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Spatial Evolutionary Game Theory: Deterministic Approximations, Decompositions, and Hierarchical Multi-scale Models

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SPATIAL EVOLUTIONARY GAME THEORY: DETERMINISTIC APPROXIMATIONS, DECOMPOSITIONS, AND HIERARCHICAL MULTI-SCALE MODELS

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ABSTRACT

SPATIAL EVOLUTIONARY GAME THEORY: DETERMINISTIC APPROXIMATIONS, DECOMPOSITIONS, AND HIERARCHICAL MULTI-SCALE MODELS

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Evolutionary game theory has recently emerged as a key paradigm in various behavioral science disciplines. In particular it provides powerful tools and a conceptual framework for the analysis of the time evolution of strategic interdependence among players and its consequences, especially when the players are spatially distributed and linked in a complex social network. We develop various evolutionary game models, analyze these models using appropriate techniques, and study their applications to complex phenomena.

In the second chapter, we derive integro-differential equations as deterministic approximations of the microscopic updating stochastic processes. These generalize the known mean-field ordinary differential equations and provide powerful tools to

investigate the spatial effects on the time evolutions of the agents' strategy choices. The deterministic equations allow us to identify many interesting features of the evolution of strategy profiles in a population, such as standing and traveling waves, and pattern formation, especially in replicator-type evolutions.

We introduce several methods of decomposition of two player normal form games in the third chapter. Viewing the set of all games as a vector space, we exhibit explicit orthonormal bases for the subspaces of potential games, zero-sum games, and their orthogonal complements which we call anti-potential games and anti-zero-sum games, respectively. Perhaps surprisingly, every anti-potential game comes either from *Rock-paper-scissors* type games (in the case of symmetric games) or from *Matching Pennies* type games (in the case of asymmetric games). Using these decompositions, we prove old (and some new) cycle criteria for potential and zero-sum games (as orthogonality relations between subspaces).

We illustrate the usefulness of our decompositions by (a) analyzing the generalized Rock-Paper-Scissors game, (b) completely characterizing the set of all null-stable games, (c) providing a large class of strict stable games, (d) relating the game decomposition to the Hodge decomposition of vector fields for the replicator equations, (e) constructing Lyapunov functions for some replicator dynamics, (f) constructing Zeeman games - games with an interior asymptotically stable Nash equilibrium and a pure strategy ESS.

The hierarchical modeling of evolutionary games provides flexibility in addressing the complex nature of social interactions as well as systematic frameworks in which one can keep track of the interplay of within-group dynamics and between-group competitions. For example, it can model husbands and wives' interactions, playing an asymmetric game with each other, while engaging coordination problems with the likes in other families. In the fourth chapter, we provide hierarchical stochastic models of evolutionary games and approximations of these processes, and study their applications

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INTRODUCTION

1.1 Overview

1.1.1 Evolutionary game theory

Evolutionary game theory has recently emerged as a key paradigm in various behavioral science disciplines. Originally evolutionary game theory, pioneered by the biologist John Maynard Smith, was introduced to study the behaviors of animals, explaining sex ratio, animal distribution, and contest behavior and reciprocal altruism among animals (Maynard Smith, 1982). In recent years, the key idea that games are played among a large number of myopic agents has been adopted by behavioral scientists and applied successfully in explaining the important social phenomena.

Anthropologists and biologists have adopted evolutionary frameworks to explain early human cooperation (Bowles, 2006; Boyd, Gintis, and Bowles, 2010; Boyd and Mathew, 2007; Hauert, Trauslen, Brandt, and Nowak, 2007). Political scientists have examined the role of norms in the context of evolutionary games. For example, Robert Axelrod (1986) investigates the emergence and stability of behavioral norms and shows that the employment of meta-norms − the willingness to punish someone who did not enforce a norm − plays an important role in explaining the evolution and stability of various social norms. In economics, since the seminal work by Young (1993), evolutionary games have been applied frequently in explaining the evolution of social institutions, conventions and contracts (See Young (1998); for evolutionary game theory see Hofbauer and Sigmund (1998); Weibull (1995); Sandholm (2010b); Gintis (2009); Cressman (2003); Nowak (2006); Hofbauer and Sigmund (2003); Szabo (2007)).

Classical game theory supposes that each agent (or player) has a well-defined utility function that she tries to maximize given her counterpart's choice of a strategy. The strategic interdependence among players typically arises from the fact that one's objective function is dependent on another's choice as well as her own choice. The Nash equilibrium of a game, introduced by John Nash, has an influential role in predicting the outcome of the game; every player plays a "best response" to each other at equilibrium.

However, the justification of the Nash equilibrium concept − e.g., how players know that a Nash equilibrium will be played − has been questioned on various grounds. Binmore (1987, 1988) suggests that a very strong informational assumption is made in the backward induction (deductive reasoning) argument in the repeated games. The assumption that an agent can evaluate expected payoffs from the complex interactions is also challenged on various empirical settings including lap experiments (See Bowles, 2004). Agents typically have non-negligible cognitive limitations and experience high costs of gathering information about possible outcomes.

In contrast to classical game theory, evolutionary game theory addresses the above limitation by relaxing the rationality assumptions substantially. In evolutionary frameworks, an agent adopts an inductive method of reasoning typically relying on trial-and-error methods. Departing from the assumption of highly intelligent and forward looking behaviors, evolutionary game theory explains the interactions among myopic agents and their consequences in the complex situations. In this way it provides successful and powerful tools for the analysis of strategic interdependence among anonymous and heterogeneous agents who seek to improve their payoffs.

1.1.2 Spatial stochastic processes

The importance of space and spatial interactions in explaining the social behaviors of agents has been well recognized. As early as 1930s, Hotelling (1929) emphasized that a market, rather than being a unified entity, is divided into regions in which sellers enjoy quasi-monopolistic positions owing to their spatial locations. Thomas Schelling (1971) studied how the spatial segregation in residential areas might arise even if most agents prefer to live in integrated neighborhoods. We study spatial stochastic processes where individuals are located at the vertices of a graph and update their strategies upon receiving a strategy revision opportunity. The updating rules are flexible enough to encompass various behavioral assumptions, ranging from imitative behaviors to perturbed best responses.

In the second chapter, we study the deterministic approximations of the stochastic dynamics as a first step toward the understanding of the behavior of spatial stochastic processes. Following methods developed in statistical physics, known as scaling limit or meso-scopic limit approaches (Kipnis and Landim, 1999; Presutti, 2009), we show that the spatial strategy revision processes converge to deterministic limits under suitable scaling and deterministic limits satisfy non-local partial differential equations (PDE), called integro-differential equations (IDE).

Equilibrium selection from among multiple Nash equilibria has been one of the main topics in game theory. Using the derived deterministic equations, we study the effect of spatial structures and individual decision rules on equilibrium selection. Other interesting phenomena include spatial patterns of agents' choice of strategy. For example, the complete segregation of residential area even though people prefer racial integration has posed intriguing questions to social scientists: how do locally homogeneous behaviors lead to globally heterogeneous pattern? The spatial differential equations provide natural and handy settings to study these purely spatial phenomena and to explore the condition under which the segregation of choices of strategy may develop and persist in spatial models. We investigate such problems using the analytical and numerical analysis of the spatial differential equations.

Conceptually, there are deep connections between evolutionary games and statistical mechanics. At the "microscopic" level game theory is an extremely powerful tool to formulate and model simple rules with which individual agents interact. The evolutionary (and population) version of game theory uses a dynamic and statistical approach to connect individual interactions to "macroscopic", global, and long-time behavior of large populations.

The updating rules of agents in evolutionary games are related to the transition rules of particles in interacting particle models. For example, perturbed best-response dynamics in economics are related to (and generalizes) the Gibbs sampler or Metropolis dynamics in statistical mechanics. Under this correspondence, the Ising model can be regarded as a spatial evolutionary game where agents play a two strategy coordination game (ferromagnetic case) or a Hawk-Dove game (anti-ferromagnetic case).

Important techniques in analyzing the equilibrium state in statistical mechanics often involve a special function, called a potential function or an energy function. The analysis of such functions provides the understanding of the important properties of the dynamics, whether these be stochastic processes or deterministic differential equations. In game theory it is well-known that a certain class of games admits such a potential function in a properly defined dynamic, hence these games are called potential games and have received growing attention among game theorists (Monderer and Shapley, 1996; Sandholm, 2010a).

Based on these observations, in the third chapter, we first consider a method of decomposing a given game into a potential game part and the remaining part, called an "anti-potential" game. We characterize the set of all anti-potential games by a special class of games, namely the Rock-paper-scissor games or the Matching pennies games. Along with the first decomposition, we develop two additional methods of decomposing games: (1) the decomposition of a game into a zero-sum game and the remaining part, called "anti-zero" sum game and (2) the decomposition of a game into a game with a dominant strategy and the remaining part.

In non-equilibrium statistical mechanics, the studies of irreversible systems and stationary non-equilibrium states adopt concepts like entropy production, currents, and flux. Lebowitz and Spohn (Lebowitz and Spohn, 1999) show that the sample path entropy production has a large deviation principle and the rate function has a symmetry of Gallavotti-Cohen type. At the stationary state of irreversible dynamics, the entropy production rate is always positive; the system is reversible if and only if the entropy production rate is zero (Jiang, Qian, and Qian, 2004). Spatial stochastic processes whose underlying game are potential games and whose updating rules are specified by either Gibbs sampler or Metropolis dynamic are reversible. Thus the developed methods of decomposition show that only anti-potential part of a given game contributes to entropy production. Since the decomposition results hold for a general normal form game (with more than two strategies and more than two players), the same observation holds for various classes of interacting particle models in the statistical mechanics (e.g., Currie Weiss Pott model and multi-body interactions). In this way, we reveal the underlying source and structure of entropy production in the classes of irreversible dynamics in statistical mechanics.

Another important aspect of social interactions is the complex interconnections between different interactions at various levels. For example, interactions within a group and interactions between groups are clearly related and mutually constraining. Important behavioral traits − such as altruistic behavior − may have developed because they bring benefits at a group level, even though those adopting this behavior might do worse within a group than fellow group members who behave differently. To address these important aspects of social interactions, we develop hierarchical models of evolutionary games using coarse-graining methods in the final chapter. Starting from the microscopic level, we define a coarse cell containing microscopic sites. Here coarse cells can be regarded as groups. And then we aggregate the microscopic stochastic process into a process defined at coarse cell levels under appropriate conditions. In this way, we obtain group-level stochastic processes.

The hierarchical modeling of evolutionary games provides flexibility in addressing the complex nature of social interactions. It also provides systematic frameworks in which one can keep track the interplay of within-group dynamics and between-group competitions. For example, it can model husbands and wives' interactions, playing an asymmetric game with each other, while engaging coordination problems with the likes in other families. The influence from the outside members can be regarded as conformism effect or social norm propagation. In addition, the social phenomena usually involve variables evolving at multi-scales; the propagation of social norm is extremely slow, whereas the individual's updating of the strategy may be relatively fast.

Using hierarchical models we consider a situation in which one group size is relatively large to the other. We derive a coupled system of dynamics where the large group dynamics are governed by deterministic evolutions, while the small group dynamics are subject to stochastic randomness.

1.2 Main Results

1.2.1 Deterministic approximations of spatial stochastic processes

A normal form game consists of players, strategies, and payoff functions. When there are two players and the payoffs of two players are symmetric, a given game is succinctly described by a matrix whose dimension equals to the number of strategies. For example, a matrix A specifies a game where player 1 obtains a payoff of $a(i,j)$ by playing strategy i against player 2's strategy j. Here we denote by S the set of all strategies, so $i, j \in S$.

In the second chapter, we study the deterministic approximations of the spatial stochastic processes with focus on a local density function $f(u,i)$, describing the proportion of the population with strategy i around spatial location u. The main result is that local mean-field stochastic processes are approximated, on finite time intervals and in the limit of infinite population, by equations of the following type:

$$\frac{\partial}{\partial t} f_t(u,i) = \sum_{k \in S} \mathbf{c}(u,k,i,f_t) f_t(u,k) - f_t(u,i) \sum_{k \in S} \mathbf{c}(u,i,k,f_t) \quad \text{for } i \in S\,. \tag{1.1}$$

The term $\mathbf{c}(u,k,i,f)$ describes the rate at which agents at spatial location u switch from strategy k to i. This rate depends on the strategies of agents at other spatial locations and the explicit form varies with the behavioral rules of agents. A typical example of the rate is

$$\begin{aligned}\mathbf{c}(u,k,i,f) &= F(\sum_{l\in S} a(i,l)\mathcal{J}*f(u,l) - \sum_{l\in S} a(k,l)\mathcal{J}*f(u,l)),\\ \text{where } \mathcal{J}*f(u,i) &:= \int \mathcal{J}(u-v)f(v,i)dv\,.\end{aligned}$$

Here F is a non negative and increasing function and $\mathcal{J}*f$ is the convolution product of $\mathcal{J}$ with f. The function $\mathcal{J}(u)$ is a non-negative probability kernel which describes the interaction strength between players whose relative distance is u. When $\mathcal{J}$ is a constant function equation (1.1) reduces to ordinary differential equations such as replicator dynamics, Brown-von Neumann-Nash dynamics, and logit dynamics, which have been well-known to evolutionary game theorists. Note that the rate of increases in f_t at u depends on $f_t(v,i)$ for all v in the spatial domain and that equation (1.1) is an *integro-differential equation* (IDE).

