2013年12月 运筹学学报 第17卷 第4期
Dec., 2013 Operations Research Transactions Vol.17 No.4

An optimal solution for economic production quantity models with deteriorating items and time-varying production cost*

BAI Qingguo[1,2,†] XU Jianteng[2] ZHANG Yuzhong[2] XU Xianhao[1]

Abstract A generalized economic production quantity model with deteriorating items over a finite planning horizon is considered in this paper. The unit production cost, the production rate and the demand rate are assumed to be known and continuous functions of time, and the forgetting effect of setup cost is incorporated into the problem. Shortages are not allowed in this model. A mixed-integer constraint optimization mathematical model in which the objective is to minimize the total cost is established and the conditions of the optimal solution for this problem are derived. A discrete variable in the total cost function is relaxed to the continuous variable and this technique is used to prove the uniqueness and optimality of the optimal solution for a special case. In addition, the optimal solution of the special case is regarded as the initial condition to simplify the search process of finding the optimal solution of the generalized problem. Finally, a numerical example is provided to illustrate the results.

Keywords operations research, inventory model, convex function, deterioration, time-varying cost

Chinese Library Classification O227

2010 Mathematics Subject Classification 90B05, 90C11, 90C39

带时变生产成本的易变质经济批量模型的最优策略分析*

柏庆国[1,2,†] 徐健腾[2] 张玉忠[2] 徐贤浩[1]

摘要 考虑了具有时变生产成本的易变质产品经济批量模型. 有限计划期内，单位生产成本、生产率以及需求率假定为时间的连续函数，生产固定成本则具有遗忘效应现象. 当不允许缺货时，建立了以总成本最小为目标的混合整数优化模型并证明了此问题最优解的相关性质. 对于此问题的特殊情形，将成本函数中的离散型变量松弛为连续型变量, 通过分析其最优解的存在性及唯一性, 求解了此最优解，将其作为初始值设计了求取一般情形最优解的有效算法. 最后通过算例验证了理论结果的有效性.

收稿日期：2011年11月3日

* Supported by the State Key Program of National Natural Science Foundation of China (No. 71131004), National Natural Science Foundation of China (Nos. 11071142, 71371107), Shandong Province Natural Science Foundation (Nos. BS2013SF016, ZR2011AL017)

1. School of Management, Huazhong University of Science and Technology, Wuhan 430074, China; 华中科技大学管理学院, 武汉 430074

2. School of Management, Qufu Normal University, Rizhao 276826, Shandong, China; 曲阜师范大学管理学院, 山东日照 276826

† 通讯作者 Corresponding author, E-mail: qfnubaiqg@163.com

关键词 运筹学, 库存模型, 凸函数, 易变质, 时变成本
中图分类号 O227
2010 数学分类号 90B05, 90C11, 90C39

0 Introduction

The traditional economic production quantity (EPQ) model is widely used by practitioners as a decision-making tool for inventory control. It plays an important role in determining the production quantity on a single facility so as to meet the deterministic demand over an infinite planning horizon. Currently, Many researchers focus their study on the generalized EPQ model for deteriorating items. Deterioration is defined as decay, evaporation, obsolescence, and loss of quality marginal value of commodity that result in decreasing usefulness from its original condition. In reality, a lot of products deteriorate during storage, like vegetables, milks, and fruits. Misra[1] initially studied the EPQ model for deteriorating items with both varying and constant rate of deterioration. Pasandideh and Niaki[2] expanded the EPQ model by assuming that the orders may be delivered discretely in the form of multiple pallets. They formulated the problem as a non-linear-integer-programming model and proposed a genetic algorithm to solve it. Teng and Chang[3] studied the optimal replenishment policies in the EPQ model under two levels of trade credit policies. More information related to this issue can be found in [4–9].

All of the above-mentioned literatures assumed that the demand rate or deterioration rate varies with time and did not incorporate the dynamic unit cost or the setup cost into their models. In fact, in time-based competition today, it is a quite natural phenomenon that the unit cost or the setup cost of products varies with time. For instance, seasonal variations may cause the increase or decrease in the unit production cost of certain commodity. Consequently, the EPQ problem with time-varying cost has been studied by researchers. Teng et al.[10] extended the EPQ model without deterioration in which the demand rate and the unit production cost are positive and fluctuating with time. In addition, owing to the increasing emphasis on time-based competition, the importance of learning and forgetting effects on production has been widely recognized. Some researchers extended the EPQ model by incorporating the forgetting effect into setup cost or the production rate. The forgetting effect is mainly caused by a break between two consecutive production runs and leads to retrogression in learning. Carlson and Rowe[11] firstly presented the forgetting curve equation to describe the forgetting or interruption portion of the learning cycle by improving the learning cure equation. Other extensions can be found in [12–14]. However, to our best knowledge, few researchers have considered the EPQ model for deteriorating items with forgetting effect of setup cost and time-varying unit production cost over the finite planning horizon.

In view of the above arguments, this paper incorporates the forgetting effect of setup cost and time-varying unit production cost into the EPQ model for deteriorating items over a finite planning horizon. The production rate and demand rate are time-varying and the unit production cost is continuous, differentiable and non-increasing function with time in this generalized EPQ model. A mixed-integer non-linear cost minimization model is formulated to determine the optimal production quantity, start time and end time over the finite planning horizon.

The remainder of this paper is organized as follows. Section 1 defines the notations used throughout the paper and states the basic assumptions. The generalized EPQ model is derived in Section 2. Section 3 proves that the optimal production schedule exists uniquely and the total cost is a convex function of the replenishment times. A simple and efficient algorithm to find the optimal solution is also developed. In addition, the search process of the optimal solution is simplified by proving some optimal properties of the special case as a possible start value in this section. A numerical example is given to illustrate the algorithm and results in Section 4. Conclusion and future research suggestions are provided in Section 5.

