环形腔内双层薄液层热毛细对流的渐近解 1)

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摘要　为了解水平温度梯度作用下环形腔内双层薄液层热毛细对流的基本特性，采用渐近线方法获得了热毛细对流的近似解．环形腔外壁被加热，内壁被冷却，上、下壁面绝热．结果表明，当环形腔宽度与内半径比趋于零时，环形腔退化为矩形腔，所得到的主流区速度场和温度场的表达式演化为 Nepomnyashchy 等得到的矩形腔内的结果；与数值模拟结果的比较发现，在主流区渐近解与数值解吻合较好．

关键词　热毛细对流，环形腔，双层薄液层，水平温度梯度，渐近解

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引　言

多层流体中的对流流动及界面现象广泛存在于自然界和工程技术领域，因此，引起了人们的极大兴趣 [1,2]．Villers 等 [3] 实验研究了水平温度梯度作用下矩形腔内两层流体的热对流，测量了水 / 庚醇两层流体系统的速度分布，发现液 - 液界面处流体的运动随着液层相对厚度的变化而明显改变．随后，他们建立了一个简单的理论模型并得到了二层流一维近似理论解 [4]．Doi 等 [5] 对水平叠置的两层流体的稳态热毛细对流进行了研究，得到了水平无限大的双液层内速度分布和温度分布的分析解，并且发现，随着液 - 液界面张力温度系数和自由表面张力温度系数比 λ 的不同，可能出现 4 种流型，当 $\lambda = 0.5$ 时，下层流体的流动被完全抑制，即下层流体几乎处于静止状态，此时的抑制效果最佳；同时，为了研究垂直边壁的影响，进行了二维数值模拟，结果表明，由于边壁效应的影响，只有在较低的 Marangoni 数或毛细雷诺数下 $\lambda = 0.5$ 时的抑制效果最佳，随着 Marangoni 数或毛细雷诺数的增加，最佳抑制效果时的 λ 值将减小． Liu 等 [6,7] 及 Gupta 等 [8,9] 对矩形腔内双层流体稳态热毛细对流和浮力对流进行了数值模拟，前者假定自由表面和液 - 液交界面均不变形，后者考虑了变形的影响，他们的结果都表明：两层流体内的对流流型同上下液层的物理和几何参数比有关，液封层的厚度和黏度对下层流体的对流运动有很大影响，薄的液封层厚度和高的黏度有助于减弱下层流体的对流运动． Madruga 等 [10] 采用线性稳定性方法研究了矩形腔内无自由表面、水平叠置的两流体层内热毛细 – 浮力对流，同样，当温度梯度较小时，基本流动是稳定的二维流动，随着界面张力系数的不同，可能呈现出 4 种流型；当温度梯度增加到某一临界值后，流动将会失去其稳定性转化成复杂的非稳态三维振荡流动，在不同的相对液层厚度下，可能出现 3 种流型，分别是：从冷区向热区运动的热波、从热区向冷区运动的热波或稳定的轴向滚胞．后来， Nepomnyashchy 等 [11] 借助于非线性稳定性方法研究了同样的问题，在水平方向分别采用了周期性边界条件和绝热边界条件，结果发现，波的运动方向不仅与液层厚度有关，还与 Marangoni 数的大小有关．

目前，对双层液体系统内热对流的研究主要集中在水平无限大的双液体层或矩形腔内的双液体层，而对于环形双层液体系统内热毛细对流的研究很少．最近， Li 等 [12] 给出了一组环形单层液池内热毛细对流的渐近解，本文则主要研究环形双层薄液层内的热毛细对流，采用匹配渐近展开法得到了主流区的温度分布和速度分布．

1　物理数学模型

物理模型如图 1 所示，内、外半径分别为 r_i 和

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$r_o = r_i + l$ 的环形腔内盛有深度均为 h 的两层不相混溶的流体，顶部和底部均为绝热固壁，内、外壁分别维持恒温 T_c 和 T_h ($T_h > T_c$). 用 $\rho_i, \kappa_i, \nu_i, \mu_i$ 和 λ_i 分别表示两流体的密度、热扩散率、运动黏度、动力黏度和导热系数，下标 $i = 1$ 或 2 分别代表下层和上层流体. 定义深宽比 $\varepsilon = h/l$ 、几何尺寸参量 $\delta = r_i/h$, 环形腔曲率用 $\varGamma = r_i/l$ 表示. 假定流体为不可压缩的牛顿流体，物性为常数，温差 $\Delta T = T_h - T_c$ 较小，流动为轴对称二维稳定流动.