These non-local PDEs are similar to reaction diffusion equations known as Allen-Cahn type PDEs (Cahn, Elliott, and Novik-Cohen, 1996) or Glauber IDEs (Presutti, 2009). For example, an IDE based on the updating rule of "Gibbs sampler" yields a "logit dynamic":

$$\frac{\partial}{\partial t} f_t(u,i) = \frac{\exp(\sum_l a(i,l)\mathcal{J}*f_t(u,l))}{\sum_k \exp(\sum_l a(k,l)\mathcal{J}*f_t(u,l))} - f_t(u,i), \tag{1.2}$$

When the number of strategies are two, (1.2) becomes the well-known Glauer mesoscopic equation via the change of the variable, $f \mapsto 2f-1 := u$. When $a(1,1) = a(2,2) > 0$ and $a(1,2) = a(2,1) = 0$, the existence of a unique standing wave was proved and when $a(1,1) \neq a(2,2)$, the existence of traveling wave was proved (Orlandi and Triolo, 1997; Chen, 1997). By adopting various tools and techniques such as linear stability analysis and numerical simulations, we study dynamics of interfaces, the existence of traveling wave solutions, and pattern formations (see figure 1).

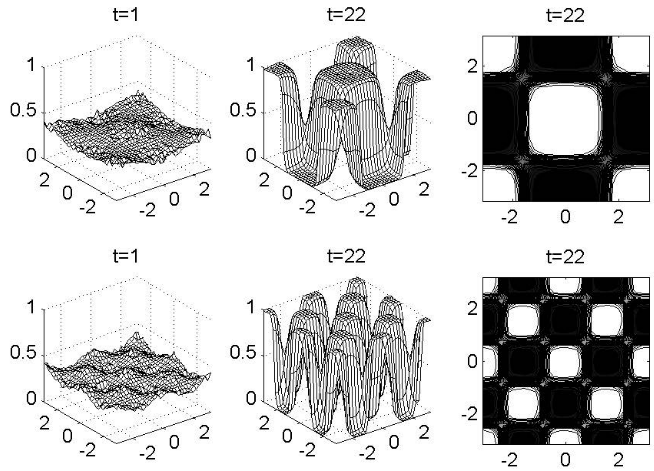


Figure 1. Pattern formations in the replicator dynamics **We consider two player coordination game with payoffs,** $a_{11} = 2/3, a_{22} = 1/3,\ a_{12} = a_{21} = 0$**. Left and Middle panels show the time evolutions of population densities using strategy 1 in the spatial domain** $T^d = [-\pi, \pi]^2$**. The heights of surfaces represent the densities at a given location** $u \in T^d$**. The number of nodes is** 64 **for the simulation and the time step is** 0.0175 **which was determined by the stability analysis of the numerical method. The initial conditions are** $1/3 + \text{rand} \cos(x) \cos(y)$ **(upper panel) and** $1/3 + \text{rand} \cos(2x) \cos(2y)$ **(lower panel), where rand denotes a realization of uniform random variable** $[0, 1]$ **at each node. For the interaction kernel, we use** $J(r) = \exp\left(-bx^2\right) / \int \exp(-bx^2) dx, b = 15$**. The right panels show the contours of the densities at** $t = 22$**.**

1.2.2 Decompositions of normal form games and statistical mechanics

Special classes of games such as potential games, zero-sum games, and stable games have received growing attention because of their respective analytical advantages. For instance, in potential games, every player's motivation to choose or deviate from a strategy can be described by a *single* function, called a potential function. In the third chapter of the dissertation, we develop various methods of decomposing normal form games, viewing the set of all games as a vector space and exhibiting bases of important subspaces : e.g., the subspaces of potential games and zero-sum games.

Given a game A, Nash equilibria are invariant with respect to the following payoff transformation: the transformation that adds the same constants to any column of A. This is because the transformation keeps the payoff difference between two strategies of a player unchanged for a given strategy of her counterpart. Then the invariance property induces equivalent classes in the space of normal form games and we study potential games and zero-sum games defined on such equivalent classes.

In particular, we show that every game A, up to this equivalence relation $\sim$, can be decomposed into a part belonging to potential games and another part belonging to a special class of zero-sum games, namely zero-sum games whose row sums are all zeros (called anti-potential games):

$$A \sim S + N, \tag{1.3}$$

where S is a symmetric matrix (so a potential game) and N is an antisymmetric matrix (so a zero-sum game) whose row sums are all zeros.

Then we proceed to show that when the number of strategies is three, the Rock, Paper and Scissors game is the only anti-potential game. When the number of strategies is more than three, every anti-potential game can be written as a linear combination of the "extended" Rock, Paper, and Scissor games − a game whose three strategies give the payoffs of the Rock, Paper, and Scissors games with other strategies giving zero payoffs. We further decompose a potential game (e.g., S in (1.3)) into a game with a dominant strategy and its remaining part. And the remaining parts turn out to be a special class of potential games - potential games whose row sums are all zeros. In sum, a given game A can be decomposed, up to equivalence, into three parts: a potential game whose row sums are all zeros, a game with a dominant strategy, and a zero-sum game whose row sums are all zeros:

$$A \sim K + D + N, \tag{1.4}$$

where K is a symmetric matrix whose row sums are all zeros (a special potential game), D is a matrix whose elements in each row are constant (so has a dominant strategy), and N is an anti-symmetric game whose row sums are all zeros (an anti-potential game). For example, one can decompose

$$\begin{pmatrix} 10 & -5 & 3 \\ 4 & 13 & -3 \\ -2 & 1 & 12 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{pmatrix} + \begin{pmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix}$$

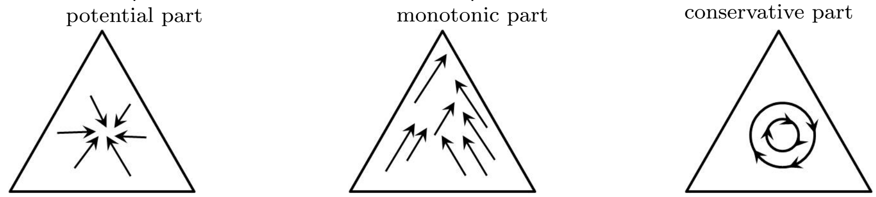
$$\sim \underbrace{\begin{pmatrix} 9 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{pmatrix}}_{S} + \underbrace{\begin{pmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -3 & 3 & 0 \end{pmatrix}}_{N}$$

$$\begin{pmatrix} 9 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{pmatrix} = \begin{pmatrix} 7 & -4 & -3 \\ -4 & 9 & -5 \\ -3 & -5 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 3 \\ 2 & 4 & 3 \\ 2 & 4 & 3 \end{pmatrix}$$

$$\sim \underbrace{\begin{pmatrix} 7 & -4 & -3 \\ -4 & 9 & -5 \\ -3 & -5 & 8 \end{pmatrix}}_{K} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}}_{D}$$

The decomposition in (1.4) immediately implies the decomposition of vector fields of the replicator ordinary differential equations, since the replicator dynamics are linear with respect to the matrix of a game:

$$F_i(x) = \underbrace{x_i((Kx)_i - x^T Kx)}_{\text{potential part}} + \underbrace{x_i((Dx)_i - x^T Dx)}_{\text{monotonic part}} + \underbrace{x_i N x}_{\text{conservative part}}$$



where x denotes the population fractions of each strategy and F_i denotes the ith element of the replicator vector field. We also consider other applications in chapter 3.

1.2.3 Hierarchical multi-scale models: Coarse-grained Markov chains

In the final chapter, we derive coarse-grained Markov chains from the spatial stochastic evolutionary models. The derived coarse-grained models generalize the matrix models of evolutionary games with two groups of individuals introduced by Taylor (1979). Of particular interest is a hybrid system which models interactions among groups with different scales.

For example, we consider two populations of size N_1 and N_2 such that $N_1 \ll N_2$, so the size of group 2, N_2, is much greater than that of group 1, N_1. In this case, by considering the scaling limit, $N_2 \to \infty$, we obtain a hybrid system where the dynamic of the group of size, N_2, follows deterministic evolution, while that of the group of size, N_1, remain a stochastic process.

More concretely, consider a two strategy game with η and ρ being the population fractions using the first strategy of a game in each group, respectively. Then η takes a value from a discrete set, $\{0, \frac{1}{N_1}, .. \frac{N_1-1}{N_1}, 1\}$, while ρ takes a value from a continuum set, namely the unit interval. When the strategy revision rate is imitative, from the coarse graining of the original microscopic system we obtain the birth-death rates:

$$c_+(\eta,\rho) := [\beta_1(\eta-\zeta_1)+\beta_2(\rho-\zeta_2)]_+ , \qquad c_-(\eta,\rho) := [\beta_1(\zeta_1-\eta)+\beta_2(\zeta_2-\rho)]_+$$

where $\beta_i > 0$ and $0 < \zeta_i < 1$. Then the coarse-grained stochastic processes yield

$$\begin{cases} L_\rho f(\eta) = (1-\eta)\eta c_+(\eta,\rho)(f(\eta+\frac{1}{N_1})-f(\eta)) + \eta(1-\eta)c_-(\eta,\rho)(f(\eta-\frac{1}{N_1})-f(\eta)) \\ \frac{d\rho}{dt} = \beta_3\rho(1-\rho)(\rho-\zeta_3)+\beta_4\rho(1-\rho)(\eta-\zeta_4) \end{cases} \tag{1.5}$$

where f is a function defined on $\{0, \frac{1}{N_1}, .. \frac{N_1-1}{N_1}, 1\}$.

Note that if $N_1 \to \infty$, (1.5) becomes the replicator ordinary differential equations accounting within-group and between-group interactions. This, in particular, illustrates that the replicator ordinary differential equations may be a good approximation when two group sizes are equally large, but would be a poor approximation when the size of one group is relatively small to that of the other. Interesting social interactions usually involve such asymmetry between the sizes of groups; a small number of sellers competes for a large number of customers who are spatially located (e.g., telecommunication providers). Thus, in those instances, the hybrid model (1.5) provides the better approximation of the original stochastic process by retaining the microscopic fluctuations of smaller group's behavior.

1.3 Future Research Agendas

The future research agendas related to the dissertation projects include :

- **The rigorous treatment of traveling wave solutions**. One of widely studied ordinary differential equations among game theorists and biologists is called the replicator dynamics, which can be derived from the imitative updating rule. The spatial version of the replicator dynamics for a two strategy game is given by

$$\frac{\partial}{\partial t} f_t = (1-f)\mathcal{J} * f[\beta(\mathcal{J} * f - \zeta)]_+ - f(1-\mathcal{J} * f)[\beta(\zeta - \mathcal{J} * f)]_+ \quad (1.6)$$

where ζ is a constant function such that $0 < \zeta < 1$ and $[t]_+ = \max\{t, 0\}$. Unlike the logit dynamics, there is no existing rigorous result on the existence of standing or traveling wave solutions, though we observe such phenomena in the numerical simulations.

- **Pattern Formation and Meta-stability**. In numerical simulations, we frequently observe the formations of patterns or the metastable states where both strategies coexist for a long run in the replicator dynamics. The metastable behavior for scalar reaction-diffusion equations was studied by Carr and Pego (1989)(See also Duncan, Grinfeld, and Stoleriu (2000); Otto and Reznikoff (2007)). Similarly to questions on traveling wave solutions, there is no existing rigorous study on the conditions for the pattern formations and metastability of the replicator equations (equation (1.6)).

- **Large Deviation**. Large deviation tools provide a powerful method to compute asymptotically small probabilities on an exponential scale to study fluctuation around deterministic path. To investigate the role of stochasticity in the spatial stochastic processes, one can derive large deviation functionals for the system and use Freidlin-Wentzell theory to account for stochasticity at the meso-scopic level.

- **Stochastic PDE (SPDE) Approximation**. SPDE approximation was used to study phase transitions of interacting particle models such as the long-range contact processes and voter processes (Durrett, 1999; Mueller and Tribe, 1995). This approximation is better than the deterministic one in the sense that it retains the stochastic fluctuations, still reducing the complexities of the original system.