1 Assumptions and notation

The generalized EPQ model considered in this paper is on the basis of the following assumptions:

1) The forgetting phenomenon for the setup cost in production is characterized by the forgetting curve equation proposed in [11].

2) shortages are not allowed.

3) lead time is zero and replenishment is instantaneous.

4) The planning horizon is finite and is taken as H time units. The initial and final inventory levels of the planning horizon are both zero.

5) A constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory.

6) The unit production cost is predetermined at the beginning of each production cycle.

For convenience, the following notations are used throughout this paper:

$f(t)$ the demand rate at time t. Without loss of generality, we assume that $f(t)$ is continuous in the planning horizon $[0, H]$, and $f(t) > 0$ over $(0, H]$.

$K(t)$ the production rate at time t, which satisfies $K(t) > f(t)$ for any t. We assume that $K(t)$ is continuous in the planning horizon $[0, H]$, and $K(t) > 0$ over $(0, H]$.

$I(t)$ the inventory level at time t.

$c(t)$ the unit production cost at time t. We assume that $c(t)$ is positive and non-increasing in the planning horizon $[0, H]$.

θ the deterioration rate.

c_1 the holding cost per unit product per unit time.

c_2 the deterioration cost per unit deteriorated product.

ϕ the forgetting rate. It indicates increment speed of setup cost.

b the forgetting coefficient, $b = -\frac{ln\phi}{ln2}$, $0 < b < 1$.

n the total number of replenishments over $[0, H]$ (a decision variable).

A_i the setup cost for the ith production cycle, $A_1 \leqslant \cdots \leqslant A_n, i = 1, \cdots, n$. Following the forgetting effect curve, A_i can be characterized by $A_i = A_1 i^b$ for $i = 1, \cdots, n$.

s_i the start time of the ith production cycle with $s_1 = 0$ and $s_{n+1} = H$ (a decision variable).

t_i the end time of the ith production cycle. It is also the time when inventory level reaches the maximum in the ith production cycle (a decision variable).

2 Mathematical model

The objective of the generalized EPQ problem is to determine the number of replenishments n, and the start time $\{s_i\}$ and end time $\{t_i\}$ of the production over a finite horizon $[0, H]$ in order to minimize the total relevant cost. The ith production cycle starts at time s_i and ends at time t_i, $t_i > s_i$. Since the production rate is higher than the demand rate, a portion of products is used to meet the current demand and the rest is accumulated as inventory. As a result, the inventory level is gradually increasing from s_i to t_i. After the production stops at t_i, the accumulated inventory then gradually decreases due to demand until it reduces to zero at s_{i+1}. Consequently, the inventory level for the generalized EPQ problem with forgetting effect of setup cost can be shown in Fig. 1.

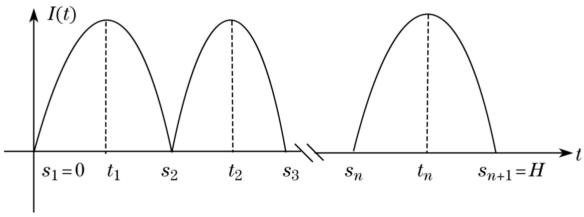

Fig. 1 Graphical representation of inventory level

Since the inventory is depleted by the combined effect of production, demand and deterioration, the inventory level at time t during the production run $[s_i, t_i]$ is governed by the following differential equation

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = K(t) - f(t) - \theta I(t), \quad s_i \leqslant t \leqslant t_i, \tag{2.1}$$

with the boundary condition $I(s_i) = 0$. Solving the differential equation (2.1), we have

$$I(t) = \int_{s_i}^{t} [K(u) - f(u)]\mathrm{e}^{\theta(u-t)}\mathrm{d}u, \quad s_i \leqslant t \leqslant t_i. \tag{2.2}$$

As a result, we obtain the time-weighted inventory during the production run $[s_i, t_i]$ as

$$I_i^1 = \int_{s_i}^{t_i} I(t)\mathrm{d}t = \frac{1}{\theta}\int_{s_i}^{t_i} [K(t) - f(t)][1 - \mathrm{e}^{\theta(t-t_i)}]\mathrm{d}t. \tag{2.3}$$

Similarly, the inventory level at time t during $[t_i, s_{i+1}]$, $I(t)$ can be represented by the following differential equation

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -f(t) - \theta I(t), \quad t_i \leqslant t \leqslant s_{i+1}, \tag{2.4}$$

with the boundary condition $I(s_{i+1}) = 0$. Solving the differential equation (2.4), we have

$$I(t) = \int_t^{s_{i+1}} \mathrm{e}^{\theta(u-t)} f(u)\mathrm{d}u, \quad t_i \leqslant t \leqslant s_{i+1}. \tag{2.5}$$

Thus, the cumulative inventory during $[t_i, s_{i+1}]$ is

$$I_i^2 = \int_{t_i}^{s_{i+1}} I(t)\mathrm{d}t = \frac{1}{\theta}\int_{t_i}^{s_{i+1}} [\mathrm{e}^{\theta(t-t_i)} - 1] f(t)\mathrm{d}t. \tag{2.6}$$

Noting the continuity of $I(t)$ at time t_i, from (2.2) and (2.5) we have

$$\int_{s_i}^{t_i} [K(t) - f(t)]\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t = \int_{t_i}^{s_{i+1}} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t, \tag{2.7}$$

which is followed by

$$\int_{s_i}^{t_i} K(t)\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t = \int_{s_i}^{s_{i+1}} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t. \tag{2.8}$$