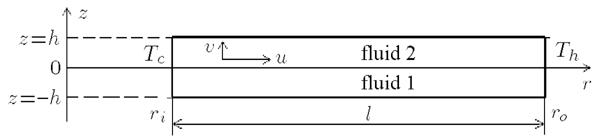

图 1 物理模型

Fig.1 Physical model

分别以 $\varepsilon\nu_1/h$, $\rho_1\nu_1^2/h^2$, $T_h - T_c$ 和 h 作为无量纲参考速度、压力、温度和长度，并引进无因次流函数 ψ 和涡量 ω

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad v = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad \omega = \frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}$$

则无因次控制方程为 (为简化起见，除物性参数外，所有无量纲参数上标 * 均省略)

$$\frac{\varepsilon}{r}\Big(\frac{\partial\psi_i}{\partial r}\frac{\partial\omega_i}{\partial z} - \frac{\partial\psi_i}{\partial z}\frac{\partial\omega_i}{\partial r} + \frac{\omega_i}{r}\frac{\partial\psi_i}{\partial z}\Big) = \frac{\nu_i}{\nu_1}\Big(\frac{\partial^2\omega_i}{\partial r^2} + \frac{1}{r}\frac{\partial\omega_i}{\partial r} - \frac{\omega_i}{r^2} + \frac{\partial^2\omega_i}{\partial z^2}\Big) \tag{1}$$

$$\frac{1}{r}\frac{\partial^2\psi_i}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi_i}{\partial r} + \frac{1}{r}\frac{\partial^2\psi_i}{\partial z^2} = -\omega_i \tag{2}$$

$$\frac{Pr\varepsilon}{r}\Big(\frac{\partial\psi_i}{\partial r}\frac{\partial\theta_i}{\partial z} - \frac{\partial\psi_i}{\partial z}\frac{\partial\theta_i}{\partial r}\Big) = \frac{\kappa_i}{\kappa_1}\Big(\frac{\partial^2\theta_i}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_i}{\partial r} + \frac{\partial^2\theta_i}{\partial z^2}\Big) \tag{3}$$

其中， $Pr = \nu_1/\kappa_1$ 为普朗特数.

相应的无因次边界条件为

$$z = -1: \ \psi_1 = \frac{\partial\psi_1}{\partial z} = \frac{\partial\theta_1}{\partial z} = 0 \tag{4}$$

$$z = 1: \ \psi_2 = \frac{\partial\psi_2}{\partial z} = \frac{\partial\theta_2}{\partial z} = 0 \tag{5}$$

$$\left.\begin{aligned} z = 0: \quad & \psi_1 = \psi_2 = 0, \quad \frac{\partial\psi_1}{\partial z} = \frac{\partial\psi_2}{\partial z} \\ & \theta_1 = \theta_2, \quad \frac{\partial\theta_1}{\partial z} = \lambda^*\frac{\partial\theta_2}{\partial z} \\ & \frac{1}{r}\Big(\frac{\partial^2\psi_1}{\partial z^2} - \mu^*\frac{\partial^2\psi_2}{\partial z^2}\Big) = Re\frac{\partial\theta_1}{\partial r} \end{aligned}\right\} \tag{6}$$

$$r = \delta: \ \psi_i = \partial\psi_i/\partial r = 0, \ \theta_i = 0 \tag{7}$$

$$r = \delta + 1/\varepsilon: \ \psi_i = \partial\psi_i/\partial r = 0, \ \theta_i = 1 \tag{8}$$

其中， $Re = \gamma_{1\text{-}2}l(T_h - T_c)/(\mu_1\nu_1)$ 为毛细雷诺数，$\gamma_{1\text{-}2} = -\partial\sigma/\partial T$ 为液 - 液界面的界面张力温度系数，$\mu^* = \mu_2/\mu_1$ 为上下两层流体的黏度比， $\lambda^* = \lambda_2/\lambda_1$ 为两层流体的导热系数比.

2 中心区域的解

当 $\varepsilon \to 0$ 时， $\varGamma = \varepsilon\delta = r_i/l = O(1)$. 为方便起见，将变量 r 用 x 替代，即 $r = \delta + x$, $1/r = 1/(\delta + x) = \varepsilon/(\varGamma + \varepsilon x)$, 同时做变换 $\hat{x} = \varepsilon x$, $\hat{\psi} = \varepsilon\psi$, $\hat{\omega} = \omega$, $\hat{\theta} = \theta$, 并将各物理量按下式展开 $(\hat{\theta}_i, \hat{\psi}_i, \hat{\omega}_i) = \sum_{j=0}^{N}\varepsilon^j(\hat{\theta}_{ij}, \hat{\psi}_{ij}, \hat{\omega}_{ij})$, 代入无因次方程中，可得到各级方程组和边界条件.

