{1}{(1-w)^2 (R+\underline{p})} > 0$. Hence, if we can show that $A > 0$ for $k = \bar{\bar{k}}$ and $w = 0$, we are done. For $w = 0$, we have $\bar{\bar{k}} = \frac{R}{R+\underline{p}}$, which gives us $A = \underline{p} - R\bar{p} + 2\left(\frac{R}{R+\underline{p}}\right)R\bar{p} + R$. For $R > \underline{p}$, we have $\frac{R}{R+\underline{p}} > \frac{1}{2}$ so that $A > \underline{p} + R > 0$.

Hence, we have $\frac{\partial h(k,w)}{\partial k} > 0$.

Part (ii): From the indifference equation (11), we have $E_k\left[h\left(k, w^*\right)\right] = 0$. Thus, we have

$$\frac{\partial E\left[h\left(k, w^*\right)\right]}{\partial w} \cdot \frac{dw^*}{d\Delta} + \frac{\partial E\left[h\left(k, w^*\right)\right]}{\partial \Delta} = 0. \tag{45}$$

We already showed that $\left(\frac{\partial E[h(k,w^*)]}{\partial w}\right) < 0$, so that

$$sign\left(\frac{dw^*}{d\Delta}\right) = sign\left(\frac{\partial E\left[h\left(k, w^*\right)\right]}{\partial \Delta}\right). \tag{46}$$

Hence, we need to show that $\left(\frac{\partial E[h(k,w^*)]}{\partial \Delta}\right) < 0$. Note that the price $p^*(k)$ given in equation (8) is continuous in Δ. Hence, $h\left(k, w^*\right)$ is continuous in Δ. Thus, using Leibnitz's rule, we can show that

$$\frac{\partial E\left(h\right)}{\partial \Delta} = \int_{k=0}^{1} \left(\frac{\partial h}{\partial \Delta}\right) f(k) dk. \tag{47}$$

Next, we analyze each of the four regions of k given in equation (8). Note that for $k \leqslant \bar{k}$, price $p(k, w)$ is independent of Δ. This, in turn, implies that for $k \leqslant \bar{k}$, $h\left(\cdot\right)$ is independent of Δ.

For $k \in (\bar{k}, \bar{\bar{k}}]$, we have $p(k, w) = \underline{p}$, where $\frac{\partial \underline{p}}{\partial \Delta} = -1 < 0$. For $k \in [\bar{k}, \bar{\bar{k}}]$, we have

$$h\left(k, w^*\right) = -(1-k)\left(\frac{R\bar{p}}{\underline{p}}\right) - k\underline{p} - R + 1, \text{ which gives us } \frac{\partial h}{\partial \underline{p}} = (1-k)\left(\frac{R\bar{p}}{\underline{p}^2}\right) - k. \tag{48}$$

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Note that $\frac{\partial h}{\partial \underline{p}}$ is decreasing in k. Hence, if $\frac{\partial h}{\partial \underline{p}} \geqslant 0$ for $k = \overline{\overline{k}}$, then $\frac{\partial h}{\partial \underline{p}} > 0$ for all $k \in [\overline{k}, \overline{\overline{k}})$. Furthermore, we have $\frac{\partial h}{\partial \underline{p}} = (1-k)\left(\frac{R\overline{p}}{\underline{p}^2}\right) - k = 0$ when $k = \hat{k} = \frac{R\overline{p}}{R\overline{p}+\underline{p}^2}$. Hence, if $\overline{\overline{k}} < \hat{k}$, then $\frac{\partial h}{\partial \underline{p}} > 0$ for all $k \in [\overline{k}, \overline{\overline{k}}]$. We have $\overline{\overline{k}} < \hat{k}$ if and only if

$$\frac{(1-w)R+w}{(1-w)(R+\underline{p})} < \frac{R\overline{p}}{R\overline{p}+\underline{p}^2},$$

which holds if and only if $w < \hat{w} = \frac{R\underline{p}\Delta}{R\underline{p}\Delta+R\overline{p}+\underline{p}^2}$.

This, combined with the fact that $\frac{\partial \underline{p}}{\partial \Delta} < 0$, gives us $\frac{\partial h}{\partial \Delta} < 0$, for $k \in (\overline{k}, \overline{\overline{k}}]$.

For $k > \overline{\overline{k}}$, the price p^* is independent of Δ as all the funds within the liquid insiders and arbitrageurs are exhausted. Hence, for $k > \overline{\overline{k}}$, $\frac{\partial h}{\partial \Delta} = 0$.