From (2.3) and (2.6), the holding cost for the ith cycle is

$$HC_i = c_1(I_i^1 + I_i^2), \quad i = 1, \cdots, n. \tag{2.9}$$

Using partial integration, we have

$$HC_i = c_1\Big\{\frac{1}{\theta}\int_{s_i}^{t_i} [K(t) - f(t)][1 - \mathrm{e}^{\theta(t-t_i)}]\mathrm{d}t + \frac{1}{\theta}\int_{t_i}^{s_{i+1}} [\mathrm{e}^{\theta(t-t_i)} - 1] f(t)\mathrm{d}t\Big\}. \tag{2.10}$$

Similarly, the deterioration cost in the ith cycle is

$$\begin{aligned} DC_i &= c_2\theta(I_i^1 + I_i^2) \\ &= c_2\Big\{\int_{s_i}^{t_i} [K(t) - f(t)][1 - \mathrm{e}^{\theta(t-t_i)}]\mathrm{d}t + \int_{t_i}^{s_{i+1}} [\mathrm{e}^{\theta(t-t_i)} - 1] f(t)\mathrm{d}t\Big\}. \end{aligned} \tag{2.11}$$

Recall that the unit production cost is predetermined at the beginning of each production cycle in assumption 6, we obtain the unit production cost in the ith cycle is $c(s_i)$ and the production cost in the ith cycle is

$$PC_i = Kc(s_i)(t_i - s_i). \tag{2.12}$$

The total cost in the finite planning horizon can be expressed as

$$\begin{aligned} TC(n, \{s_i\}, \{t_i\}) &= \sum_{i=1}^n A_i + \sum_{i=1}^n PC_i + \sum_{i=1}^n (HC_i + DC_i) \\ &= \sum_{i=1}^n A_i + \sum_{i=1}^n c(s_i)\int_{s_i}^{t_i} K(t)\mathrm{d}t + \sum_{i=1}^n \frac{(c_1 + \theta c_2)}{\theta} \\ &\cdot\Big\{\int_{s_i}^{t_i} K(t)[1 - \mathrm{e}^{\theta(t-t_i)}]\mathrm{d}t + \int_{s_i}^{s_{i+1}} [\mathrm{e}^{\theta(t-t_i)} - 1] f(t)\mathrm{d}t\Big\}. \end{aligned} \tag{2.13}$$

Equation (2.13) shows that the total cost depends on a discrete variable n, and continuous variables s_i and t_i. Then the optimal production schedule of the generalized EPQ problem with $s_1 = 0$ and $s_{n+1} = H$ can be solved via the following optimization model

$$\begin{aligned} &\min \ TC(n, \{s_i\}, \{t_i\}) \\ &\text{s.t.} \int_{s_i}^{t_i} K(t)\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t = \int_{s_i}^{s_{i+1}} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t, \quad i = 1, \cdots, n, \qquad (2.14) \\ &\quad s_i \leqslant t_i \leqslant s_{i+1}, \quad i = 1, \cdots, n. \qquad (2.15) \end{aligned}$$

3 The theoretical results and solution procedure

3.1 The theoretical results

Since the total cost $TC(n, \{s_i\}, \{t_i\})$ is a mixed-integer non-linear programming where the decision variable n is an integer and the decision variables s_i and t_i are continuous real values. Usually, it is difficult to obtain a closed form solution for a mixed-integer non-linear programming problem. Thus, we solve the problem using the following two-stage procedure:

(1) for a given value of n, we present an algorithm which can be used to determine the optimal s_i^*, and t_i^*;

(2) obtain the optimal value of n which minimizes the total cost $TC(n, \{s_i\}, \{t_i\})$. For any given value of n, if we ignore (2.15), the problem is reduced to an equality-constrained problem in which the Lagrange is

$$\begin{aligned} L(n, \{s_i\}, \{t_i\}, \{\lambda_i\}) = TC(n, \{s_i\}, \{t_i\}) + \sum_{i=1}^{n} \lambda_i \Big\{ \int_{s_i}^{t_i} K(t)\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t \\ - \int_{s_i}^{s_{i+1}} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t \Big\}, \qquad (3.1) \end{aligned}$$

where λ_i, $i = 1, \cdots, n$, is the Lagrangian multiplier.

By taking the first partial derivatives of $L(n, \{s_i\}, \{t_i\}, \{\lambda_i\})$ with respect to s_i, t_i and λ_i, and setting the result to zero. After simplification, we obtain the following optimality conditions

$$\lambda_i = -c(s_i), \qquad (3.2)$$

$$\begin{aligned} c'(s_i)\int_{s_i}^{t_i} K(t)\mathrm{d}t - c(s_i)K(s_i) + \frac{(c_1 + \theta c_2)}{\theta}\{[\mathrm{e}^{\theta(s_i - t_i)} - 1]K(s_i) + [\mathrm{e}^{\theta(s_i - t_{i-1})} \\ -\mathrm{e}^{\theta(s_i - t_i)}]f(s_i)\} + c(s_i)\mathrm{e}^{\theta(s_i - t_i)}[K(s_i) - f(s_i)] + c(s_{i-1})\mathrm{e}^{\theta(s_i - t_{i-1})} f(s_i) = 0 \end{aligned} \qquad (3.3)$$

and

$$\int_{s_i}^{t_i} K(t)\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t = \int_{s_i}^{s_{i+1}} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t. \qquad (3.4)$$

From (3.2)–(3.4), we can obtain the following theorem.