零级方程组和边界条件分别为

$$\frac{\partial^2\hat{\omega}_{i0}}{\partial z^2} = 0 \tag{9}$$

$$\frac{1}{\varGamma + \hat{x}}\frac{\partial^2\hat{\psi}_{i0}}{\partial z^2} = -\hat{\omega}_{i0} \tag{10}$$

$$\frac{\partial^2\hat{\theta}_{i0}}{\partial z^2} = 0 \tag{11}$$

$$z = -1: \ \hat{\psi}_{10} = \frac{\partial\hat{\psi}_{10}}{\partial z} = \frac{\partial\hat{\theta}_{10}}{\partial z} = 0 \tag{12}$$

$$z = 1: \ \hat{\psi}_{20} = \frac{\partial\hat{\theta}_{20}}{\partial z} = \frac{\partial\hat{\psi}_{20}}{\partial z} = 0 \tag{13}$$

$$\left.\begin{aligned} z = 0: \quad & \hat{\psi}_{10} = \hat{\psi}_{20} = 0, \quad \frac{\partial\hat{\psi}_{10}}{\partial z} = \frac{\partial\hat{\psi}_{20}}{\partial z} \\ & \hat{\theta}_{10} = \hat{\theta}_{20}, \quad \frac{\partial\hat{\theta}_{10}}{\partial z} = \lambda^*\frac{\partial\hat{\theta}_{20}}{\partial z} \\ & \frac{\partial^2\hat{\psi}_{10}}{\partial z^2} - \mu^*\frac{\partial^2\hat{\psi}_{20}}{\partial z^2} = 0 \end{aligned}\right\} \tag{14}$$

解方程组，并由边界条件可得以下零级近似解

$$\hat{\psi}_{i0} = \hat{\omega}_{i0} = 0 \tag{15}$$

$$\hat{\theta}_{i0} = g_{i0}(\hat{x}, \varGamma) \tag{16}$$

用同样的方法,通过一级近似方程和边界条件,可以得到以下一级近似解

$$\hat{\psi}_{11} = -c_{11}\Big(\frac{z^3}{6} + \frac{z^2}{3} + \frac{z}{6}\Big) \tag{17}$$

$$\hat{\omega}_{11} = \frac{c_{11}}{\varGamma + \hat{x}}\Big(z + \frac{2}{3}\Big) \tag{18}$$

$$\hat{\theta}_{11} = \hat{\theta}_{21} = g_{11}(\hat{x}, \varGamma) \tag{19}$$

$$\hat{\psi}_{21} = -c_{11}\Big(\frac{z^3}{6} - \frac{z^2}{3} + \frac{z}{6}\Big) \tag{20}$$

$$\hat{\omega}_{21} = \frac{c_{11}}{\Gamma+\hat{x}}\left(z - \frac{2}{3}\right) \tag{21}$$

其中， c_{11} 为仅与 $\hat{x}$ 相关的参量.

将前面的结果代入二级能量等式，并由边界条件可得

$$\frac{\partial^2\hat{\theta}_{22}}{\partial z^2} - \frac{2}{3}\frac{\mu^*+1}{\Gamma+\hat{x}}\frac{1}{Re}\frac{\partial c_{11}}{\partial \hat{x}} = 0 \tag{22}$$

$$\frac{\partial^2\hat{\theta}_{12}}{\partial z^2} - \frac{2}{3}\frac{\mu^*+1}{\Gamma+\hat{x}}\frac{1}{Re}\frac{\partial c_{11}}{\partial \hat{x}} = 0 \tag{23}$$

将式 (22) 两边从 $z = 0 \sim 1$, 式 (23) 两边从 $z = -1 \sim 0$ 分别进行积分得

$$\left.\frac{\partial\hat{\theta}_{12}}{\partial z}\right|_{z=0} = \frac{2}{3}\frac{\mu^*+1}{\Gamma+\hat{x}}\frac{1}{Re}\frac{\partial c_{11}}{\partial \hat{x}} \tag{24}$$

$$\left.\frac{\partial\hat{\theta}_{22}}{\partial z}\right|_{z=0} = -\frac{2}{3}\frac{\mu^*+1}{\Gamma+\hat{x}}\frac{1}{Re}\frac{\partial c_{11}}{\partial \hat{x}} \tag{25}$$