CHAPTER 2

DETERMINISTIC EQUATIONS FOR SPATIAL EVOLUTIONARY GAMES

2.1 Spatial Games and and Strategy-Revision Processes

In models of spatial evolutionary games, agents are located at the sites of a graph and play a normal form game with their neighbors. The graph Λ is assumed here to be a subset of the integer lattice $\mathbb{Z}^d$. We focus on a single population playing a normal form game, but the generalization to multiple population games is straightforward. A normal form game is specified by a finite set of strategies S and a payoff function $a(i,j)$ which gives the payoff for a player using strategy $i \in S$ against strategy $j \in S$.

The strategy of the agent at site $x \in \Lambda$ is $\sigma_\Lambda(x) \in S$, and we denote by $\sigma_\Lambda = \{\sigma_\Lambda(x) : x \in \Lambda\}$ the configuration of strategies for every agent in the population. With these notations, the state space, i.e., the set of all possible configurations, is S^Λ. The subscript of σ_Λ will be suppressed, whenever no confusion arises. As in Young (1998, chapter 6), positive weights $\mathcal{W}(x-y)$ are assigned to any two sites x and y to capture the importance or intensity of the interaction among neighbors. Note that this assumes that these weights depend only on the relative location $x-y$ between the players (i.e., translation invariance). It is convenient to assume that total weight that site x attaches to all its neighbors is normalized to 1, i.e.,

$$\sum_{y \in \Lambda} \mathcal{W}(x-y) \approx 1. \tag{2.1}$$

The site y is called a neighbor of x whenever $\mathcal{W}(x-y) > 0$. An individual agent, at site x with strategy i given a configuration σ, receives an average payoff

$$u(x,\sigma,i) := \sum_{y \in \Lambda} \mathcal{W}(x-y) a(i,\sigma(y)). \tag{2.2}$$

If the weight $\mathcal{W}$ is interpreted as the probability with which an agent samples his neighbors, then $u(x,\sigma,i)$ is the expected payoff for an agent at x choosing strategy i if the population strategy profile is σ. Or one may think that an agent receives an instantaneous payoff flow from her interactions with other neighbors (Blume, 1993; Young, 1998; Young and Burke, 2001). The specific examples of such weights are as follows:

Example. The following weight specifies uniform interactions where agents attach an equal weight to every interaction with their neighbors.

$$\mathcal{W}(x-y) = \frac{1}{n^d} \text{ for all } x \neq y,$$

where n^d is the total number of individuals in the system. When the weight are given by the following formula

$$\mathcal{W}(x-y) = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\| = 1 \\ 0 & \text{otherwise} \end{cases},$$

the interactions are called the nearest neighbor interactions in which interactions only arise between nearest sites (Blume, 1995; Szabo, 2007).

In this chapter we concentrate on long range interactions where each agent interacts with as many other agents as in the mean-field case, but the interaction is not uniform. This limit is known as "local mean field model" (Comets, 1987) or "Kac potential" (Lebowitz and Penrose, 1966; DeMasi, Orlandi, Presutti, and Triolo, 1994; Presutti, 2009). More specifically, let $\mathcal{J}(x)$ be a non-negative, compactly supported, and integrable function such that $\int \mathcal{J}(x)dx = 1$. We assume that $\mathcal{W}$ has the form:

$$\mathcal{W}^{\gamma}(x-y) = \gamma^d \mathcal{J}(\gamma(x-y))\,, \tag{2.3}$$

and we will take the limit $\Lambda \nearrow \mathbb{Z}^d$ and $\gamma \to 0$ in such a way that $\gamma^{-d} \approx |\Lambda| \approx n^d$. Here n^d is the size of the population and $|\;|$ denotes the cardinality. Hence the factor γ^d is chosen in such a way that $\sum \mathcal{W}^{\gamma}(x-y) \approx \int \mathcal{J}(x)dx = 1$, so $\mathcal{W}^{\gamma}(x-y)$ indeed represents the intensity of interactions. Note that in (2.3) the interaction vanishes when $\|x-y\| \geq R\gamma^{-1}$ if $\mathcal{J}$ is supported on the ball of radius R. So as $\gamma \to 0$, an agent interacts very weakly but with a growing number of neighbors in the population.

The time evolution of the system is given by a continuous time Markov process $\{\sigma_t\}$ with state space S^{Λ}, in which each agent receives, independently of all the other agents, a strategy revision opportunity in response to his own exponential "alarm clock" with rate 1, and then updates his strategy according to a rate $c(x, \sigma, k)$ – the rate with which agent x switches to strategy k when the configuration is σ. This process is then characterized by a generator :

$$(Lg)\,(\sigma) = \sum_{x\in\Lambda}\sum_{k\in S} c(x,\sigma,k)\left(g(\sigma^{x,k}) - g(\sigma)\right) \tag{2.4}$$

where g is a bounded function on S^{Λ} and

$$\sigma^{x,k}(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ k & \text{if } y = x \end{cases}$$

represents a configuration where the agent at site x switches from his current strategy $\sigma(x)$ to a new strategy k.

If the stochastic process can introduce a new strategy that is not currently used in the population, this case is called *innovative* (Szabo, 2007). When a strategy which is not present in the population does not appear under the dynamics, the dynamics are called *non-innovative.* Furthermore if, upon switching, agents only consider the payoff of the new strategy we call the dynamics *targeting.* In contrast, when agents' decision depends on the payoff difference between the current strategy and the new strategy the dynamics are called *comparing.* To define the rate, let us introduce

$$w(x, \sigma, k) := \sum_{y \in \Lambda} \mathcal{W}(x - y)\delta(\sigma(y), k)$$

where $\delta(i, j) = 1$ if $i = j$ and 0 otherwise; $w(x, \sigma, k)$ can be interpreted as the probability for an agent at site x to find a neighbor with strategy k, provided the neighbors are sampled with the probability distribution $\mathcal{W}(x - y)$. Let also F denote a non-negative and non-decreasing function. We have the following examples of rates.

- **Targeting and Innovative**: This case arises if $c(x, \sigma, k) = F(u(x, \sigma, k))$ and $F > 0$. For example,

$$c_{n,\beta}(x, \sigma, k) = \exp(\beta u(x, \sigma, j))$$

where β is a non-negative constant. If

$$c(x, \sigma, k) = \frac{\exp(\beta u(x, \sigma, k))}{\sum_l \exp(\beta u(x, \sigma, l))} \tag{2.5}$$

the rate is called "logit choice rule" in the game theory literature, and it is a generalization of the "Gibbs sampler" in statistics and of the "Glauber dynamics "of physics. Here the inverse of β captures the noisy level; $\beta = 0$ means the uniform randomization of strategies, while the choice rule tends to the best response rule as β approaches the infinity. In particular, when the normal form game is a potential game, the corresponding Markov chain satisfies detailed balance and its invariant distribution can be explicitly expressed as a Gibbs distribution. Next, let $BR(x,\sigma) := \arg\max_{i\in S} u(x,\sigma,i)$ be the set of best response strategies for individual x when the configuration is σ. Then another version of perturbed best response can be written as

$$c(x,\sigma,k) = \begin{cases} \frac{\beta}{1+\beta}\frac{1}{|BR|} & \text{if } k \in BR(x,\sigma) \\ \frac{1}{1+\beta}\frac{1}{|S|} & \text{if } k \notin BR(x,\sigma) \end{cases} .$$

• **Comparing and Innovative**: The rate is $c(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$ and is comparing and innovative provided $F > 0$. When

$$c(x,\sigma,k) = \min\left\{1, \exp(\beta\left[u(x,\sigma,k) - u(x,\sigma,\sigma(x))\right]_+)\right\},$$

the rate corresponds to a generalization of the well-known Metropolis algorithm. The Markov process, in this case too, satisfies the detailed balance for potential games and has the same Gibbs invariant distribution as Glauber dynamics (Szabo, 2007). Here, the parameter β plays the similar role to the logic choice rule. When $\beta = 0$,the strategy revising agent is indifferent between his current strategy $\sigma(x)$ and new strategy k regardless of the payoffs, while as $\beta \to 0$ the importance of the payoffs becomes significant. More generally, one can consider

$$c(x,\sigma,j) = G(u(x,\sigma,j) - u(x,\sigma,\sigma(x)))$$

with

Metropolis $\quad G_M(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ \exp(\beta t) & \text{if } t < 0 \end{cases}$

Baker $\quad G_B(t) = \dfrac{1}{1+\exp(-\beta t)}$

$$\textbf{Generalized Metropolis}\qquad G_\gamma(t) = \begin{cases} \frac{1}{1+\exp(-\beta t)}\left(1+2\left(\frac{1}{2}\exp(-\beta t)\right)^{\gamma}\right) & \text{if } t \geq 0 \\ \frac{1}{1+\exp(-\beta t)}\left(1+2\left(\frac{1}{2}\exp(\beta t)\right)^{\gamma}\right) & \text{if } t < 0 \end{cases}$$

for $\gamma \geq 1$. We note that when $\gamma = 1$ $G_\gamma(t) = G_M(t)$ and as $\gamma \to \infty, G_\gamma(t) \to G_B(t)$ for $t \neq 0$.

• **Comparing and Non-innovative:** Suppose that when an individual receives a revision opportunity, she chooses to switch strategies with a probability that is linearly increasing in her neighbor's payoff. For example, individuals might revise when their neighbors' payoffs reach a certain threshold (emulation) level. The strategy revision rate (called the imitation of success) is

$$c(x, \sigma, j) = w(x, \sigma, j)F(u(x, \sigma, j) - K),$$

where K is some positive constant. Alternatively, one can consider has the following imitation rate

$$c(x, \sigma, k) = w(x, \sigma, k)F(u(x, \sigma, k) - u(x, \sigma, \sigma(x))) \,. \tag{2.6}$$

Here, the first factor $w(x, \sigma, k)$ is the probability for an agent at x to choose an agent with strategy k and the second factor $F(u(x, \sigma, k) - u(x, \sigma, \sigma(x)))$ gives the rate at which the new strategy k is adopted (Weibull, 1995; Benaim and Weibull, 2003; Hofbauer and Sigmund, 2003). The standard example is

$$c(x, \sigma, k) = w(x, \sigma, k)\left[u(x, \sigma, k) - u(x, \sigma, \sigma(x))\right]_+ \tag{2.7}$$

where $[s]_+ = \max\{s, 0\}$. As a variant of (2.6), one can simply assume that the rate of imitating is proportional to the current payoff difference between the imitating person and the imitated person *regardless whether doing so actually increases one's payoff or not*. In this case the following strategy revision rule arises:

$$c(x, \sigma, k) = \sum_{y\in\Lambda} \mathcal{W}(x-y)\delta(\sigma(y), k)F(u(y, \sigma, k) - u(x, \sigma, \sigma(x))) \tag{2.8}$$

The rate (2.7), in the mean-field case, gives rise to the famous replicator ODEs as the deterministic approximation. More generally if F in (2.6) satisfies

$$F(s) - F(-s) = s\,, \tag{2.9}$$

then the corresponding mean field ODE is the replicator dynamics. Note that $[s]_+$ satisfies condition (2.9). In the chapter we frequently use

$$F_\kappa\,(s) := \frac{1}{\kappa}\log(\exp(\kappa s) + 1) \tag{2.10}$$

and it is easily seen that the function (2.10) satisfies (2.9) and converges uniformly to $[s]_+$ as $\kappa \to \infty$; hence (2.10) can serve as a smooth regularization of (2.7).

2.2 Meso-scopic Limits

2.2.1 Basic setup

We consider the limit $\gamma \to 0$ in equation (2.3); i.e., the interaction range $\frac{1}{\gamma}$ becomes infinite and the agent at x interacts with a growing number of agents. In order to obtain a limiting equation, we rescale space and take a continuum limit. Let $\mathbb{A} \subset \mathbb{R}^d$ (meso-scopic domain) and $\mathbb{A}^\gamma := \gamma^{-1}\mathbb{A} \cap \mathbb{Z}^d$ (microscopic domain). If $\mathbb{A}$ is a smooth region in $\mathbb{R}^d$, then $\mathbb{A}^\gamma$ contains $\gamma^{-d}|\mathbb{A}|$ lattice sites and as $\gamma \to 0$, $\gamma\mathbb{A}^\gamma$ approximates $\mathbb{A}$.

At the meso-scopic scale the state of the system is described by the *strategy profile* function $f_t(u,i)$ – the density of agents with strategy i at u. The bridge between microscopic and meso-scopic scale is given by the *empirical measure* π^γ_σ defined as follows. For $(v,j) \in \mathbb{A} \times S$, let $\delta_{(v,j)}$ denote the Dirac delta measure at (v,j).