Theorem 3.1 *For any given* n*, the optimal values of* $\{s_i\}$ *and* $\{t_i\}$ *are uniquely determined by* (3.3) *and* (3.4).

Proof $TC(n, \{s_i\}, \{t_i\})$ is a continuous and differentiable function minimized over the compact set $[0, H]^{2n}$. Hence there exists a global minimum. The optimal value of s_i cannot be on the boundary since $TC(n, \{s_i\}, \{t_i\})$ increases when any one of the s_i is shifted to the end points 0 or H. Consequently, any optimal solution to the problem must be an interior point, which implies that there exists at least one solution to (3.3) and (3.4) simultaneously. Next, we prove the solution of (3.3) and (3.4) is unique. For any given t_{i-1} and s_i, we let

$$\begin{aligned}F(x) = {} & c'(s_i)\int_{s_i}^{x} K(t)\mathrm{d}t - c(s_i)K(s_i) + \frac{(c_1+\theta c_2)}{\theta}\{[\mathrm{e}^{\theta(s_i-x)}-1]K(s_i) \\ & +[\mathrm{e}^{\theta(s_i-t_{i-1})}-\mathrm{e}^{\theta(s_i-x)}]f(s_i)\} + c(s_i)\mathrm{e}^{\theta(s_i-x)}[K(s_i)-f(s_i)] + c(s_{i-1}) \\ & \times \mathrm{e}^{\theta(s_i-t_{i-1})}f(s_i). \end{aligned} \tag{3.5}$$

Since $c(t)$ is a positive and non-increasing function in $[0, H]$ and $K(t) > f(t)$, we have

$$F'(x) = c'(s_i)K(x) + [c_1+\theta c_2+\theta c(s_i)][f(s_i)-K(s_i)]\mathrm{e}^{\theta(s_i-x)} < 0. \tag{3.6}$$

In addition,

$$\begin{aligned}F(s_i) & = \frac{(c_1+\theta c_2)}{\theta}[\mathrm{e}^{\theta(s_i-t_{i-1})}-1]f(s_i) - c(s_i)f(s_i) + c(s_{i-1})\mathrm{e}^{\theta(s_i-t_{i-1})}f(s_i) \\ & \geqslant \Big[\frac{(c_1+\theta c_2)}{\theta} + c(s_i)\Big][\mathrm{e}^{\theta(s_i-t_{i-1})}-1]f(s_i) > 0 \end{aligned} \tag{3.7}$$

and $F(+\infty) < 0$, therefore, for any given t_{i-1} and s_i, there exists a unique $t_i^*(> s_i)$ such that $F(t_i^*) = 0$, which implies that solution to (3.3) uniquely exists. Similarly, let

$$G(x) = \int_{s_i}^{t_i} K(t)\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t - \int_{s_i}^{x} \mathrm{e}^{\theta(t-t_i)}f(t)\mathrm{d}t. \tag{3.8}$$

We then obtain $G(t_i) = \int_{s_i}^{t_i} \mathrm{e}^{\theta(t-t_i)}[K(t)-f(t)]\mathrm{d}t > 0$, $G(+\infty) < 0$ and $G'(x) = -\mathrm{e}^{\theta(x-t_i)}f(x) < 0$. As a result, we know that there exists a unique $s_{i+1}^*(> t_i)$ such that $G(s_{i+1}^*) = 0$. Thus, the solution to (3.4) uniquely exists.

The result of Theorem 3.1 reduces the $2n$-dimensional problem of finding s_i^* and t_i^* to a one-dimensional problem. Since $s_1 = 0$, we only need to find t_1^* to generate s_2^* by (3.4), and then the rest of $\{s_i^*\}$ and $\{t_i^*\}$ uniquely by repeatedly using (3.3) and (3.4). For any chosen t_1^*, if $s_n^* = H$, then t_1^* is chosen correctly. Otherwise, we can easily find the optimal t_1^* by standard search techniques. For any given value of n, the solution procedure for finding s_i^* and t_i^* can be obtained by the algorithm in [5] with $L = 0$ and $U = H/4n$ or any standard search method.

We employ the following approximation for the term of the setup cost in the objective function of the problem (for example, [15, 16]).

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} A_1 i^b \approx \int_0^n A_1 i^b \mathrm{d}i = \frac{A_1}{b+1}n^{b+1}. \tag{3.9}$$

Next, by applying Bellman's principle of optimality[17], we prove the total cost $TC(n, \{s_i\}, \{t_i\})$, $TC(n)$ for short, is a convex function of n. Therefore, the search for the optimal value of n is reduced to find local minimum.

Theorem 3.2 *$TC(n)$ is convex in n.*

Proof The proof is similar to that of [10], in which deterioration, time-varying production rate and forgetting effect of setup cost are not considered. For simplicity, let

$$TC(n) = \sum_{i=1}^{n} A_i + T(n, 0, H), \tag{3.10}$$

where $\sum\limits_{i=1}^{n} A_i = \frac{A_1}{b+1} n^{b+1}$ and $T(n, 0, H) = \sum\limits_{i=1}^{n} (PC_i + HC_i + DC_i)$. It is easy to know that $\frac{A_1}{b+1} n^{b+1}$ is an increasing convex function of n. Next, by Bellman's principle of optimality, we know that the minimum value of $T(n, 0, H)$ can be calculated by

$$T^*(n, 0, H) = \min_{t \in [0,H]} \{T^*(n-1, 0, t) + T(1, t, H)\}. \tag{3.11}$$