由式 (24),(25) 及边界条件 $z = 0$: $\partial\hat{\theta}_{12}/\partial z = \lambda^*\partial\hat{\theta}_{22}/\partial z$ 可得： $\partial c_{11}/\partial\hat{x} = 0$. 故 c_{11} 是一个常数. 由一级边界条件得

$$\hat{\theta}_{i0} = -\frac{2c_{11}}{3}\frac{\mu^*+1}{Re}\ln(\Gamma+\hat{x}) + c_{11}^* \tag{26}$$

同理，可以通过二级和三级方程和边界条件得到

$$\hat{\psi}_{1j} = -c_{1j}\left(\frac{z^3}{6} + \frac{z^2}{3} + \frac{z}{6}\right) \tag{27}$$

$$\hat{\omega}_{1j} = \frac{c_{1j}}{\Gamma+\hat{x}}\left(z + \frac{2}{3}\right) \tag{28}$$

$$\hat{\theta}_{1j-1} = \hat{\theta}_{2j-1} = -\frac{2c_{1j}}{3}\frac{\mu^*+1}{Re}\ln(\Gamma+\hat{x}) + c_{1j}^* \tag{29}$$

$$\hat{\psi}_{2j} = -c_{1j}\left(\frac{z^3}{6} - \frac{z^2}{3} + \frac{z}{6}\right) \tag{30}$$

$$\hat{\omega}_{2j} = \frac{c_{1j}}{\Gamma+\hat{x}}\left(z - \frac{2}{3}\right) \quad (j = 2,3) \tag{31}$$

由三级能量等式和边界条件可解得

$$\hat{\theta}_{13} = -\frac{2c_{11}^2}{3}\frac{\mu^*+1}{(\Gamma+\hat{x})^2}\frac{Pr}{Re}\left(\frac{z^4}{24} + \frac{z^3}{9} + \frac{z^2}{12}\right) + \Theta_{13} \tag{32}$$

$$\hat{\theta}_{23} = -\frac{2c_{11}^2}{3}\frac{\mu^*+1}{(\Gamma+\hat{x})^2}\frac{1}{\kappa^*}\frac{Pr}{Re}\left(\frac{z^4}{24} - \frac{z^3}{9} + \frac{z^2}{12}\right) + \Theta_{23} \tag{33}$$

其中， $\kappa^* = \kappa_2/\kappa_1$ 为上下两层流体的热扩散率之比， Θ_{13}, Θ_{23} 和 $\hat{x}$ 相关.

由边界条件 $z = 0$: $\hat{\theta}_{13} = \hat{\theta}_{23}$ 可得 $\Theta_{13} = \Theta_{23}$.

代入前几级结果，可以得到中心区域的五级能量等式

$$\frac{\partial^2\hat{\theta}_{15}}{\partial z^2} + \frac{4M}{(\Gamma+\hat{x})^4}\left(\frac{z^4}{24} + \frac{z^3}{9} + \frac{z^2}{12}\right) + \frac{\partial^2\Theta_{13}}{\partial\hat{x}^2} + \frac{1}{\Gamma+\hat{x}}\frac{\partial\Theta_{13}}{\partial\hat{x}} = 0 \tag{34}$$

$$\frac{\partial^2\hat{\theta}_{25}}{\partial z^2} + \frac{4M}{\kappa^*(\Gamma+\hat{x})^4}\left(\frac{z^4}{24} - \frac{z^3}{9} + \frac{z^2}{12}\right) + \frac{\partial^2\Theta_{23}}{\partial\hat{x}^2} + \frac{1}{\Gamma+\hat{x}}\frac{\partial\Theta_{23}}{\partial\hat{x}} = 0 \tag{35}$$

其中， $M = -\dfrac{2(\mu^*+1)}{3}\dfrac{c_{11}^2 Pr}{Re}$.