Definition 2.2.1 (Empirical measure) *The empirical measure* $\pi^\gamma : S^{\mathbb{A}^\gamma} \to \mathcal{P}(\mathbb{A}\times S)$ *is the map given by*

$$\sigma \mapsto \pi^\gamma_\sigma := \frac{1}{|\mathbb{A}^\gamma|} \sum_{x\in\mathbb{A}^\gamma} \delta_{(\gamma x, \sigma(x))} \tag{2.11}$$

where $\mathcal{P}(\mathbb{A}\times S)$ *denotes the set of all probability measures on* $\mathbb{A}\times S$.

In addition to the empirical measure, we define a measure m on $\mathbb{A}\times S:$ for a measurable function f,

$$\int_{\mathbb{A}\times S} f(u,i)dm(u,i) := \sum_{i\in S}\int_{\mathbb{A}} f(u,i)du$$

where du is the Lebegues measure on $\mathbb{A}$. Our main result is to show that, under suitable conditions,

$$\pi^\gamma_{\sigma_t} \to f_t\, m \text{ in probability,} \tag{2.12}$$

and f_t satisfies an integro-differential equation. Since σ_t is the state of the microscopic system at time t, $\pi^\gamma_{\sigma_t}$ is a random measure, while f_t is a solution of a deterministic equation. So (2.12) is in a sense a form of a time-dependent law of large numbers. For this result to hold we need to assume that the initial distribution for σ_0 is sufficiently regular.

Definition 2.2.2 (Product measures with a slowly varying parameter) *The collection of measure* $\{\mu_\gamma\}$ *is called a family of product measures with a slowly varying parameter if* $\mu_\gamma := \bigotimes_{x\in\mathbb{A}^\gamma} \rho_x$ *on* $S^{\mathbb{A}^\gamma}$ *and there exists a profile* $f(u,i)$ *such that*

$$\rho_x(\{i\}) = f(\gamma x, i)$$

The following definition gives a bit more general initial distributions:

Definition 2.2.3 (Probability measure associated to an initial profile) *A sequence* $\{\mu^\gamma\}_{\gamma>0}$ *of probability measures on* $S^{\mathbb{A}_\gamma}$ *is associated to an initial profile*

$f \in L^\infty(\mathbb{A} \times S)$ if for every continuous function $g : \mathbb{A} \times S \to \mathbb{R}_+$ and for every $\epsilon > 0$, we have

$$\lim_{\gamma \to 0} \mu^\gamma \left\{ \sigma_{\mathbb{A}_\gamma} : \left| \frac{1}{|\mathbb{A}_\gamma|} \sum_{x \in \mathbb{A}_\gamma} g(\gamma x, \sigma_{\mathbb{A}_\gamma}(x)) - \frac{1}{|\mathbb{A}|} \int g(u,i) f(u,i) du di \right| > \epsilon \right\} = 0 \tag{2.13}$$

where $\sigma_{\mathbb{A}_\gamma} \in S^{\mathbb{A}_\gamma}$.

Then from Proposition 0.4 in Kipnis and Landim (1999), a family of product measures with a slowly varying parameter $\{\mu_\gamma\}_\gamma$ satisfies (2.13). Furthermore, we focus on two types of boundary conditions:

(a) Periodic Boundary Conditions. Let $\mathbb{A} = [0,1]^d$ and suppose that $\mathbb{A}^\gamma = \gamma^{-1}\mathbb{A} \cap \mathbb{Z}^d = [0, \frac{1}{\gamma}]^d \cap \mathbb{Z}^d$. Then we extend periodically the profile $f_t(u,i)$ and the configuration $\sigma_{\mathbb{A}^\gamma}$ on $\mathbb{R}^d$ and $\mathbb{Z}^d$. Equivalently we identify $\mathbb{A}$ with the torus $\mathbf{T}^d$ and similarly $\mathbb{A}^\gamma$ with the discrete torus $\mathbf{T}^{d,\gamma}$.

(b) Fixed Boundary Conditions. In applications it is also useful to consider the case where the configurations in some regions do not change with time. Let $\Lambda \subset \Gamma \subset \mathbb{R}^d$ be a region. Then the "boundary region" is given by $\partial\Lambda := \Gamma \backslash \Lambda$ where agents do not revise their strategies. Since $\mathcal{J}$ is compactly supported, we can take, for suitable $r > 0$

$$\Gamma := \bigcup_{u \in \Lambda} B(u,r),$$

where B denotes a ball centered at u with radius r.

2.2.2 Main results

Consider first the case with periodic boundary conditions. The assumptions on the interactions weights $\mathcal{W}^\gamma(x-y)$ are

(F) $\mathcal{W}^\gamma(x-y) = \gamma^d \mathcal{J}(\gamma(x-y))$ where $\mathcal{J}$ is nonnegative, continuous with compact support, and normalized, $\int \mathcal{J}(x)dx = 1$.

The meso-scopic strategy profiles are described by functions $f \in \mathcal{M}(\mathbf{T}^d \times S)$ where

$$\mathcal{M}(\mathbf{T}^d \times S) := \left\{ f : 0 \leq f(u,i) \leq 1,\ \sum_i f(u,i) = 1 \text{ for all } u \in \mathbf{T}^d \right\} .$$

Let $\{\sigma_t^\gamma\}_{t\geq 0}$ be the stochastic process with generator L^γ given by

$$(L^\gamma g)(\sigma) = \sum_{x \in \mathbf{T}^{d,\gamma}} \sum_{k \in S} c^\gamma(x,\sigma,k) \left(g(\sigma^{x,k}) - g(\sigma) \right) \tag{2.14}$$

for $g \in L^\infty(S^{\mathbf{T}^{d,\gamma}})$. The assumptions on the strategy revision rate $c^\gamma(x,\sigma,k)$ are that there exists a real-valued function

$$\mathbf{c}(u,i,k,\pi),\ u \in \mathbf{T}^d,\ i,k \in S,\ \pi \in \mathcal{P}(\mathbf{T}^d \times S)$$

such that

(C1) $\mathbf{c}(u,i,k,\pi)$ satisfies

$$\lim_{\gamma \to 0} \sup_{x \in \mathbf{T}^{d,\gamma}, \sigma \in S^{\mathbf{T}^{d,\gamma}}, k \in S} |c^\gamma(x,\sigma,k) - \mathbf{c}(\gamma x, \sigma(x), k, \pi_\sigma^\gamma)| = 0 ,$$

(C2) $\mathbf{c}(u,i,k,\pi)$ is uniformly bounded: i.e., there exists M such that

$$\sup_{u \in \mathbf{T}^d, i,k \in S, \pi \in \mathcal{P}(\mathbf{T}^d \times S)} |\mathbf{c}(u,i,k,\pi)| \leq M ,$$

(C3) $\mathbf{c}(u,i,k,f\,m)$ satisfies a Lipschitz condition with respect to f : i.e., there exists L such that for all $f_1,\ f_2 \in \mathcal{M}(\mathbf{T}^d \times S)$

$$\sup_{u \in \mathbf{T}^d, i,k \in S} |\mathbf{c}(u,i,k,f_1\,m) - \mathbf{c}(u,i,k,f_2\,m)| \leq L \, \|f_1 - f_2\|_{L^1(\mathbf{T}^d \times S)} .$$

Hereafter we abuse the notation by denoting $\mathbf{c}(u,i,k,f) := \mathbf{c}(u,i,k,f du\ di) = \mathbf{c}(u,i,k,\pi)$ for a measure π that is absolutely continuous with respect m (so $\pi = f\,m$ for some measurable function f). In the section below we show that all the classes of rates given in the examples in Section 2.1 and several others satisfy conditions **C1**$-$**C3**. For example, if $c^{\gamma}(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$, with $u(x,\sigma,i) = \sum_y \mathcal{W}^{\gamma}(x-y)a(i,\sigma(y))$ and the weights $\mathcal{W}^{\gamma}(x-y)$ satisfy condition **F**, then

$$\mathbf{c}(u,i,k,f) = F\left(\sum_{l\in S} a(k,l)\mathcal{J} * f(u,l) - a(i,l)\mathcal{J} * f(u,l)\right) \tag{2.15}$$

satisfies condition **C1** $-$ **C3** (recall that $\mathcal{J} * f(u,l) := \int_{\mathbf{T}^d} \mathcal{J}(u-v)f(v,l)dv$ is the convolution of $\mathcal{J}$ with f). A slight modification of (2.15) yields corresponding expressions for each choice of $c^{\gamma}(x,\sigma,k)$ previously given. Note that when f_1 and f_2 are constant over $\mathbf{T}^d$ or there is no spatial dimension, f_1 and f_2 can be regarded as points in the simplex Δ. In this case **C3** reduces to the Lipschitz continuity condition in Benaim and Weibull (2003, p.878) and, in this way, **C3** generalizes their conditions.

Theorem 2.2.4 shows that the stochastic process $\pi^{\gamma}_{\sigma_t}$ has a deterministic limit.

Theorem 2.2.4 (Long Range Interaction and Periodic Boundary Condition) *Suppose the revision rate satisfies* **C1**$-$**C**3*. Let $f \in \mathcal{M}(\mathbf{T}^d \times S)$ and assume that the initial distribution $\{\mu^{\gamma}\}_{\gamma}$ is a family of measures with a slowly varying parameter associated to the profile of f. Then for every $T > 0$*

$$\lim_{\gamma\to 0} \pi^{\gamma}_{\sigma_t} = f_t\, m \ \textit{in probability}$$

uniformly for $t \in [0,T]$ and f_t satisfies the following differential equation: for $u \in \mathbf{T}^d, i \in S$

$$\begin{aligned} \frac{\partial}{\partial t} f_t(u,i) &= \sum_{k\in S} \mathbf{c}(u,k,i,f)f_t(u,k) - f_t(u,i)\sum_{k\in S}\mathbf{c}(u,i,k,f) \\ f_0(u,i) &= f(u,i) \end{aligned} \tag{2.16}$$

Next consider fixed boundary conditions. In this case, the stochastic process,

$\{\sigma_t\}_{t\geq 0}$, is specified by the generator L^γ

$$(L^\gamma g)(\sigma_{\Gamma^\gamma}) = \sum_{x\in\Lambda^\gamma}\sum_{k\in S} c^\gamma(x,\sigma_{\Gamma^\gamma},k)(g(\sigma^{x,k}_{\Gamma^\gamma}) - g(\sigma_{\Gamma^\gamma})) \tag{2.17}$$

for $g \in L^\infty(S^{\Gamma^\gamma})$. Note that the summation in terms of x in (2.17) is taken over Λ^γ, which represents the fact that only individuals in Λ^γ revise their strategies, whereas the rate depends on the configuration in entire Γ^γ. For a given $f \in \mathcal{M}$, its restriction on Λ is defined by $f_\Lambda(u,i) : f_\Lambda(u,i) = f(u,i)$ if $u \in \Lambda$ and $f_\Lambda(u,i) = 0$ if $u \in \Lambda^C$.

Theorem 2.2.5 (Long Range Interaction and Fixed Boundary Condition) *Suppose the revision rate satisfies* **C1**−**C3**. *Let $f \in \mathcal{M}(\Gamma^d \times S)$ and assume that the initial distribution $\{\mu^\gamma\}_\gamma$ is a family of measures with a slowly varying parameter associated to the profile of f. Then for every $T > 0$*

$$\lim_{\gamma\to 0} \pi^\gamma_{\sigma_t} = \frac{1}{|\Gamma|} f_t\, m \textit{ in probability}$$

uniformly for $t \in [0,T]$ and $f_t = f_{\Lambda,t} + f_{\partial\Lambda,t}$ satisfies the following differential equation: for $u \in \Gamma, i \in S$

$$\begin{aligned} \frac{\partial}{\partial t} f_{\Lambda,t}(u,i) &= \sum_{k\in S} \mathbf{c}(u,k,i,f) f_{\Lambda,t}(u,k) - f_{\Lambda,t}(u,i) \sum_{k\in S} \mathbf{c}(u,i,k,f) \\ f_0(u,i) &= f(u,i) \end{aligned} \tag{2.18}$$

Note that $\mathbf{c}(u,k,i,f) = c(u,k,i,f_\Lambda + f_{\partial\Lambda})$ is given by the similar formula to (2.15) with $\mathcal{J} * f(u) = \int_\Gamma \mathcal{J}(u-v)f(v)dv$ for $u \in \Lambda$; so the rates depend on $f_{\partial\Lambda}$ as well as f_Λ.