Let $t = H$, we have $T^*(n-1, 0, H) > T^*(n, 0, H)$. The strict inequality follows since minimum in (3.11) occurs at an interior point. Thus $T^*(n, 0, H)$ is strictly decreasing in n. Recursive application of (3.11) yields the following relation:

$$s_i^*(n, 0, H) = s_i^*(n-j, 0, s_{n-j}^*(n, 0, H)),\ i = 1, \cdots, n-j-1,\ j = 1, \cdots, n-2, \tag{3.12}$$

where $s_n^*(n, 0, H)$ is the start time of the nth production when n production cycles are executed in $[0, H]$. To prove $T^*(n, 0, H)$ is strictly convex in n, we choose H_1 and H_2 such that

$$s_n^*(n+1, 0, H_1) = s_{n+1}^*(n+2, 0, H_2) = H, \tag{3.13}$$

and $s_1^*(n+1, 0, H_1) = s_1^*(n+2, 0, H_2) = 0$. Employing the principle of optimality on (3.13) again, we have

$$\begin{aligned} T^*(n+1, 0, H_1) &= \min_{t \in [0,H_1]} \{T^*(n, 0, t) + T(1, t, H_1)\} \\ &= T^*(n, 0, H) + T(1, H, H_1) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} T^*(n+2, 0, H_2) &= \min_{t \in [0,H_2]} \{T^*(n+1, 0, t) + T(1, t, H_2)\} \\ &= T^*(n+1, 0, H) + T(1, H, H_2). \end{aligned} \tag{3.15}$$

Since H is an optimal interior point in both $T^*(n+1, 0, H_1)$ and $T^*(n+2, 0, H_2)$, we know that

$$\frac{\partial T^*(n, 0, t)}{\partial t} + \frac{\partial T^*(1, t, H_1)}{\partial t}\Big|_{t=H} = 0 \tag{3.16}$$

and

$$\frac{\partial T^*(n+1,0,t)}{\partial t}+\frac{\partial T^*(1,t,H_2)}{\partial t}\Big|_{t=H}=0. \tag{3.17}$$

From equation (3.4), we have $\sum\limits_{i=1}^{n}\int_{s_i}^{t_i}K(t)\mathrm{e}^{\theta(t-t_i)}\mathrm{d}t=\sum\limits_{i=1}^{n}\int_{s_i}^{s_{i+1}}\mathrm{e}^{\theta(t-t_i)}f(t)\mathrm{d}t$ and we take the partial derivatives of above equation with respect to s_i, then the following equation holds.

$$K(s_i)\mathrm{e}^{\theta(s_i-t_i)}=\mathrm{e}^{\theta(s_i-t_i)}f(s_i)-\mathrm{e}^{\theta(s_i-t_{i-1})}f(s_i). \tag{3.18}$$

Utilizing the fact that

$$\begin{aligned}T(1,a,b)=c(a)\int_a^b K(t)\mathrm{d}t+\frac{(c_1+\theta c_2)}{\theta}\Big\{\int_a^v K(t)[1-\mathrm{e}^{\theta(t-v)}]\mathrm{d}t\\+\int_a^b[\mathrm{e}^{\theta(t-v)}-1]f(t)\mathrm{d}t\Big\},\end{aligned} \tag{3.19}$$

where v is the production ending time. By equations (3.16) and (3.18), we obtain

$$\begin{aligned}\frac{\partial T^*(n,0,t)}{\partial t}\Big|_{t=H}&=-\frac{\partial T^*(1,t,H_1)}{\partial t}\Big|_{t=H}\\&=\Big\{\frac{(c_1+\theta c_2)}{\theta}[1-e^{\theta(H-t_n^*(n,0,H))}]-c(H)\mathrm{e}^{\theta(H-t_n^*(n,0,H))}\\&\quad-c(s_n^*(n,0,H))\mathrm{e}^{\theta(H-t_n^*(n,0,H))}\Big\}f(H),\end{aligned} \tag{3.20}$$

where $s_n^*(n,0,H)$ and $t_n^*(n,0,H)$ are the last production starting and ending time when n production cycles are executed in $[0,H]$. Similarly, from $s_{n+1}^*(n+2,0,H_2)=H$, we have

$$\begin{aligned}\frac{\partial T^*(n+1,0,t)}{\partial t}\Big|_{t=H}&=-\frac{\partial T^*(1,t,H_2)}{\partial t}\Big|_{t=H}\\&=\Big\{\frac{(c_1+\theta c_2)}{\theta}[1-\mathrm{e}^{\theta(H-t_{n+1}^*(n+1,0,H))}]-c(H)\mathrm{e}^{\theta(H-t_{n+1}^*(n+1,0,H))}\\&\quad-c(s_{n+1}^*(n+1,0,H))\mathrm{e}^{\theta(H-t_{n+1}^*(n+1,0,H))}\Big\}f(H),\end{aligned} \tag{3.21}$$

where $s_{n+1}^*(n+1,0,H)$ and $t_{n+1}^*(n+1,0,H)$ are the last production starting and ending time when $n+1$ production cycles are executed in $[0,H]$. For simplicity, let $s_j^*(j,0,H)=s_j^*$ and $t_j^*(j,0,H)=t_j^*$ for $j=n,n+1$.