式 (34) 两边从 $z = -1 \sim 0$, 式 (35) 两边从 $z = 0 \sim 1$ 分别进行积分可以得到

$$\frac{\partial^2\Theta_{13}}{\partial\hat{x}^2} + \frac{1}{\Gamma+\hat{x}}\frac{\partial\Theta_{13}}{\partial\hat{x}} = -\frac{M}{30}\frac{1}{(\Gamma+\hat{x})^4} - \left.\frac{\partial\hat{\theta}_{15}}{\partial z}\right|_{z=0} \tag{36}$$

$$\frac{\partial^2\Theta_{23}}{\partial\hat{x}^2} + \frac{1}{\Gamma+\hat{x}}\frac{\partial\Theta_{23}}{\partial\hat{x}} = -\frac{M}{30\kappa^*}\frac{1}{(\Gamma+\hat{x})^4} + \left.\frac{\partial\hat{\theta}_{25}}{\partial z}\right|_{z=0} \tag{37}$$

由 $\Theta_{13} = \Theta_{23}$ 及边界条件 $z = 0$: $\partial\hat{\theta}_{15}/\partial z = \lambda^*\partial\hat{\theta}_{25}/\partial z$, 联立式 (36), (37) 解得

$$\Theta_{13} = \Theta_{23} = a_3'\ln(\Gamma+\hat{x}) + d_3 - \frac{M}{120}\frac{\lambda^*+\kappa^*}{\lambda^*\kappa^*+\kappa^*}\frac{1}{(\Gamma+\hat{x})^2} \tag{38}$$

所以有

$$\hat{\theta}_{13} = \frac{M}{(\Gamma+\hat{x})^2}\left(\frac{z^4}{24} + \frac{z^3}{9} + \frac{z^2}{12} - \frac{1}{120}\frac{\lambda^*+\kappa^*}{\lambda^*\kappa^*+\kappa^*}\right) + a_3'\ln(\Gamma+\hat{x}) + d_3 \tag{39}$$

$$\hat{\theta}_{23} = \frac{M}{\kappa^*(\Gamma+\hat{x})^2}\left(\frac{z^4}{24} - \frac{z^3}{9} + \frac{z^2}{12} - \frac{1}{120}\frac{\lambda^*+\kappa^*}{\lambda^*+1}\right) + a_3'\ln(\Gamma+\hat{x}) + d_3 \tag{40}$$

上述各式中， c_{1j}, c_{1j}^*, a_3' 和 d_3 可以通过与边界区域的匹配得到.

3 与边界区域的匹配

用 $\theta_i, \psi_i, \omega_i$ 表示在边界区域的无因次温度、流函数和涡量. 为保持与中心区域的一致, 令 $\bar{\psi}_i = \varepsilon\psi_i$, 则边界区域的控制方程为

$$\frac{\varepsilon}{\Gamma+\varepsilon x}\left(\frac{\partial\bar{\psi}_i}{\partial x}\frac{\partial\omega_i}{\partial z} - \frac{\partial\bar{\psi}_i}{\partial z}\frac{\partial\omega_i}{\partial x} + \frac{\varepsilon\omega_i}{\Gamma+\varepsilon x}\frac{\partial\bar{\psi}_i}{\partial z}\right) = \frac{\nu_i}{\nu_1}\left(\frac{\partial^2\omega_i}{\partial x^2} + \frac{\varepsilon}{\Gamma+\varepsilon x}\frac{\partial\omega_i}{\partial x} - \frac{\varepsilon^2\omega_i}{(\Gamma+\varepsilon x)^2} + \frac{\partial^2\omega_i}{\partial z^2}\right) \tag{41}$$

$$\frac{1}{\Gamma+\varepsilon x}\left(\frac{\partial^2\bar{\psi}_i}{\partial x^2}-\frac{\varepsilon}{\Gamma+\varepsilon x}\frac{\partial\bar{\psi}_i}{\partial x}+\frac{\partial^2\bar{\psi}_i}{\partial z^2}\right)=-\omega_i \tag{42}$$

$$\frac{Pr\varepsilon}{\Gamma+\varepsilon x}\left(\frac{\partial\bar{\psi}_i}{\partial x}\frac{\partial\theta_i}{\partial z}-\frac{\partial\bar{\psi}_i}{\partial z}\frac{\partial\theta_i}{\partial x}\right)=\frac{\kappa_i}{\kappa_1}\left(\frac{\partial^2\theta_i}{\partial x^2}+\frac{\varepsilon}{\Gamma+\varepsilon x}\frac{\partial\theta_i}{\partial x}+\frac{\partial^2\theta_i}{\partial z^2}\right) \tag{43}$$

中心区域与边界区域匹配条件为：当 $\varepsilon\to 0$ 时

$$\lim_{\hat{x}\to 0}(\hat{\theta},\hat{\psi},\hat{\omega})_{\rm core}\Leftrightarrow\lim_{x\to\infty}(\theta,\bar{\psi},\omega)_{\rm cold} \tag{44}$$

$$\lim_{\hat{x}\to 1}(\hat{\theta},\hat{\psi},\hat{\omega})_{\rm core}\Leftrightarrow\lim_{\xi\to\infty}(\theta,\bar{\psi},\omega)_{\rm hot} \tag{45}$$

其中，$\xi=1/\varepsilon-x$.