Combining these results, we get $\left(\frac{\partial E(h)}{\partial \Delta}\right) < 0.\diamondsuit$

Proof of Lemma 2: The steps of the proof are organized in a way that lays down the results for different regions of the proportion (k) of illiquid insiders.

(1) For $k \leqslant \hat{k}$, liquidity within the liquid insiders and the liquidity they can raise by issuing shares to arbitrageurs is sufficient to sustain the price for the illiquid insiders' assets at $\overline{p}$. Since $p^*(k) = \overline{p} > \underline{p}$, we have $x = 0$ and $m = \frac{k}{1-k}$. Each liquid insider issues enough equity, at $q(k) = \overline{p}$, to purchase $\frac{k}{1-k}$ units of illiquid insiders' assets at $p^*(k) = \overline{p}$. Thus, we have

$$R + s\overline{p} = \left(\frac{k}{1-k}\right)\overline{p}, \text{ which gives us:}$$

$$s = \frac{k}{1-k} - \frac{R}{\overline{p}} \quad \text{and} \quad z = \frac{1-w}{w}\left[k - \frac{(1-k)R}{\overline{p}}\right].$$

(2) For $\hat{k} < k \leqslant \bar{\bar{k}}$, liquidity within the liquid insiders and the liquidity they can raise through equity issuance from arbitrageurs is sufficient to sustain $p^*(k)$ at least at $\underline{p}$.

Since $p^*(k) \geqslant \underline{p}$, we have $x = 0$ and $m = \frac{k}{1-k}$. Each liquid insider issues enough equity, at $q(k) = \bar{p}$, to purchase $\frac{k}{1-k}$ units of illiquid insiders' assets at $p^*(k) = \frac{(1-w)(1-k)R+w}{(1-w)k}$, that is,

$$R + s\bar{p} = \left(\frac{k}{1-k}\right) p^*(k), \text{ which gives us}$$

$$s = \frac{w}{(1-w)(1-k)\bar{p}} \quad \text{and} \quad z = \frac{1}{\bar{p}}.$$

(3) For $k > \bar{\bar{k}}$, total liquidity within the liquid insiders and the liquidity they can raise through equity issuance from arbitrageurs is no longer sufficient to sustain $p^*(k)$ at $\underline{p}$. Since $p^*(k) < \underline{p}$, arbitrageurs may prefer to participate in the market for illiquid insiders' assets.

If $p^*(k) < \underline{p}$ and $\underline{p} \, q(k) > \bar{p} \, p^*(k)$, then arbitrageurs use all their funds for the asset purchase, that is $x = \frac{1}{p^*(k)}$.

If $p^*(k) < \underline{p}$ and $\underline{p} \, q(k) < \bar{p} \, p^*(k)$, then arbitrageurs use all their funds for the equity purchase, that is $z = \frac{1}{q(k)}$, and if $\underline{p} \, q(k) = \bar{q} \, p^*(k)$, arbitrageurs are indifferent between the purchase of liquid insiders' shares and the illiquid insiders' assets.

Now, let $\mu = \frac{\bar{p}}{\underline{p}}$. Whether arbitrageurs buy shares of the liquid insiders or the assets of the illiquid insiders, their entire funds w eventually end up in the asset market. Hence, for $k > \bar{\bar{k}}$, the price for illiquid insiders' assets is given as:

$$p^*(k) = \frac{(1-w)(1-k)R + w}{(1-w)k}. \tag{49}$$

If the price $q(k)$ of a share is higher then $\mu p^*(k)$, then arbitrageurs are better off buying

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the assets of illiquid insiders, rather than buying shares of the liquid insiders, that is, $z = 0$ and $x = \frac{1}{p^*(k)}$. Hence, we cannot have an equilibrium where $q(k) > \mu p^*(k)$ and $z > 0$.

Next, we show that liquid insiders need to suffer some discount when they generate funds in the capital market. Note that arbitrageurs are willing to purchase shares of liquid insiders, that is, $z > 0$, only when $q(k) \leqslant \mu p^*(k)$ and liquid insiders are willing to issue equity, that is, $s > 0$, only when $q(k) \geqslant p^*(k)$. Suppose that the market-clearing mechanism works in a way that allows the maximum possible funds to go to insiders through equity issuance, that is $q(k) = \mu p^*(k)$. Note that this allows for the highest price $q(k)$ for shares. However, even in this case, liquid insiders need to suffer some discount when they generate funds in the capital market. Hence, in equilibrium, for $k > \overline{\overline{k}}$, we have $q(k) = \mu p^*(k) < \bar{p}$. $\diamondsuit$