Subtracting equation(3.21) from equation (3.20) and using the fact that $c(t)$ is positive and no-increasing function, we have

$$\begin{aligned}\frac{\partial[T^*(n,0,t)-T^*(n+1,0,t)]}{\partial t}\Big|_{t=H}&=\Big[\frac{(c_1+\theta c_2)}{\theta}+c(a)\Big][\mathrm{e}^{\theta(H-t_n^*)}-\mathrm{e}^{\theta(H-t_{n+1}^*)}]f(H)\\&\quad+[c(s_n^*)\mathrm{e}^{\theta(H-t_n^*)}-c(s_{n+1}^*)\mathrm{e}^{\theta(H-t_{n+1}^*)}]f(H)>0,\end{aligned} \tag{3.22}$$

which implies that $T^*(n,0,H)-T^*(n+1,0,H)$ is a strictly increasing function of H. Thus,

$$T^*(n,0,H)-T^*(n+1,0,H)<T^*(n,0,H_1)-T^*(n+1,0,H_1). \tag{3.23}$$

Again, by equation (3.11) and the optimality principle, we obtain

$$T^*(n,0,H_1)-T^*(n+1,0,H_1)=\min_{t\in[0,H_1]}\{T^*(n-1,0,t)+T(1,t,H_1)\} - T^*(n,0,H)-T(1,H,H_1). \tag{3.24}$$

Taking $t=H$ into equation(3.24), we have

$$T^*(n,0,H_1)-T^*(n+1,0,H_1)<T^*(n-1,0,H)-T^*(n,0,H). \tag{3.25}$$

Therefore, we have

$$T^*(n,0,H)-T^*(n+1,0,H)<T^*(n-1,0,H)-T^*(n,0,H), \tag{3.26}$$

which implies $T^*(n,0,H)$ is convex in n. Hence, $TC(n)$ is also convex in n. This completes the proof.

3.2 A special case of the primal problem

In this subsection, we investigated the special case with $K(t)=K$, $f(t)=D$ and $c(t)=a$ and we can obtain some stronger results which may help to design the optimal algorithm for general problem.

Theorem 3.3 *The length of t_i-s_i and the length of $s_{i+1}-s_i$ in the special case of the primal problem with $K(t)=K$, $f(t)=D$ and $c(t)=a$ are equal, respectively. That is, $t_1-s_1=t_2-s_2=\cdots=t_n-s_n$ and $s_{i+1}-s_i=\frac{H}{n}$ for $i=1,\cdots,n$.*

Proof Substituting $K(t)=K, f(t)=D$ and $c(t)=a$ into (3.3) and (3.4), we have

$$K\Big(a+\frac{c_1+\theta c_2}{\theta}\Big)[\mathrm{e}^{\theta(s_i-t_i)}-1]=D\Big(a+\frac{c_1+\theta c_2}{\theta}\Big)[\mathrm{e}^{\theta(s_i-t_i)}-\mathrm{e}^{\theta(s_i-t_{i-1})}] \tag{3.27}$$

and

$$K[1-e^{\theta(s_i-t_i)}]=D[\mathrm{e}^{\theta(s_{i+1}-t_i)}-\mathrm{e}^{\theta(s_i-t_i)}]. \tag{3.28}$$

By rearranging the equations (3.27) and (3.28), we have $e^{s_{i+1}-t_i}=\mathrm{e}^{s_i-t_{i-1}}, i=2,\cdots,n$. We easily prove the function $f(x)=\mathrm{e}^{\theta x}$ is a strictly monotonic function, moreover, we have the following conclusion: for any x_1,x_2, if $f(x_1)=f(x_2)$, then $x_1=x_2$ holds. Hence we conclude that $s_{i+1}-t_i=s_i-t_{i-1},(i=2,\cdots,n)$ is unique determined. Combining the equation (3.27) or (3.28), we also can obtain $t_{i-1}-s_{i-1}=t_i-s_i$. Therefore, $s_{i+1}-s_i=(s_{i+1}-t_i)+(t_i-s_i)$, $s_{i+1}-s_i=\frac{H}{n}$ for $i=1,\cdots,n$ hold.

Utilizing the fact that $\mathrm{e}^x\approx 1+x$, as x is small, we can approximately estimate the length of production cycle. From equation (3.28), we have

$$K\theta(t_i-s_i)\approx K[\mathrm{e}^{\theta(t_i-s_i)}-1]=D[\mathrm{e}^{\theta(s_{i+1}-s_i)}-1]=D(\mathrm{e}^{\theta\frac{H}{n}}-1) \tag{3.29}$$

and

$$t_i-s_i=\frac{D}{K\theta}(\mathrm{e}^{\theta\frac{H}{n}}-1), i=1,\cdots,n. \tag{3.30}$$

Substituting $s_{i+1}-s_i=\frac{H}{n}$ and (3.30) into (2.13), we obtain

$$TC(n)=\frac{A_1}{b+1}n^{b+1}+na\frac{D}{\theta}(\mathrm{e}^{\theta\frac{H}{n}}-1)+n(c_1+\theta c_2)\frac{D}{\theta^2}(\mathrm{e}^{\theta\frac{H}{n}}-1)-(c_1+\theta c_2)\frac{DH}{\theta}. \quad (3.31)$$

We note that the only decision variable in equation (3.31) is the discrete integer variable n. On the other hand, according to the proof of Theorem 3.2, $TC(n)$ is a convex function of n. However, we also can prove this result using a different technique. By ignoring the integer constraint of the decision variable n and taking the first derivative of equation (3.31) and equating them to zero, we obtain

$$\begin{aligned}\frac{\partial TC(n)}{\partial n}&=A_1n^b+a\frac{D}{\theta}(\mathrm{e}^{\theta\frac{H}{n}}-1)-\frac{aDH}{n}\mathrm{e}^{\theta\frac{H}{n}}+(c_1+\theta c_2)\frac{D}{\theta^2}(\mathrm{e}^{\theta\frac{H}{n}}-1)\\&\quad-\frac{(c_1+\theta c_2)DH}{\theta n}\mathrm{e}^{\theta\frac{H}{n}}=0.\end{aligned} \quad (3.32)$$