同样，分别将各物理量按渐近形式近似展开，并代入到控制方程中，得到边壁区各级方程组.

由边壁区域零级方程和边界条件可得：$x=0$: $\theta_{i0}=0$(冷壁); $x=\varepsilon^{-1}$: $\theta_{i0}=1$(热壁). 由匹配条件得到

$$S_1\ln\Gamma+S_1\left[\frac{\varepsilon x}{\Gamma}-\frac{1}{2}\left(\frac{\varepsilon x}{\Gamma}\right)^2+\cdots\right]+c_{11}^*=0 \tag{46}$$

$$S_1\ln(\Gamma+1)+S_1\left[-\frac{\varepsilon\xi}{1+\Gamma}-\frac{1}{2}\left(\frac{\varepsilon\xi}{1+\Gamma}\right)^2-\cdots\right]+c_{11}^*=1 \tag{47}$$

其中，$S_1=-\dfrac{2c_{11}}{3}\dfrac{\mu^*+1}{Re}$. 解之得

$$c_{11}=\frac{3}{2\mu^*+2}\frac{Re}{\ln[\Gamma/(\Gamma+1)]},\quad c_{11}^*=\frac{\ln\Gamma}{\ln[\Gamma/(1+\Gamma)]}$$

将在冷壁区域的 θ_{i0} 代入到能量等式在冷壁区域的一级方程，可得

$$\frac{\partial^2\theta_{11}}{\partial x^2}+\frac{\partial^2\theta_{11}}{\partial z^2}=0 \tag{48}$$

$$\frac{\partial^2\theta_{21}}{\partial x^2}+\frac{\partial^2\theta_{21}}{\partial z^2}=0 \tag{49}$$

式 (48) 两边从 $z=-1\sim 0$, 式 (49) 两边从 $z=0\sim 1$ 分别进行积分并利用边界条件得

$$\int_{-1}^{0}\theta_{11}{\rm d}z+\lambda^*\int_0^1\theta_{21}{\rm d}z=a_1x$$

再结合匹配条件有

$$(\lambda^*+1)\left(S_1\frac{x}{\Gamma}+S_2\ln\Gamma+c_{12}^*\right)=a_1x \tag{50}$$

其中，$S_1\dfrac{x}{\Gamma}$ 为在零级匹配时忽略的一阶小量，$S_2=-\dfrac{2c_{12}}{3}\dfrac{\mu^*+1}{Re}$.

同理，在热壁处有

$$(\lambda^*+1)\left(-S_1\frac{\xi}{\Gamma+1}+S_2\ln(\Gamma+1)+c_{12}^*\right)=a_2\xi \tag{51}$$

解由式 (50) 和式 (51) 组成的方程组可得

$$a_1=\frac{\lambda^*+1}{\Gamma}S_1,\ a_2=-\frac{\lambda^*+1}{\Gamma+1}S_1,\ c_{12}=c_{12}^*=0$$

利用同样的方法，通过二级和三级匹配，可以得到

$$c_{13}=c_{13}^*=a_3'=d_3=0 \tag{52}$$

4 结果与讨论

将以上求得的各系数的值代入各级近似解的表达式，可得到中心区域的解，结合流函数的定义，得到温度分布和速度分布的渐近解为

$$\theta_1\approx\theta_{10}+\varepsilon\theta_{11}+\varepsilon^2\theta_{12}+\varepsilon^3\theta_{13}=\frac{\ln\dfrac{\Gamma}{\Gamma+\varepsilon x}}{\ln\dfrac{\Gamma}{\Gamma+1}}-\frac{3}{2\mu^*+2}\frac{PrRe}{\left(\ln\dfrac{\Gamma}{\Gamma+1}\right)^2}\frac{\varepsilon^3}{(\Gamma+\varepsilon x)^2}\cdot\left(\frac{z^4}{24}+\frac{z^3}{9}+\frac{z^2}{12}-\frac{1}{120}\frac{\lambda^*+\kappa^*}{\lambda^*\kappa^*+\kappa^*}\right) \tag{53}$$