Proof of Lemma 3: For small proportion of illiquid insiders in the two assets, that is, for

$$[(1 - w_1)(1 - k_1) + (1 - w_2)(1 - k_2)] R \geqslant [(1 - w_1)k_1 + (1 - w_2)k_2] \bar{p},$$ there is enough liquidity within the liquid insiders and the prices of the two assets are above the fundamental value $\bar{p}$. Given $w_1 = w_2$, this holds for $k_i \leqslant \underline{k}_i(k_j)$ for $i \neq j$, where

$$\underline{k}_i(k_j) = \frac{2R}{R + \bar{p}} - k_j = 1 - k_j, \tag{50}$$

the prices of both assets equal $\bar{p}$ and all assets are acquired by liquid insiders.

As the proportion of illiquid insiders in the two assets increase, that is, for

$$[(1 - w_1)k_1 + (1 - w_2)k_2] \underline{p} \leqslant [(1 - w_1)(1 - k_1) + (1 - w_2)(1 - k_2)] R \leqslant [(1 - w_1)k_1 + (1 - w_2)k_2] \bar{p},$$

the liquidity within the insiders cannot sustain the prices at $\bar{p}$ and we observe cash-in-the-market prices in the two assets. Note that the threshold $\underline{k}_i$ is decreasing in k_j. Hence, asset

i enters the cash-in-the-market region relatively sooner when asset j experiences higher pro-portion of failures. That is, given $w_1 = w_2$, for $\underline{k}_i(k_j) < k_i \leqslant \bar{k}_i(k_j)$ for $i \neq j$, where

$$\bar{k}_i(k_j) = \frac{2R}{R + \underline{p}} - k_j = \frac{2R}{2R - \Delta} - k_j, \tag{51}$$

even though liquid insiders can pay a price higher than $\underline{p}$ for all illiquid insiders' assets and therefore can "beat" arbitrageurs in the auction, their liquidity is not sufficient to sustain the prices at the fundamental value $\bar{p}$.

In equilibrium, liquid insiders allocate their funds in these two assets such that they make the same profit from both assets, which implies that

$$\frac{\bar{p} - p_i}{p_i} = \frac{\bar{p} - p_j}{p_j}, \text{ that is, } p_i = p_j. \tag{52}$$

Hence, for $\underline{k}_i(k_j) < k_i \leqslant \bar{k}_i(k_j)$, we obtain

$$p_i^*(k_i, k_j) = \frac{(2 - k_1 - k_2)R}{k_1 + k_2}. \tag{53}$$

For $k_i > \bar{k}_i(k_j)$, liquidity within the insiders cannot sustain the prices at $\underline{p}$ and profitable opportunities emerge for arbitrageurs. With the injection of arbitrageurs' funds, prices can be sustained at $\underline{p}$ until some critical proportion of illiquid insiders $\bar{\bar{k}}_i(k_j)$, where

$$\bar{\bar{k}}_i(k_j) = \frac{2R}{2R - \Delta} + \frac{2w}{(1 - w)(R + \underline{p})} - k_j. \tag{54}$$

For $k_i > \bar{\bar{k}}_i(k_j)$, total liquidity of insiders and the arbitrageurs is not sufficient to keep the asset prices above $\underline{p}$. In equilibrium, arbitrageurs allocate their funds in these two assets such that they make the same profit from both assets, which implies that in equilibrium

$p_i = p_j$. Note that the threshold $\overline{\overline{k}}_i$ is decreasing in k_j, that is, asset i enters the second cash-in-the-market region relatively sooner when asset j experiences more severe liquidity. For $k_i > \overline{\overline{k}}_i(k_j)$, we obtain

$$p_i^*(k_i, k_j) = \frac{(2 - k_1 - k_2)R}{k_1 + k_2} + \frac{2w}{(1 - w)(k_1 + k_2)}. \ \diamondsuit \tag{55}$$

Proof of Proposition 5: Let $w^I = w^A = w$. Define n as a random variable equal to $\frac{k_1 + k_2}{2}$. Then, the equilibrium price in the case of integrated markets (Lemma 3) can be rewritten as:

$$p^*(n) = \begin{cases} \bar{p} & \text{for} & n \leqslant \underline{k} \\[2mm] \frac{(1-n)R}{n} & \text{for} & n \in (\underline{k}, \bar{k}] \\[2mm] \underline{p} & \text{for} & n \in (\bar{k}, \overline{\overline{k}}] \\[2mm] \frac{(1-n)R}{n} + \frac{w}{(1-w)n} & \text{for} & n > \overline{\overline{k}} \end{cases}, \tag{56}$$

where $\underline{k}$, $\bar{k}$, $\overline{\overline{k}}$ are given by the same equations as in the autarky case (see equations (27), (28), and (29), respectively, in the proof of Lemma 1). Thus, the price for assets in the integrated case is the same as in the autarky case except that the state variable is n instead of k.