Rearranging above equation, we can have the approximate real solution for the optimal integer solution. In order to derive an optimal integer solution, we employ another technique to prove the convexity of the total cost. By ignoring the integer constraint and taking the second derivative of $TC(n)$ with respect to n and substituting (3.32) into it, we have

$$\begin{aligned}\frac{\partial^2 TC(n)}{\partial n^2}&=\frac{D}{n}\Big[\frac{ab}{\theta}(1-\mathrm{e}^{\frac{\theta H}{n}})+\frac{abH}{n}\mathrm{e}^{\frac{\theta H}{n}}+(c_1+\theta c_2)\frac{b}{\theta^2}(1-\mathrm{e}^{\frac{\theta H}{n}})\\&\quad+\frac{(c_1+\theta c_2)bH}{\theta n}\mathrm{e}^{\frac{\theta H}{n}}+\frac{a\theta H^2}{n^2}\mathrm{e}^{\frac{\theta H}{n}}+\frac{(c_1+\theta c_2)H^2}{n^2}\mathrm{e}^{\frac{\theta H}{n}}\Big].\end{aligned} \quad (3.33)$$

Set $\frac{\theta H}{n}=x$, we have

$$\frac{\partial^2 TC(n)}{\partial n^2}=\frac{D}{n\theta}\Big[a+\frac{(c_1+\theta c_2)}{\theta}\Big](x^2\mathrm{e}^x+bx\mathrm{e}^x-b\mathrm{e}^x+b). \quad (3.34)$$

Set $f(x)=x^2\mathrm{e}^x+bx\mathrm{e}^x-b\mathrm{e}^x+b$, we can easily verify that $f(0)=0$ and $f'(x)=\mathrm{e}^x[x^2+(b+2)x]>0$ since $0<b<1$ and $x>0$. Hence $f(x)>0$, and $\frac{\partial^2 TC(n)}{\partial n^2}>0$ which means $TC(n)$ is a convex function with respect to n.

Since n is an integer, the inequalities $TC(n_s^*)\leqslant TC(n_s^*-1)$ and $TC(n_s^*)\leqslant TC(n_s^*+1)$ can be utilized to find the optimal integer solution n_s^*. From $TC(n_s^*)\leqslant TC(n_s^*-1)$, we have

$$\begin{aligned}&\frac{A_1}{b+1}[(n_s^*)^{b+1}-(n_s^*-1)^{b+1}]+\frac{D(c_1+\theta a+\theta c_2)}{\theta^2}\Big[n_s^*\mathrm{e}^{\frac{\theta H}{n_s^*}}-(n_s^*-1)\mathrm{e}^{\frac{\theta H}{n_s^*-1}}\Big]\\&\qquad\leqslant\frac{D(c_1+\theta a+\theta c_2)}{\theta^2}.\end{aligned} \quad (3.35)$$

On the other hand, the inequality $TC(n_s^*)\leqslant TC(n_s^*+1)$ can be shown as

$$\begin{aligned}&\frac{A_1}{b+1}[(n_s^*+1)^{b+1}-(n_s^*)^{b+1}]+\frac{D(c_1+\theta a+\theta c_2)}{\theta^2}\Big[(n_s^*+1)\mathrm{e}^{\frac{\theta H}{n_s^*+1}}-(n_s^*)\mathrm{e}^{\frac{\theta H}{n_s^*}}\Big]\\&\qquad\geqslant\frac{D(c_1+\theta a+\theta c_2)}{\theta^2}.\end{aligned} \quad (3.36)$$

The following theorem can be concluded from (3.35)–(3.36).

Theorem 3.4 *The optimal number of replenishments, n_s^*, is unique in the special case with $K(t)=K,\ f(t)=D,\ c(t)=a\ (a>0)$, if the conditions* (3.35) *and* (3.36) *are satisfied.*

3.3 Solution procedure

The optimal value n^* for general model is typically obtained using a full enumeration starting with $n = 1$ and going on until the optimal n^* is located. However, with this initial value many more iterations be required before the optimal n^* is identified. Since $TC(n)$ is convex with respect to n, we take the optimal solution of a special case as a new possible initial value to speed up the search for the optimal n^*. Hence, an algorithm for determining the optimal value n^* is summarized as follows:

Algorithm A

Step 0 Choose one of special case with $K(t) = K,\ f(t) = D,\ c(t) = a\ (a > 0)$ and compute its optimal solution n_s^*.

Step 1 Set $n = n_s^*$, use a standard search method to obtain s_i^* and t_i^*, and compute the corresponding $TC(n)$ and $TC(n-1)$, respectively.

Step 1.1 If $TC(n) \geqslant TC(n-1)$, compute $TC(n-2),\ TC(n-3), \cdots$, until there is some k with $TC(k) < TC(k-1)$. Set $n^* = k$, stop. Otherwise, go to Step2.

Step 1.2 If $TC(n) < TC(n-1)$, compute $TC(n+1),\ TC(n+2), \cdots$, until we find some k with $TC(k) < TC(k+1)$. Set $n^* = k$, stop. Otherwise, go to Step2.

Step 2 Compute the value of $TC(n)$ for $n = 1, 2, \cdots, n_s^* - 1$ until find $TC(k) < TC(k-1)$ and $TC(k) < TC(k+1)$. Set $n^* = k$, stop.

4 Numerical example

In this section, a numerical example is provided to illustrate the applicability of the algorithm developed in Section 3.