$$\theta_2\approx\theta_{20}+\varepsilon\theta_{21}+\varepsilon^2\theta_{22}+\varepsilon^3\theta_{23}=\frac{\ln\dfrac{\Gamma}{\Gamma+\varepsilon x}}{\ln\dfrac{\Gamma}{\Gamma+1}}-\frac{3}{2\mu^*\kappa^*+2\kappa^*}\frac{PrRe}{\left(\ln\dfrac{\Gamma}{\Gamma+1}\right)^2}\frac{\varepsilon^3}{(\Gamma+\varepsilon x)^2}\cdot\left(\frac{z^4}{24}-\frac{z^3}{9}+\frac{z^2}{12}-\frac{1}{120}\frac{\lambda^*+\kappa^*}{\lambda^*+1}\right) \tag{54}$$

$$u_1\approx\frac{3Re}{2\mu^*+2}\frac{1}{\ln\dfrac{\Gamma}{\Gamma+1}}\frac{\varepsilon}{\Gamma+\varepsilon x}\left(\frac{z^2}{2}+\frac{2z}{3}+\frac{1}{6}\right) \tag{55}$$

$$u_2\approx\frac{3Re}{2\mu^*+2}\frac{1}{\ln\dfrac{\Gamma}{\Gamma+1}}\frac{\varepsilon}{\Gamma+\varepsilon x}\left(\frac{z^2}{2}-\frac{2z}{3}+\frac{1}{6}\right) \tag{56}$$

当 $r_i\to\infty$, 即 $\Gamma\to\infty$ 时，上述结果简化为

$$\theta_1=\varepsilon x-\frac{3}{2\mu^*+2}PrRe\varepsilon^3\cdot\left(\frac{z^4}{24}+\frac{z^3}{9}+\frac{z^2}{12}-\frac{1}{120}\frac{\lambda^*+\kappa^*}{\lambda^*\kappa^*+\kappa^*}\right) \tag{57}$$

$$\theta_2=\varepsilon x-\frac{3}{2\mu^*\kappa^*+2\kappa^*}PrRe\varepsilon^3\cdot\left(\frac{z^4}{24}-\frac{z^3}{9}+\frac{z^2}{12}-\frac{1}{120}\frac{\lambda^*+\kappa^*}{\lambda^*+1}\right) \tag{58}$$

$$u_1=\frac{3\varepsilon Re}{2\mu^*+2}\left(\frac{z^2}{2}+\frac{2z}{3}+\frac{1}{6}\right) \tag{59}$$

$$u_2=\frac{3\varepsilon Re}{2\mu^*+2}\left(\frac{z^2}{2}-\frac{2z}{3}+\frac{1}{6}\right) \tag{60}$$

从物理模型来看，当 $\Gamma\rightarrow\infty$ 时，环形液池退化为矩形液池，从解的结果看，此时，式 (57)~(60) 与 Nepomnyashchy 等 [13] 得到的矩形腔内的分析解完全一致.

为了进一步检验解的正确性，进行了数值模拟、并与渐近解进行了比较. 利用有限差分法对基本方程进行离散，扩散项采用中心差分，对流项采用 QUICK 格式，压力 - 速度修正采用 SIMPLE 方法，网格为 $102^r\times 53^z$. 程序的正确性和网格的收敛性已被证实 [13]. 流体为硅油 (Silicon oil 5cSt 上层) 和氟化液 (HT-70 下层)，物性参数同文献 [10]，结果如图 2 和图 3 所示. 显然，温度分布两者基本一致，径向速度分布在中心区域吻合较好，但靠近壁面处误差增大，主要原因是现在的渐近解中没有考虑边界层效应.