2.2.3 Heuristic derivation of the differential equations

In this section we justify, heuristically, the IDEs obtained in Theorems 2.2.4 and 2.2.5. For simplicity we assume periodic boundary conditions but the other case is similar. The differential equations (2.16) and (2.18) are examples of input-output equations. In particular, by summing over the strategy set, it is easy to see that $\sum_{i\in S} f_t(u,i)$ is independent of t and therefore if $f_0 \in \mathcal{M}$, then $f_t \in \mathcal{M}$ for all t. Also the space $\mathcal{M}$ can be thought of as a product over the space of the standard strategy simplex Δ of game theory, i.e., $\mathcal{M} = \prod_{u\in\mathbf{T}^d} \Delta$. As shown in evolutionary game theory textbooks (Weibull, 1995; Sandholm, 2010b) one can derive heuristically the ODEs from corresponding stochastic processes. The main assumption used there is that the rates depend only on the average proportion of players with a given strategy. In this section a similar heuristic derivation from microscopic processes in the case of the spatial IDE (2.16) is provided where global average is replaced by spatially localized averages as expressed in the limit of the empirical measure (2.11).

For microscopic sites x and y, let us denote by $u = \gamma x$ and $v = \gamma y$ the corresponding spatial positions at the meso-scopic level. For the sake of exposition let us suppose that $c^\gamma(x,\sigma,k)$ is given by

$$c^\gamma(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x))\,.$$

For any continuous function g on $\mathbf{T}^d \times S$, by the definition of the empirical measure (2.11) we have the identity

$$\frac{1}{|\mathbf{T}^{d,\gamma}|}\sum_{x\in\mathbf{T}^{d,\gamma}} g(\gamma x,\sigma(x)) = \int_{\mathbf{T}^d\times S} g(u,i)d\pi^\gamma_\sigma(u,i)\,.$$

Since $\left|\mathbf{T}^{d,\gamma}\right| \approx \gamma^{-d}$ and if we *assume* that $\pi^\gamma(\sigma) \to f du\, di$, we obtain

$$\lim_{\gamma\to 0}\sum_{x\in\mathbf{T}^{d,\gamma}} \gamma^d g(\gamma x,\sigma(x)) = \int_{\mathbf{T}^d\times S} g(u,i)f(u,i)dm(u,i). \tag{2.19}$$

Using (2.19), we find

$$\begin{aligned}\lim_{\gamma\to 0}\sum_{x\in\mathbf{T}^{d,\gamma}}\gamma^d\mathcal{J}\ \ (\gamma(x-y))\,a(k,\sigma(y)) &= \int_{\mathbf{T}^d\times S} a(k,l)\mathcal{J}(u-v)f(v,l)dm(v,l)\\ &= \sum_{l\in S} a(k,l)\mathcal{J}*f(u,l).\end{aligned}$$

Therefore if $\sigma(x)=i$ we then obtain

$$\begin{aligned} c^\gamma(x,\sigma,k) &= F(u(x,\sigma,k)-u(x,\sigma,\sigma(x)))\\ &= F\left(\sum_{y\in\mathbf{T}^{d,\gamma}}\gamma^d\mathcal{J}\ \ (\gamma x-\gamma y)\,a(k,\sigma(y))-\sum_{y\in\mathbf{T}^{d,\gamma}}\gamma^d\mathcal{J}\ \ (\gamma x-\gamma y)\,a(\sigma(x),\sigma(y))\right)\\ &\underset{\gamma\to 0}{\longrightarrow} F\left(\sum_{l\in S}a(k,l)\mathcal{J}*f(u,l)-\sum_{l\in S}a(i,l)\mathcal{J}*f(u,l)\right)=\mathbf{c}(u,i,k,f),\end{aligned}$$

and this gives equation (2.15). After having identified rates, we can now explain how to derive the IDE (2.16). We write

$$\langle\pi_\sigma^\gamma,g\rangle := \int_{\mathbf{T}^d\times S} g(u,i)d\pi_\sigma^\gamma,\qquad \langle f,g\rangle := \int_{\mathbf{T}^d\times S} g(u,i)f(u,i)dm(u,i),$$

where we view $\langle\pi_\sigma^\gamma,g\rangle$ as a function of the configuration σ. The action of the generator on this function is

$$L_\gamma\left\langle\pi_\sigma^\gamma,g\right\rangle=\sum_{k\in S}\int_{\mathbf{T}^d\times S}\mathbf{c}(u,i,k,\pi_\sigma^\gamma)\left(g(u,k)-g(u,i)\right)d\pi_\sigma^\gamma(u,i)\,.$$

From the martingale representation theorem for Markov processes (for example see Ethier and Kurtz, 1986) there exists a martingale $M_t^{g,\gamma}$ such that

$$\left\langle\pi_{\sigma_t}^\gamma,g\right\rangle=\left\langle\pi_{\sigma_0}^\gamma,g\right\rangle+\int_0^t ds\sum_{k\in S}\int_{\mathbf{T}^d\times S}\mathbf{c}(u,i,k,\pi_{\sigma_s}^\gamma)\left(g(u,k)-g(u,i)\right)d\pi_{\sigma_s}^\gamma(u,i)+M_t^{g,\gamma}. \tag{2.20}$$

As $\gamma\to 0$, one proves that $M_t^{g,\gamma}\to 0$. Thus if $\pi_{\sigma_t}^\gamma=f_t\,m$ as $\gamma\to 0$, equation (2.20) becomes

$$\langle f_t,g\rangle=\langle f_0,g\rangle+\int_0^t ds\sum_{k\in S}\int_{\mathbf{T}^d\times S}\mathbf{c}(u,i,k,f_s)\left(g(u,k)-g(u,i)\right)f_s(u,i)dm(u,i)$$

and upon differentiating with respect to time, we find

$$< \frac{\partial f_t}{\partial t}, g >= \sum_{k \in S} \int_{\mathbf{T}^d \times S} \mathbf{c}(u, i, k, f_t) \left(g(u, k) - g(u, i) \right) f_t(u, i) dm(u, i) \qquad (2.21)$$

which is the weak formulation of the IDE (2.16) obtained by integrating over u and i. In the next section, we collect the existing results that are used in the proof of Theorems 2.2.4 and 2.2.5.

2.3 Existing Results

Consider a bounded function $F : \mathbb{R}_+ \times E \to \mathbb{R}$

(1) $F(\cdot, x)$ is twice differentiable for each x in E

(2) There exists C such that

$$\sup_{(s,x)} \left| (\partial_s^j F)(s, x) \right| \leq C \text{ for } j = 1, 2$$

Suppose that we have a continuous time Markov chain $\{X_t\}$ with a generator L. We define $M^F(t), N^F(t)$ by

$$\begin{aligned} M^F(t) &= F(t, X_t) - F(0, X_0) - \int_0^t ds (\partial s + L) F(s, X_s) \\ N^F(t) &= (M^F(t))^2 - \int_0^t ds \left\{ LF(x, X)^2 - 2F(s, X_s) LF(s, X_s) \right\} \end{aligned}$$

Lemma 2.3.1 (Kipnis & Landim (1999) p.330) *Denote by $\{\mathcal{F}_t : t \geq 0\}$ the filtration induced by the Markov process. The processes $M^F(t)$ and $N^F(t)$ are $\mathcal{F}_t-$martingale.*

Let E be a complete separable metric space. Then $D([0, T], E)$-valued random variable (or random element) is a stochastic process with sample paths in

$D([0,T],E)$ (or E-valued stochastic process). Let $\{X_n\}$ be a family of stochastic processes with sample paths in (or random elements taking values from) $D([0,T],E)$. Let $\{P_n\} \subset \mathcal{P}(D([0,T],E))$ be the family of associated probability distributions, i.e. $P_n(B) = P\{\omega : X_n \in B\}$ for all B in the Borel σ-algebra of $D([0,T],E)$. We say that $\{X_n\}$ is relatively compact if $\{P_n\}$ is relatively compact; i.e. if the closure of $\{P_n\}$ in $\mathcal{P}(D([0,T],E))$ is compact. In our case, $E = \mathcal{P}(\Lambda \times S)$, the set of all probability measures on $\Lambda \times S$. Because of the following theorem, it is enough we consider the case $E = \mathbb{R}$.

Proposition 2.3.2 (Kipnis & Landim (1999) p.54) *Let $\{g_k : k \geq 1\}$ be a dense subfamily of $C(\Lambda \times S)$ with $g_1 = 1$. A family of probability measures $\{P_n\}$ on $D([0,T],\mathcal{P}(\Lambda \times S))$ is relatively compact if for every positive integer k the family $P_n g_k^{-1}$ of probabilities on $D([0,T],\mathbb{R})$ is relatively compact. Here $P_n g_k^{-1}$ is defined by*

$$P_n g_k^{-1}(A) = P_n\{\pi_\cdot \in D : \langle \pi_\cdot, g_k\rangle \in A\} \text{ for all Borel sets } A \text{ of } D([0,T],\mathbb{R})$$

where $\langle \pi_\cdot, g_k\rangle : [0,T] \to \mathbb{R}, t \mapsto \langle \pi_t, g_k\rangle$.

Next we provide the Prohorov criteria for the relative compactness for $\{P_n\}$ on $D([0,T],\mathbb{R})$. First we define a modified modulus of continuity in $D = D([0,T],\mathbb{R})$. For $x \in D$ and $T_0 \subset [0,T]$, put

$$w_x(T_0) = \sup\{|x(s) - x(t)| : s,t \in T_0\}$$

and

$$w_x(\delta) = \sup_{0 \leq t \leq T-\delta} w_x[t, t+\delta]$$

For $0 < \delta < T$, put

$$w'_x(\delta) = \inf_{\{t_i\}} \max_{0 < i \leq r} w_x[t_{i-1}, t_i)$$

where the infimum extends over the finite sets $\{t_i\}$ of points satisfying

$$\begin{cases} 0 = t_0 < t_1 < \cdots < t_r = 1 \\ t_i - t_{i-1} > \delta, \quad i = 1, 2, \cdots, r \end{cases}$$

Then we see that $\lim_{\delta \to 0} w'_x(\delta) = 0$ is necessary and sufficient for x to lie in D.

Theorem 2.3.3 (Prohorov; Billingsley (1968) p.125) *The sequence $\{P_n\}$ is relatively compact if and only if the following two conditions hold:*

(i) For each positive η, there exists an a such that

$$P_n\{x : \sup_t |x(t)| > a\} \leq \eta, \ n \geq 1$$

(ii) For each positive ϵ and η, there exist a δ, $0 < \delta < 1$, and an integer n_0 such that

$$P_n\{x : w'_x(\delta) \geq \epsilon\} \leq \eta, \ n \geq n_0$$

For the condition (ii) in Theorem 2.3.3, we have the following sufficient condition due to Aldous (1978). Let $\{\tau_n, \delta_n\}$ be such that

(i) for each n, τ_n is a stopping time on the process $\{X_n(t) : 0 \leq t \leq T\}$ with respect to the natural filtration

(ii) for each n, δ_n is a constant, $0 \leq \delta_n \leq T$ and $\delta_n \to 0$ as $n \to \infty$.

Proposition 2.3.4 (Aldous (1978)) *A sequence of probability measures $\{P_n\}$ on D satisfies condition (ii) of Prohorov theorem if an associated sequence of random elements $\{X_n\}$ satisfies*

$$P\{\omega : |X_n(\tau_n + \delta_n) - X_n(\tau_n)| > \epsilon\} \to 0 \text{ for all } \epsilon > 0$$

for $\{\tau_n, \delta_n\}$ satisfying (i) and (ii)

Note that we abuse notations by writing $\tau_n + \delta_n = \tau_n + \delta_n \wedge T$, $\tau_n = \tau_n \wedge T$. Finally we use the following proposition to conclude the theorem. We write $P_n \Rightarrow P$ when P_n converge weakly to P. Then $P_n \Rightarrow P$ implies $P_n h^{-1} \Rightarrow P h^{-1}$ when h is continuous, where $P_n h^{-1}(A) = P_n \{x : h(x) \in A\}$. Let D_h be the set of discontinuity of h. Then D_h is measurable.

Theorem 2.3.5 (Billingsley (1968) p.30) *If $P_n \Rightarrow P$ and $P(D_h) = 0$, then $P_n h^{-1} \Rightarrow P h^{-1}$*

The proof of Theorem 2.2.4 and Theorem 2.2.5, which we present in the next section, is a variation on the proof given in Comets (1987), Kipnis and Landim (1999), and Katsoulakis, Plechac, and Tsagkarogiannis (2005). Unlike these papers, in the case of non-innovative dynamics studied here there is no detailed balance condition, however the meso-scopic limit of the type (2.12) can still be carried out in the Kac scaling (2.3). Using the martingale representation (2.20), we show that $\{\mathbf{Q}^\gamma\}_\gamma$, a sequence of probability laws of $\{\pi^\gamma_{\sigma_t}\}_\gamma$, is relatively compact. We then show that all the limit points are concentrated on the weak solutions of (2.18) and on measures absolutely continuous with respect to Lebesgue measure. Finally we demonstrate that the weak solutions of (2.18) are unique so that we conclude the convergence of $\mathbf{Q}^\gamma$ to the Dirac measure concentrated on the solution of (2.18).