Example Let $f(t) = 100 + 150t$, $c(t) = 20 + 100\mathrm{e}^{-5t}$, $K(t) = 300 + 60t,\ c_1 = 50,\ c_2 = 10,\ A_1 = 200,\ \theta = 0.09,\ H = 1$ and $b = -\frac{ln(0.9)}{ln(2)}$.

We firstly study the special case in which $K(t) = 350, f(t) = 100, c(t) = 120$. Applying (3.35) and (3.36), we get the optimal replenishment times, n_s^*. By substituting above parameters into the left hand of inequality (3.35) or inequality (3.36), we have $\frac{D(c_1+\theta a+\theta c_2)}{\theta^2} = 761\,728.4$. Let $g(n)$ represents the right hand side of the inequality (3.35), obviously, $g(n+1)$ are the right hand side of the inequality (3.36). Set $n = 1, \cdots, 6$, the values of $g(n)$ and $g(n+1)$ are shown in Table 1. From Theorem 3.4, the optimal number of replenishment n_s^* is 4, the corresponding optimal total cost is 13 634.4 for the special case.

Table 1 The results for the special case

n	1	2	3	4	5	6
$g(n)$	833 637.2	760 326.8	761 430.9	761 708.6	761 823.4	761 883.5
$g(n+1)$	760 326.8	761 430.9	761 708.6	761 823.4	761 883.5	761 920.0

Next, we start the search for the optimal replenishment times n^* from $n = 4$. The algorithm A ends at $n = 6$. Therefore the optimal number of replenishment $n^* = 6$ and the total cost is 50 262.0. The values of s_i^* and t_i^* are given in Table 2.

Table 2 Optimal solution for the general case

i	1	2	3	4	5	6	7
s_i	0	0.208 2	0.392 8	0.560 9	0.716 7	0.862 6	1.000 0
t_i	0.080 1	0.293 6	0.481 5	0.651 9	0.809 1	0.956 0	

5 Conclusions

An generalized EPQ problem for deteriorating items with forgetting consideration of setup cost and time-varying unit production cost, production rate and demand rate is proposed. A mixed-integer non-linear optimization model is formulated. Via analysis, we find that the optimal production schedule uniquely exists, the total cost is a convex function with respect to replenishment times. Furthermore, the search process of optimal solution can be simplified by providing a better initial value for the replenishment times. Future research could consider models with effects such as shortages, quantity discounts and probabilistic demand.

References

[1] Misra R B. Optimum production lot size model for a system with deteriorating inventory [J]. *International Journal of Production Research*, 1975, **13**(2): 495-505.

[2] Pasandideh S H R, Niaki S T A. A genetic algorithm approach to optimize a multi-products EPQ model with discrete delivery orders and constrained space [J]. *Applied Mathematics and Computation*, 2008, **195**(2): 506-514.

[3] Teng J T, Chang C T. Optimal manufacturer's replenishment policies in the EPQ model under two level of trade credit policy [J]. *EuropeanJournal of Operational Research*, 2009, **196**(2): 177-185.

[4] Zhou Y W. Effect of inflation on the inventory replenishment policy for deteriorating items with time-vaying demand (in Chinese) [J]. *Operations Research Transactions*, 1998, **2**(1): 43-50.

[5] Yang H L, Teng J T, Chern M S. Deterministic inventory lot-size models under inflation with shortages and deterioration for fluctuating demand [J]. *Naval Research logistics*, 2001, **48**: 144-158.

[6] Lin G C, Kroll D E, Lin C J. Determining a common production cycle time for an economic lot scheduling problem with deteriorating items [J]. *European Journal of Operational Research*, 2006, **173**(2): 669-682.

[7] Wang C X. Deteriorating rate based integrated optimization of dynamic pricing and order quantity for seasonal products (in Chinese) [J]. *Operations Research Transactions*, 2009, **13**(4): 71-82.

[8] Liao G L, Sheu S H. Economic production quantity model for randomly failing production process with minimal repair and imperfect maintenance [J]. *International Journal of Production Economics*, 2011, **130**(1): 118-124.

[9] Luo C L. Risk aversion in inventory management with bayesian information updating (in Chinese) [J]. *Operations Research Transactions*, 2013, **17**(1): 59-68.

[10] Teng J T, Ouyang L Y, Chang C T. Deterministic economic production quantity models with time-varying demand and cost [J]. *Applied Mathemacitcs Modelling*, 2005, **29**(10): 987-1003.

[11] Carlson J G, Rowe R G. How much does forgetting cost? [J]. *Industrial Enginerring*, 1976, **8**: 40-47.

[12] Chiu H N, Chen H M. An optimal algorithm for solving the dynamic lot-sizing model with learning and forgetting in setups and production [J]. *International Journal of Production Economics*, 2005, **95**(2) 179-193.

[13] Alamri A A, Balkhi Z T. The effects of learning and forgetting on the optimal production lot size for deteriorating items with time varying demand and deterioration rates [J]. *Internation Journal of Production Economics*, 2007, **107**(1): 125-138.

[14] Jaber M Y, Bonney M, Moualek I. Lot sizing with learning, forgetting and entropy cost [J]. *Internation Journal of Production Economics*, 2009, **118**(1): 19-25.

[15] Cheng T C E. An EOQ model with learning effect on setups [J]. *Production and Inventory Management Journal*, 1991, **32**: 83-84.

[16] Jaber M Y, Bonney M. Economic manufacture quantity (EMQ) model with lot-size dependent learning and forgetting rates [J]. *Internatiaonal Journal of Production Economic*, 2007, **108**(1-2): 359-367.

[17] Bellman R E. *Dynamic Programming* [M]. Princeton: Princeton University Press, 1957.