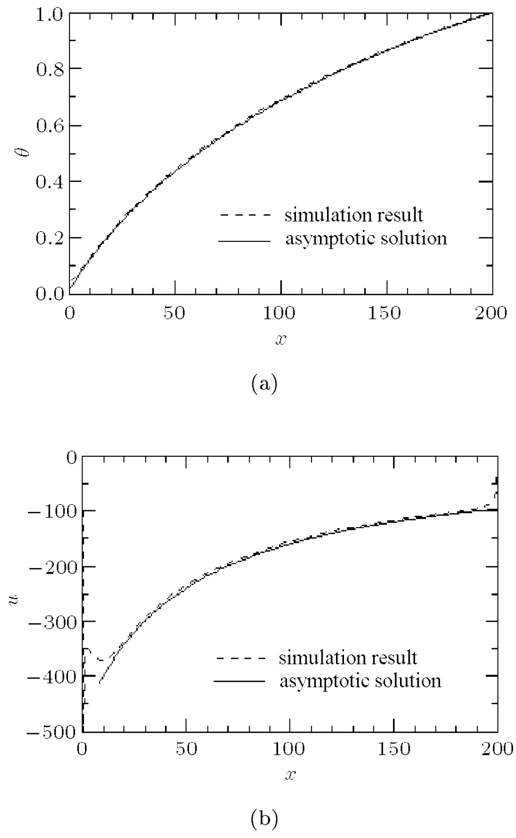

图 2 $\varepsilon=0.005$ 和 $Re=10^6$ 时液 - 液界面处的温度分布 (a) 和速度分布 (b)

Fig.2 The radial distributions of temperature (a) and radial velocity (b) at the interface at $\varepsilon=0.005$, $Re=10^6$

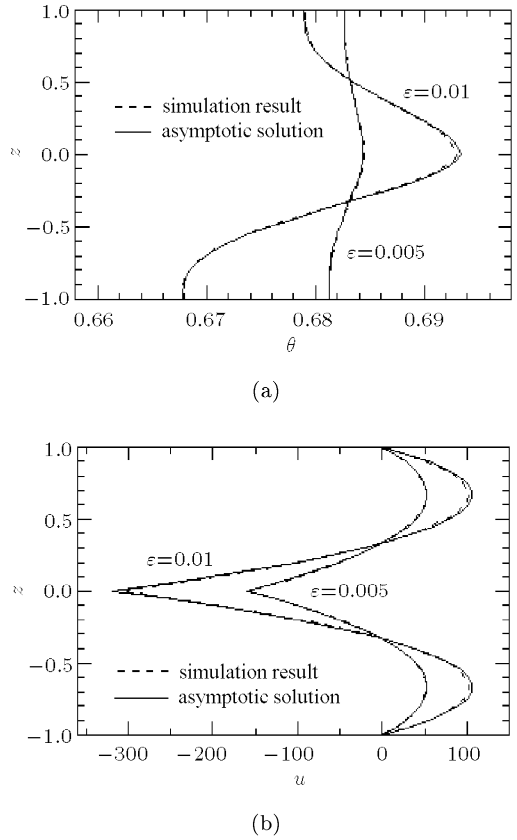

图 3 $\Gamma=0.25$ 和 $Re=10^6$ 时中间截面温度 (a) 和速度 (b) 的垂直分布

Fig.3 Distributions of temperature (a) and radial velocity (b) in the middle part as a function of z for the case of $\Gamma=0.25$ and $Re=10^6$

5 结 论

采用渐近线方法得到了微重力下环形腔内双层薄液层热毛细对流的温度和速度分布的渐近解，结果发现：(1) 当环形池曲率 $\Gamma\rightarrow\infty$ 时，解演变为 Nepomnyashchy 等 [11] 得到的矩形腔内的分析解，说明结果是正确的；(2) 与数值解的比较表明，中心区域两者吻合较好，壁面附近边界层效应对解的影响较大.

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ASYMPTOTIC SOLUTION OF THERMOCAPILLARY CONVECTION OF THIN TWO-LAYER SYSTEM IN AN ANNULAR CAVITY [1)]

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Abstract　The convection phenomena in two-layer liquid systems have attracted great attention in the past two decades, mainly owing to their relevance in nature and in many engineering applications. Many works have been carried out to investigate the thermocapillary convection in two-layer liquid systems in rectangular cavities or in infinite horizontal layers. However, few studies focused on the convection phenomena in two-layer liquid systems in the annular cavity. In order to understand the basic characteristics of thermocapillary convection of the thin two superposed horizontal liquid layers subjected to a radial temperature gradient in an annular cavity, an approximate analytical solution is obtained using asymptotical analysis. The cavity is heated from the outer cylindrical wall and cooled at the inner wall. Bottom and top surfaces are adiabatic. Results show that the expressions of velocity and temperature field in the core region are the same as the results obtained by Nepomnyashchy et al (Physics of Fluids, 2006, 18: 032105) when thin annular pool approaches to thin two-dimensional slot. The numerical experiments are also carried out to compare with the asymptotic solution. It is found that there is a good agreement between the asymptotic solution and numerical result in the core region.

Key words　thermocapillary convection, annular cavity, thin two-layer system, radial temperature gradient, asymptotic solution

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