2.4 The proof of Theorem 2.2.4

We first show that condition **C1−C3** are satisfied for the following strategy revision rates:

- $c^{\gamma}(x,\sigma,k) = F(u(x,\sigma,k))$

$\mathbf{c}(u,i,k,f) = F\left(\sum_l a(i,l)\,\mathcal{J} * f(u,l)\right)$

- $c^{\gamma}(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$

$\mathbf{c}(u,i,k,f) = F\left(\sum_l [a(k,l) - a(i,l)]\mathcal{J} * f(u,l)\right)$

- $c^{\gamma}(x,\sigma,k) = \sum_y w(x,y,\sigma,k) F(u(x,\sigma,k))$

$\mathbf{c}(u,i,k,f) = \mathcal{J} * f(u,k)\, F\left(\sum_l a(k,l)\,\mathcal{J} * f(u,l)\right)$

- $c^{\gamma}(x,\sigma,k) = \sum_y w(x,y,\sigma,k) F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$

$\mathbf{c}(u,i,k,f) = \mathcal{J} * f(u,k)\, F\left(\sum_l [a(k,l) - a(i,l)]\mathcal{J} * f(u,l)\right)$

- $c^{\gamma}(x,\sigma,k) = \frac{\exp(u(x,\sigma,k))}{\sum_l \exp(u(x,\sigma,l))}$

$\mathbf{c}(u,i,k,f) = \frac{\exp(\mathcal{J}*f(u,k))}{\sum_l \exp(\mathcal{J}*f(u,l))}$

if F satisfies the global Lipschitz condition: i.e., for all $x, y \in Dom(F)$, there exists $L > 0$ such that $|F(x) - F(y)| \leq L|x-y|$. Note that the list above is far from being exhaustive; one can easily invent various other rates which satisfy **C1**$-$**C3**. Since the verifications of the conditions are similar, we will check the conditions for the following rate (2.22) in the periodic domain.

$$c^{\gamma}(x,\sigma,k) = \sum_{\substack{y\in\Lambda_\gamma \\ \{\sigma(y)=k\}}} \gamma^d \mathcal{J}(\gamma x, \gamma y) F\left(\sum_{z\in\mathbb{Z}^d} \gamma^d a(k,\sigma(z))\mathcal{J}(\gamma y,\gamma z) - \sum_{z\in\mathbb{Z}^d} \gamma^d a(\sigma(x),\sigma(z))\mathcal{J}(\gamma x,\gamma z)\right) \tag{2.22}$$

Lemma 2.4.1 *The rate given by (2.22) satisfies* **C1** $-$ **C3**.

Proof. In the proof we set $a_{i,j} := a(i,j)$.Let

$$\begin{aligned} G_\gamma(u,v,i,k,\sigma) &:= F\Big(\sum_{z\in\mathbb{Z}^d}\gamma^d a_{k,\sigma(z)}\mathcal{J}(v,rz) - \sum_{z\in\mathbb{Z}^d}\gamma^d a_{i,\sigma(z)}\mathcal{J}(u,rz)\Big) \\ G(u,v,i,k,\pi) &:= F\Big(|\Gamma|\int_{\Gamma\times S} a_{kl}\mathcal{J}(v,w)\pi(dwdl) - |\Gamma|\int_{\Gamma\times S} a_{il}\mathcal{J}(u,w)\pi(dwdl)\Big) \\ c(u,i,k,\pi) &:= |\Gamma|\int_\Gamma G(u,v,i,k,\pi)\mathcal{J}(u,v)\pi(dv,\{k\}) \end{aligned}$$

First we show that

$$\sup_{\substack{y\in\Lambda_\gamma \\ \sigma_{\Gamma_\gamma}\in S^{\Gamma_\gamma} \\ k\in S}} \left| \sum_{z\in\mathbb{Z}^d}\gamma^d a_{k,\sigma_{\Gamma_\gamma}(z)}\mathcal{J}(\gamma y,\gamma z) - |\Gamma|\int_{\Gamma\times S} a_{k,l}\mathcal{J}(ry,w)\pi_{\sigma_{\Gamma_\gamma}}(dwdl)\right| \to 0$$

This follows from

$$\begin{aligned} &\left| \sum_{z\in\Gamma_\gamma}\gamma^d a_{k,\sigma_{\Gamma_\gamma}(z)}\mathcal{J}(\gamma y,\gamma z) - |\Gamma|\int_{\Gamma\times S} a_{k,l}\mathcal{J}(ry,w)\pi_{\sigma_{\Gamma_\gamma}}(dwdl)\right| \le \\ &\left| \sum_{z\in\Gamma_\gamma}\gamma^d a_{k,\sigma_{\Gamma_\gamma}(z)}\mathcal{J}(\gamma y,\gamma z) - \frac{|\Gamma|}{|\Gamma_\gamma|}\sum_{z\in\Gamma_\gamma} a_{k,\sigma_{\Gamma_\gamma}(z)}\mathcal{J}(\gamma y,\gamma z)\right| \le \\ &\left|\gamma^d - \frac{|\Gamma|}{|\Gamma_\gamma|}\right| \sum_{z\in\Gamma_\gamma} a_{k,\sigma(z)}\mathcal{J}(\gamma y,\gamma z) \le \\ &\left|\gamma^d|\Gamma_\gamma| - |\Gamma|\right| M \to 0 \text{ uniformly in } y,\sigma_{\Gamma_\gamma},k \end{aligned}$$

where $M := \sup_{i,k,u,v} a_{i,j}\mathcal{J}(u,v)$. Next we have

$$\begin{aligned} &\left|c_\gamma(x,\sigma_{\Gamma_\gamma},k) - c(\gamma x,\sigma_{\Gamma_\gamma}(x),k,\pi_{\sigma_{\Gamma_\gamma}})\right| \\ = &\Bigg| \sum_{y\in\Lambda_\gamma,\{\sigma(y)=k\}}\gamma^d\mathcal{J}(\gamma x,\gamma y)G_\gamma(\gamma x,\gamma y,\sigma_{\Lambda_\gamma}(x),k,\sigma_{\Lambda_\gamma}) - \\ &-|\Gamma|\int_\Gamma \mathcal{J}(\gamma x,v)G(\gamma x,v,\sigma_{\Gamma_\gamma}(x),k,\pi_{\sigma_{\Lambda_\gamma}})\pi_{\sigma_{\Lambda_\gamma}}(dv,\{k\})\Bigg| \\ \le &\left| \sum_{y\in\Lambda_\gamma,\{\sigma(y)=k\}}\gamma^d\mathcal{J}(\gamma x,\gamma y)\Big(G_\gamma(\gamma x,\gamma y,\sigma_{\Lambda_\gamma}(x),k,\sigma_{\Lambda_\gamma}) - G(\gamma x,v,\sigma_{\Gamma_\gamma}(x),k,\pi_{\sigma_{\Lambda_\gamma}})\Big)\right| \\ &+\left|\left(\gamma^d - \frac{|\Gamma|}{|\Gamma_\gamma|}\right)\sum_{y\in\Lambda_\gamma,\{\sigma(y)=k\}}\mathcal{J}(\gamma x,\gamma y)G(\gamma x,v,\sigma_{\Gamma_\gamma}(x),k,\pi_{\sigma_{\Lambda_\gamma}})\right| \\ := &\ I + II \end{aligned}$$

It is easily seen that $II \to 0$ uniformly in $x, \sigma_{\Gamma_\gamma}, k$ since G is uniformly bounded. For the estimate of I, we have

$$
\begin{aligned}
& \left| G_\gamma(\gamma x, \gamma y, \sigma_{\Lambda_\gamma}(x), k, \sigma_{\Lambda_\gamma}) - G(\gamma x, v, \sigma_{\Gamma_\gamma}(x), k, \pi_{\sigma_{\Lambda_\gamma}}) \right| \leq \\
& \left| F(\sum_{z \in \mathbb{Z}^d} \gamma^d a_{k,\sigma(z)} \mathcal{J}(\gamma y, \gamma z) - \sum_{z \in \mathbb{Z}^d} \gamma^d a_{\sigma_{\Lambda_\gamma}(x),\sigma(z)} \mathcal{J}(\gamma x, \gamma z)) \right. \\
& \left. -F(|\Gamma| \int_{\Gamma \times S} a_{k,l} \mathcal{J}(\gamma y, w) \pi_{\sigma_{\Lambda_\gamma}}(dwdl) - |\Gamma| \int_{\Gamma \times S} a_{\sigma_{\Lambda_\gamma}(x),l} \mathcal{J}(\gamma x, w) \pi_{\sigma_{\Lambda_\gamma}}(dwdl) \right| \\
\leq \; & L \left| \sum_{z \in \Gamma_\gamma} \gamma^d a_{k,\sigma_{\Gamma_\gamma}(z)} \mathcal{J}(\gamma y, \gamma z) - |\Gamma| \int_{\Gamma \times S} a_{k,l} \mathcal{J}(\gamma y, w) \pi_{\sigma_{\Lambda_\gamma}}(dwdl) \right| + \\
& L \left| \sum_{z \in \Gamma_\gamma} \gamma^d a_{\sigma_{\Lambda_\gamma}(x),\sigma_{\Gamma_\gamma}(z)} \mathcal{J}(\gamma x, \gamma z) - |\Gamma| \int_{\Gamma \times S} a_{\sigma_{\Lambda_\gamma}(x),l} \mathcal{J}(\gamma x, w) \pi_{\sigma_{\Lambda_\gamma}}(dwdl) \right| \\
\leq \; & L \sup_{\substack{y \in \Lambda_\gamma \\ \sigma_{\Gamma_\gamma} \in S^{\Gamma_\gamma} \\ k \in S}} \left| \sum_{z \in \Gamma_\gamma} \gamma^d a_{k,\sigma_{\Gamma_\gamma}(z)} \mathcal{J}(\gamma y, \gamma z) - |\Gamma| \int_{\Gamma \times S} a_{k,l} \mathcal{J}(\gamma y, w) \pi_{\sigma_{\Lambda_\gamma}}(dwdl) \right| + \\
& L \sup_{\substack{x \in \Lambda_\gamma \\ \sigma_{\Gamma_\gamma} \in S^{\Gamma_\gamma} \\ k \in S}} \left| \sum_{z \in \Gamma_\gamma} \gamma^d a_{\sigma_{\Lambda_\gamma}(x),\sigma_{\Gamma_\gamma}(z)} \mathcal{J}(\gamma x, \gamma z) - |\Gamma| \int_{\Gamma \times S} a_{\sigma_{\Lambda_\gamma}(x),l} \mathcal{J}(\gamma x, w) \pi_{\sigma_{\Lambda_\gamma}}(dwdl) \right| \\
\to \; & 0 \text{ uniformly in } x, y, \sigma_{\Lambda_\gamma}, k
\end{aligned}
$$

by 1. Hence the result follows. **C2** follows from the fact that $G(u, v, i, k, \pi)$ is uniformly bounded. For **C3**, we have

$$
\begin{aligned}
& |c(u, i, k, f_1 dudi) - c(u, i, k, f_2 dudi)| \leq \\
& \left| |\Gamma| \int_\Gamma G(u, v, i, k, f_1 dudi) \mathcal{J}(u, v) \left(f_1(v, k) - f_2(v, k) \right) dv \right| + \\
& \left| |\Gamma| \int_\Gamma (G(u, v, i, k, f_1 dudi) - G(u, v, i, k, f_2 dudi)) \mathcal{J}(u, v) f_2(v, k) dv \right| \\
: \; & = I + II
\end{aligned}
$$

Since G is uniformly bounded,

$$I \leq C \left\| f_1 - f_2 \right\|_{L^1}$$

Also

$$\begin{aligned}
& \left| G(u,v,i,k,f_1 dudi) - G(u,v,i,k,f_2 dudi) \right| \leq \\
& \Big| F(|\Gamma| \int_{\Gamma\times S} a_{kl} \mathcal{J}(v,w) f_1(w,l) dwdl - |\Gamma| \int_{\Gamma\times S} a_{il} \mathcal{J}(u,w) f_1(w,l) dwdl) \\
& - F(|\Gamma| \int_{\Gamma\times S} a_{kl} \mathcal{J}(v,w) f_2(w,l) dwdl - |\Gamma| \int_{\Gamma\times S} a_{il} \mathcal{J}(u,w) f_2(w,l) dwdl) \Big| \\
\leq \ & C \left\| f_1 - f_2 \right\|_{L^1}
\end{aligned}$$

using the Lipschitz condition for F. So we have $II \leq C \left\| f_1 - f_2 \right\|_{L^1}$. ■

We use the following notations in the proof of Theorem 2.2.4 and Theorem 2.2.5.