If we denote as y_a^i the amount of asset i acquired per unit of arbitrage capital, then we obtain that

$$y_a(n) = \begin{cases} 0 & \text{for} & n \leqslant \bar{k} \\[2mm] \frac{(1-w)}{w}\left[n - \frac{(1-n)R}{\underline{p}}\right] & \text{for} & n \in (\bar{k}, \overline{\overline{k}}] \\[2mm] \frac{(1-w)n}{(1-w)(1-n)R+w} & \text{for} & n > \overline{\overline{k}} \end{cases}. \tag{57}$$

Again, this is exactly the same as arbitrageurs' acquisition of assets in the autarky case with n (average proportion of illiquid insiders) replacing k (proportion of illiquid insider for each asset).

Then, each arbitrageur's profits from each asset can be expressed as

$$E(\pi_a^I) = \int_{\bar{\bar{k}}}^{1} y_a(n) \left[\underline{p} - p^*(n) \right] f^I(n) dn. \tag{58}$$

Substituting for variable of integration z as k, and comparing to arbitrageur profits in autarky case, we obtain that

$$E(\pi_a^A) - E(\pi_a^I) = \int_{\bar{\bar{k}}}^{1} y_a(k) \left[\underline{p} - p^*(k) \right] \cdot \left[f^A(k) - f^I(k) \right] dn. \tag{59}$$

Since $p^*(k) = \underline{p}$ for $k \in (\bar{k}, \bar{\bar{k}}]$, it follows that whenever $f^A(k) \geq f^I(k)$ for $k > \bar{k}$ (and strictly greater for some k), we obtain that for $w^I = 2w^A$, $E(\pi_a^A) > E(\pi_a^I)$. In words, arbitrageur profits are greater in autarky than in the integrated case if total arbitrage capital per asset is the same in the two cases.

Next, we compare overall welfare in the two cases. Note that overall welfare can be stated intuitively as arbitrage capital plus the expected cash flow from insiders' assets if they were to be in the hands of efficient users (namely insiders) also in the second period, minus the expected misallocation costs on assets acquired and run by arbitrageurs. Denote the expected cash flows per asset in the integrated case as $E(\Pi^I)$ and in autarky as $E(\Pi^A)$. Then,

$$E(\Pi^A) - E(\Pi^I) = - \int_{\bar{k}}^{1} w y_a(k) \left(\bar{p} - \underline{p} \right) \cdot \left[f^A(k) - f^I(k) \right] dn \ , \tag{60}$$

since misallocation per asset is $(\bar{p} - \underline{p})$ and it arises whenever arbitrageurs acquire assets, which is when $n > \bar{k}$ in the integrated case and $k > \bar{k}$ in the autarky case (even though arbitrageurs

make profits only above the threshold $\bar{\bar{k}}$). Then, it follows that whenever $f^A(k) \geq f^I(k)$ for $k > \bar{k}$ (and strictly greater for some k), we obtain that for $w^I = w^A$, $E(\Pi^A) < E(\Pi^I)$. In words, overall welfare from each asset is smaller in autarky than in the integrated case if total arbitrage capital for each asset is the same in the two cases.

Denoting insider profits in the two cases as $E(\pi^A)$ and $E(\pi^I)$, and since $\Pi - 1 = (1 - w)E(\pi) + wE(\pi_a)$, it also follows that $E(\pi^A) < E(\pi^I)$, when the condition in Proposition 5 is satisfied. That is, insider profits are smaller in autarky than in integrated case if total arbitrage capital per asset is the same in two cases.

Now, at equilibrium level of arbitrage capital in the integrated case, we must have $E(\pi_a^I) = E(\pi^I)$. Similarly, at equilibrium level of arbitrage capital for autarky, $E(\pi_a^A) = E(\pi^A)$. But if $w^I = w^A$, we just showed that $E(\pi_a^I) < E(\pi_a^A)$ and $E(\pi^I) > E(\pi^A)$, so that $E(\pi_a^I) < E(\pi^I)$. Thus, it must be the case that in equilibrium of the integrated case $w^I < w^A$, whenever the condition of Proposition 5 is satisfied. $\diamond$