- $\{\Sigma_t^\gamma\}$ is the stochastic process taking values σ_t with generator L^γ given in equation (2.17) and the sample space $D\left([0,T], S^{\Gamma^\gamma}\right)$.

- $\{\Pi_t^\gamma\}$ is the stochastic process for the empirical measure taking values π_t with the sample space $D([0,T], \mathcal{P}(\Lambda \times S))$ and we denote by $\mathbf{Q}^\gamma$ the law of the process $\{\Pi_t^\gamma\}$ and by $\mathbf{P}$ the probability measure in the underlying probability space. The stochastic process σ_t induces a measure-valued stochastic process $\pi_t^\gamma := \pi_{\sigma_t}^\gamma$ for the empirical given in equation (2.11). The proof of Theorems 2.2.4 and 2.2.5 are so similar that we only prove Theorem 2.2.5.

For $g \in C\left(\Gamma \times S\right)$ we set

$$h\left(\sigma\right) := \langle \pi_\sigma^\gamma, g \rangle = \frac{1}{|\Gamma^\gamma|} \sum_{y \in \Gamma^\gamma} g(\gamma y, \sigma(y)) \tag{2.23}$$

We define $M_t^{g,\gamma}, \langle M_t^{g,\gamma} \rangle$ as follows: for $g \in C(\Gamma \times S)$

$$M_t^{g,\gamma} = \langle \Pi_t^\gamma, g \rangle - \langle \Pi_0^\gamma, g \rangle - \int_0^t L^\gamma \langle \Pi_s^\gamma, g \rangle \, ds, \langle M_t^{g,\gamma} \rangle = \int_0^t \left[L^\gamma \langle \Pi_s^\gamma, g \rangle^2 - 2 \langle \Pi_s^\gamma, g \rangle L^\gamma \langle \Pi_s^\gamma, g \rangle \right] ds \tag{2.24}$$

Since h is measurable, so $M_t^{g,\gamma}$ and $\langle M_t^{g,\gamma}\rangle$ are $\mathcal{F}_t-$martingale with respect to $\mathbf{P}$, where $\mathcal{F}_t$ is the filtration generated by $\{\Sigma_t\}$ (Ethier and Kurtz, 1986; Darling and Norris, 2008).

Lemma 2.4.2 *For $g \in C(\Gamma \times S)$ there exist C such that*

$$|L^\gamma \langle \pi^\gamma, g\rangle| \leq C, \quad \left|L^\gamma \langle \pi^\gamma, g\rangle^2 - 2\langle \pi^\gamma, g\rangle L^\gamma \langle \pi^\gamma, g\rangle\right| \leq \gamma^d C$$

Proof. For h in (2.23),we have

$$h(\sigma^{x,k}) - h(\sigma) = \frac{1}{|\Gamma^\gamma|}\left(g(\gamma x, k) - g(\gamma x, \sigma(x)\right)$$

and so we have equation (2.25) below. Now let $q(\sigma) := \langle \pi_\sigma^\gamma, g\rangle^2$. Then

$$\begin{aligned} q(\sigma^{x,k}) - q(\sigma) &= \frac{1}{|\Gamma^\gamma|^2}(\sum_{y\in\Lambda^\gamma} g(\gamma y, \sigma^{x,k}(y)))^2 - \frac{1}{|\Gamma^\gamma|^2}(\sum_{y\in\Lambda^\gamma} g(\gamma y, \sigma(y)))^2 \\ &= \frac{1}{|\Gamma^\gamma|^2}\left(g(\gamma x, k) - g(\gamma x, \sigma(x))\right)^2 \\ &\quad + \frac{2}{|\Gamma^\gamma|^2}\left(g(\gamma x, k) - g(\gamma x, \sigma(x))\right)\sum_{y\in\Lambda^\gamma} g(\gamma y, \sigma(y)) \end{aligned}$$

Thus we have

$$L^\gamma \langle \pi^\gamma, g\rangle = \frac{1}{|\Gamma^\gamma|}\sum_{k\in S}\sum_{x\in\Lambda^\gamma} c^\gamma(x, \sigma(x), k)\left(g(\gamma x, k) - g(\gamma x, \sigma(x)\right) \tag{2.25}$$

$$L^\gamma \langle \pi^\gamma, g\rangle^2 - 2\langle \pi^\gamma, g\rangle L^\gamma \langle \pi^\gamma, g\rangle = \frac{1}{|\Gamma^\gamma|^2}\sum_{k\in S}\sum_{x\in\Lambda^\gamma} c^\gamma(x, \sigma(x), k)\left(g(\gamma x, k) - g(\gamma x, \sigma(x)\right)^2 \tag{2.26}$$

Therefore from $\mathbf{C1}-\mathbf{C2}$, $|\Gamma^\gamma| \approx |\Gamma|\,\gamma^{-d}$,and $|\Lambda^\gamma| \approx |\Lambda|\,\gamma^{-d}$, the results follow. ■

We proceed by proving a series of lemmas. First we define $M_t^{g,\gamma}, \langle M_t^{g,\gamma}\rangle$ as follows: for $g \in C(\Gamma \times S)$

$$\begin{aligned} M_t^{g,\gamma} &= \langle \Pi_t^\gamma, g\rangle - \langle \Pi_0^\gamma, g\rangle - \int_0^t L_\gamma \langle \Pi_s^\gamma, g\rangle\, ds \\ \langle M_t^{g,n}\rangle &= \int_0^t \left[L_n \langle \Pi_s^n, g\rangle^2 - 2\langle \Pi_s^\gamma, g\rangle L_\gamma \langle \Pi_s^\gamma, g\rangle\right] ds \end{aligned}$$

Then $h : \mathbb{R}_+ \times \Gamma_\gamma \to \mathbb{R}$, $(t, \sigma) \mapsto h(\sigma) := \langle \pi^\gamma, g \rangle = \frac{1}{|\Gamma_\gamma|} \sum_{y \in \Gamma_\gamma} g(\gamma y, \sigma(y))$ satisfies the condition for lemma 2.3.1, so $M_t^{g,\gamma}$ and $\langle M_t^{g,\gamma} \rangle$ are $\mathcal{F}_t-$Martingale with respect to $\mathbf{P}^\gamma$.

Lemma 2.4.3 *Let $g \in C(\Gamma \times S)$. Then we have*

$$L_\gamma \langle \pi^\gamma, g \rangle = \frac{1}{|\Gamma_\gamma|} \sum_{k \in S} \sum_{x \in \Lambda_\gamma} c_\gamma(x, \sigma(x), k) \left(g(\gamma x, k) - g(\gamma x, \sigma(x)) \right) \tag{2.27}$$

$$\begin{aligned} & L_\gamma \langle \pi^\gamma, g \rangle^2 - 2 \langle \pi^\gamma, g \rangle L_\gamma \langle \pi^\gamma, g \rangle \\ = \; & \frac{1}{|\Gamma_\gamma|^2} \sum_{k \in S} \sum_{x \in \Lambda_\gamma} c_\gamma(x, \sigma(x), k) \left(g(\gamma x, k) - g(\gamma x, \sigma(x) \right)^2 \end{aligned} \tag{2.28}$$

Proof. Let $h(\sigma) := \langle \pi^\gamma, g \rangle = \frac{1}{|\Gamma_\gamma|} \sum_{x \in \Gamma_\gamma} g(\gamma x, \sigma(x))$. Then

$$h(\sigma^{x,k}) - h(\sigma) = \frac{1}{|\Gamma_\gamma|} \left(g(\gamma x, k) - g(\gamma x, \sigma(x)) \right)$$

Hence equation (2.27) follows. Now let $h(\sigma) := \langle \pi^\gamma, g \rangle^2$. Then

$$\begin{aligned} h(\sigma^{x,k}) - h(\sigma) &= \frac{1}{|\Gamma_\gamma|^2} \left(\sum_{y \in \Lambda_\gamma} g(\gamma y, \sigma^{x,k}(y)) \right)^2 - \frac{1}{|\Gamma_\gamma|^2} \left(\sum_{y \in \Lambda_\gamma} g(\gamma y, \sigma(y)) \right)^2 \\ &= \frac{1}{|\Gamma_\gamma|^2} \left(g(\gamma x, k) - g\left(\gamma x, \sigma(x)\right) \right) \\ &\quad \times \left(2 \sum_{y \in \Lambda_\gamma} g(\gamma y, \sigma^{x,k}(y)) + g(\gamma x, k) - g(\gamma x, \sigma(x)) \right) \\ &= \frac{1}{|\Gamma_\gamma|^2} \left(g(\gamma x, k) - g(\gamma x, \sigma(x)) \right)^2 \\ &\quad + \frac{2}{|\Gamma_\gamma|^2} \left(g(\gamma x, k) - g(\gamma x, \sigma(x)) \right) \sum_{y \in \Lambda_\gamma} g(\gamma y, \sigma(y)) \end{aligned}$$

Therefore equation (2.28) follows. ■

Proposition 2.4.4 *Let $g \in C(\Gamma \times S)$ and τ^γ and δ^γ such that*

(1) τ^γ is a stopping time on the process $\{\Pi_t^\gamma : 0 \leq t \leq T\}$ with respect to the filtration $\mathcal{F}_t$.

(2) δ^γ is a constant, $0 \le \delta^\gamma \le T$ and $\delta^\gamma \to 0$ as $\gamma \to 0$.

Then for $\epsilon > 0$, there exists C such that

$$\text{(i)}\ \ \mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| \ge \epsilon \right\} \le \frac{\gamma^d C T}{\epsilon^2} \quad \text{and (ii)}\ \ \mathbf{P}\left\{\omega : \left|M^{g,\gamma}_{\tau^\gamma+\delta^\gamma} - M^{g,\gamma}_{\tau^\gamma}\right| \ge \epsilon\right\} \le \frac{\gamma^d C \delta^\gamma}{\epsilon^2}$$

and there exists γ_0 such that for $\gamma < \gamma_0$

$$\text{(iii)}\ \ \mathbf{P}\left\{\omega : \left|\int_{\tau^\gamma}^{\tau^\gamma+\delta^\gamma} L^\gamma \left\langle \Pi_s^\gamma, g\right\rangle ds\right| \ge \epsilon\right\} = 0$$

Proof. We first show (iii). Let C as in Lemma 2.4.2. Since $\delta^\gamma \to 0$, there exists γ_0 such that $\delta^\gamma < \frac{\epsilon}{2C}$ for $\gamma \le \gamma_0$. Then by Lemma 2.4.2

$$\left|\int_{\tau^\gamma}^{\tau^\gamma+\delta^\gamma} L^\gamma \left\langle \Pi_s^\gamma, g\right\rangle ds\right| \le \delta^\gamma C < \frac{\epsilon}{2},\ \text{for } \gamma \le \gamma_0.$$

For (i), let γ be fixed first. Since $(M_0^{g,\gamma})^2 - \langle M_0^{g,\gamma}\rangle = 0, \mathbf{P}$ a.e. and $(M_t^{g,\gamma})^2 - \langle M_t^{g,\gamma}\rangle$ is $\mathcal{F}_t-$martingale, by martingale inequality and Lemma 2.4.2, we have,

$$\mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| > \epsilon \right\} \le \frac{1}{\epsilon^2}\mathbf{E}\left[(M_T^{g,\gamma})^2\right] = \frac{1}{\epsilon^2}\mathbf{E}\left[\langle M_T^{g,\gamma}\rangle\right] \le \frac{\gamma^d C T}{\epsilon^2}$$

For (ii), by Lemma 2.4.2, Chevyshev inequality, and Doob's optional stopping, we have

$$\mathbf{P}\left\{\omega : \left|M^{g,\gamma}_{\tau^\gamma+\delta^\gamma} - M^{g,\gamma}_{\tau^\gamma}\right| > \epsilon\right\} \le \frac{1}{\epsilon^2}\mathbf{E}\left[(M^{g,\gamma}_{\tau^\gamma+\delta^\gamma} - M^{g,\gamma}_{\tau^\gamma})^2\right] = \frac{1}{\epsilon^2}\mathbf{E}\left[\left\langle M^{g,\gamma}_{\tau^\gamma+\delta^\gamma}\right\rangle - \left\langle M^{g,\gamma}_{\tau^\gamma}\right\rangle\right] \le \frac{\gamma^d C \delta^\gamma}{\epsilon^2}$$

■

Next we prove an exponential estimate. We let $r_\theta(x) = e^{\theta|x|} - 1 - \theta|x|$ and $s_\theta(x) = e^{\theta x} - 1 - \theta x$ for x, $\theta \in \mathbb{R}$. We define

$$\phi(\sigma,\theta) := \sum_{k\in S}\sum_{x\in\Lambda^\gamma} c^\gamma(x,\sigma,k) r_\theta(h(\sigma^{x,k}) - h(\sigma)),\ \ \psi(\sigma,\theta) := \sum_{k\in S}\sum_{x\in\Lambda^\gamma} c^\gamma(x,\sigma,k) s_\theta(h(\sigma^{x,k}) - h(\sigma))$$

Then, from Proposition 8.8 in Darling and Norris (2008), we have for $M_T^{g,\gamma}$ in (2.24)

$$Z_t^{g,\gamma} := \exp\left\{\theta M_t^{g,\gamma} - \int_0^t \psi(\Sigma_s^\gamma, \theta) ds\right\}$$

is a supermartingale for $\theta \in \mathbb{R}$. Now we let $C_g := 2\sup|g(u,i)|,\ C_c := \sup|c^\gamma(x,\sigma,k)|$.

Lemma 2.4.5 (Exponential Estimate) *There exist C depends on C_g, C_c, S and ϵ_0 such that for all $\epsilon \leq \epsilon_0$ we have*

$$\mathbf{P}\left\{\sup_{t\leq T} |M_t^{g,\gamma}| \geq \epsilon\right\} \leq 2e^{-\frac{|\Lambda^\gamma|\,\epsilon^2}{TC}}$$

Proof. We choose $\epsilon_0 \leq \frac{1}{2}|S|\,C_g C_c T$ and let $A = \frac{1}{|\Lambda^\gamma|}|S|\,C_g^2 C_c e$, $\theta = \frac{\epsilon}{AT}$. Then since r_θ is increasing in $\mathbb{R}_+$,

$$r_\theta\left(h(\sigma^{x,k}) - h(\sigma)\right) \leq r_\theta\left(\frac{1}{|\Lambda^\gamma|}C_g\right) \leq \frac{1}{2}\left(\frac{1}{|\Lambda^\gamma|}C_g\theta\right)^2 e^{\frac{1}{|\Lambda^\gamma|}\theta C_g} \text{ for all } \sigma \in S^{\Lambda\gamma}$$

where in the last line we used $e^x - 1 - x \leq \frac{1}{2}x^2 e^x$ for all $x > 0$. Also for $\epsilon \leq \epsilon_0$,

$$\frac{1}{|\Lambda^\gamma|}\theta C_g = \frac{1}{|\Lambda^\gamma|}\frac{\epsilon}{AT}C_g \leq \frac{1}{|\Lambda^\gamma|}\frac{1}{2}\frac{|S|\,C_g^2 C_c}{A} \leq \frac{1}{2e} < 1$$

Thus

$$\int_0^T \phi(\Sigma_t^\gamma, \theta)dt \leq |S|\,|\Lambda^\gamma|\,\frac{1}{|\Lambda^\gamma|^2}\frac{1}{2}C_g^2\theta^2 e^{\frac{1}{|\Lambda^\gamma|}\theta C_g}C_c T \leq \frac{1}{2}\frac{1}{|\Lambda^\gamma|}|S|\,C_g^2 C_c e\theta^2 T = \frac{1}{2}A\theta^2 T \quad \text{for all } \omega \in \Omega$$

So, since $\psi(\sigma,\theta) \leq \phi(\sigma,\theta)$,

$$\begin{aligned}
\mathbf{P}\left\{\sup_{t\leq T} M_t^{g,\gamma} > \epsilon\right\} &= \mathbf{P}\left\{\sup_{t\leq T} Z_t^{g,\gamma} > \exp[\theta\epsilon - \int_0^T \psi(\Sigma_t^\gamma,\theta)dt]\right\} \\
&\leq \mathbf{P}\left\{\sup_{t\leq T} Z_t^{g,\gamma} > \exp[\theta\epsilon - \frac{1}{2}A\theta^2 T]\right\} \\
&\leq e^{\frac{1}{2}A\theta^2 T - \theta\epsilon} = e^{-\frac{|\Lambda^\gamma|\,\epsilon^2}{TC}}
\end{aligned}$$

where we choose $C := 2\,|S|\,C_g^2 C_c e$. Since the same inequality holds for $-M_t^{g,\gamma}$, we obtain the desired result.

■

Lemma 2.4.6 (Relative Compactness) *The sequence $\{\mathbf{Q}^\gamma\}$ in $\mathcal{P}\left(D\left([0,T];\mathcal{P}(\Lambda\times S)\right)\right)$ is relatively compact.*

Proof. By Proposition 1.7 in Kipnis and Landim (1999, p.54), we show that $\{\mathbf{Q}^\gamma g^{-1}\}$ is relatively compact in $\mathcal{P}\left(D([0,T];\mathbb{R})\right)$ for each $g \in C(\Lambda \times S)$, where the definition of $\mathbf{Q}^\gamma g^{-1}$ is as follows: for any Borel set A in $D([0,T];\mathbb{R})$

$$\mathbf{Q}^\gamma g^{-1}(A) := \mathbf{Q}^\gamma \left\{\pi. \in D\left([0,T];\mathcal{P}(\Lambda \times S)\right) : \langle \pi., g\rangle \in A\right\}$$

So, from Theorem 1 in Aldous (1978) and Prohorov Theorem in Billingsley (1968, p.125), it is enough to show that

(i) for $\eta > 0$, there exists a such that

$$\mathbf{Q}^\gamma g^{-1}\left\{x \in D\left([0,T];\mathbb{R}\right) : \sup_t |x(t)| > a\right\} \leq \eta \text{ for } \gamma \leq 1$$

(ii)

$$\mathbf{P}\left\{\omega : \left|\left\langle \Pi^\gamma_{\tau^\gamma+\delta^\gamma}, g\right\rangle - \langle \Pi^\gamma_{\tau^\gamma}, g\rangle\right| > \epsilon\right\} \to 0$$

for all $\epsilon > 0$, for $(\tau^\gamma, \delta^\gamma)$ satisfying the condition (1) and (2) of Proposition 2.4.4. For (i), since g is bounded, it is enough to choose $a = 2\sup|g(u,i)|$; i.e., $\mathbf{Q}^\gamma g^{-1}\{x \in D\left([0,T];\mathbb{R}\right) : \sup_t |x(t)| > a\} = \mathbf{Q}^\gamma\{\pi. : \sup_t |\langle \pi_t, g\rangle| > a\} = 0$ since $|\langle \pi., g\rangle| < a$ for all π. For (ii)

$$\begin{aligned}
& \mathbf{P}\left\{\omega : \left|\left\langle \Pi^\gamma_{\tau^\gamma+\delta^\gamma}, g\right\rangle - \langle \Pi^\gamma_{\tau^\gamma}, g\rangle\right| > \epsilon\right\} \\
\leq\ & \mathbf{P}\left\{\omega : \left|M^{g,\gamma}_{\tau^\gamma+\delta^\gamma} - M^{g,\gamma}_{\tau^\gamma}\right| > \frac{\epsilon}{2}\right\} + \mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M^{g,\gamma}_t| > \frac{\epsilon}{2}\right\} \\
\leq\ & \frac{\gamma^d C \delta^\gamma}{\epsilon^2} \quad \text{for } \gamma \leq \gamma_0 \text{ chosen in Proposition 2.4.4}
\end{aligned}$$

■

Let $\mathbf{Q}^*$ be a limit point of $\{\mathbf{Q}^\gamma\}$ and choose a subsequence $\{\mathbf{Q}^{\gamma_k}\}$ converging weakly to $\mathbf{Q}^*$. Hereafter we denote the stochastic process defined on Λ^γ by $\left\{\Sigma^{\Lambda^\gamma}\right\}$

and its restriction on Γ^γ by $\left\{\Sigma^{\Gamma^\gamma}\right\}$. With these notations, equation (2.24) becomes

$$\left\langle \Pi_t^{\Gamma^\gamma}, g\right\rangle = \left\langle \Pi_0^{\Gamma^\gamma}, g\right\rangle + \frac{|\Lambda^\gamma|}{|\Gamma^\gamma|}\int_0^t ds \sum_{k\in S}\int_{\mathbf{\Lambda}\times S} c(u,i,k,\Pi_s^{\Gamma^\gamma})\left(g(u,k)-g(u,i)\right) d\Pi_s^{\Lambda^\gamma}(u,i) + M_t^{g,\gamma} \tag{2.29}$$

Let $\pi \in \mathcal{P}(\Gamma \times S)$ and we define $d\pi_\Lambda := \mathbf{1}_{\Lambda\times S}\ d\pi$.

Lemma 2.4.7 (Characterization of Limit Points) *For all* $\epsilon > 0$,

$$\mathbf{Q}^*\left\{\pi. : \sup_{t\in[0,T]} \langle \pi_t, g\rangle - \langle \pi_0, g\rangle - \int_0^t ds \sum_{k\in S}\left[\int_{\Lambda\times S} c(u,i,k,\pi_s)(g(u,k)-g(u,i))d\pi_{\Lambda,s}\right] > \epsilon\right\} = 0,$$

i.e. the limiting process is concentrated on the weak solutions of the IDE (2.18).

Proof. First we define $\Phi : D([0,T], \mathcal{P}(\Lambda\times S)) \to \mathbb{R}$

$$\pi. \mapsto \left|\sup_{t\in[0,T]} \langle \pi_t, g\rangle - \langle \pi_0, g\rangle - \int_0^t ds \sum_{k\in S}\left[\int_{\Lambda\times S} c(u,i,k,\pi_s)(g(u,k)-g(u,i))d\pi_{\Lambda,s}\right]\right|$$

Then Φ is continuous, hence $\Phi^{-1}((\epsilon,\infty))$ is open. From the weak convergence of $\{\mathbf{Q}^{\gamma_k}\}$ to $\mathbf{Q}^*$,

$$\mathbf{Q}^*\left\{\pi. : \Phi(\pi.) > \epsilon\right\} \le \liminf_{l\to\infty} \mathbf{Q}^{\gamma_l}\left\{\pi. : \Phi(\pi.) > \epsilon\right\}$$

Also,

$$\mathbf{Q}^\gamma\left\{\pi. : \Phi(\pi.) > \epsilon\right\} = \mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| > \epsilon\right\} \le \frac{\gamma^d C T}{\epsilon^2} \text{ (by Proposition 2.4.4) for } \gamma < \gamma_0$$

The first equality follows from (2.29) and the following equality:

$$\Pi_{\Lambda,s} = \frac{1}{|\Gamma^\gamma|}\sum_{x\in\Gamma^\gamma\cap\Lambda} \delta_{(\gamma x, \Sigma_s^{\Gamma^\gamma}(x))} = \frac{1}{|\Gamma^\gamma|}\sum_{x\in\Lambda^\gamma} \delta_{(\gamma x, \Sigma_s^{\Lambda^\gamma}(x))} = \frac{|\Lambda^\gamma|}{|\Gamma^\gamma|}\Pi_s^{\Lambda^\gamma}.$$

■

Lemma 2.4.8 (Absolutely Continuity) *We have*

$$\mathbf{Q}^*\{\pi. : \pi_t \textit{ is absolutely continuous with respect to } m \textit{ for all } t \in [0,T]\} = 1\,.$$

Proof. We define $\Phi : D([0,T]\,;\mathcal{P}(\Gamma\times S))\to\mathbb{R}, \pi_\cdot \mapsto \sup_{t\in[0,T]}|\langle \pi_t, g\rangle|$. Then Φ is continuous. Also

$$|\langle \pi^\gamma, g\rangle| \leq \frac{1}{|\Gamma^\gamma|}\sum_{x\in\Gamma^\gamma}|g(\gamma x,\sigma(x))| \leq \sum_{l\in S}\frac{1}{|\Gamma^\gamma|}\sum_{x\in\Gamma^\gamma}|g(\gamma x,l)|$$

Thus

$$\sup_{t\in[0,T]}|\langle \pi^\gamma_t, g\rangle| \leq \sum_{l\in S}\frac{1}{|\Gamma^\gamma|}\sum_{x\in\Gamma^\gamma}|g(\gamma x,l)|$$

We write $\pi^*_\cdot$ be a trajectory on which all $\mathbf{Q}^*$'s are concentrated. Then $\Pi^\gamma_\cdot \xrightarrow{\mathcal{D}} \pi^*_\cdot$ (convergence in distribution), so $\mathbf{E}\left(\Phi(\Pi^\gamma_\cdot)\right)\to\mathbf{E}\left(\Phi(\pi^*_\cdot)\right)$. Also $\frac{1}{|\Gamma^\gamma|}\sum_{x\in\Gamma^\gamma}|g(\gamma x,l)|\to \int_\Lambda |g(u,l)|\,du$ for all l by the Riemann sum approximation. Thus,

$$\sup_{t\in[0,T]}|\langle \pi^*_t, g\rangle| = \Phi(\pi^*_\cdot) = \lim_{\gamma\to 0}\mathbf{E}\left(\Phi(\Pi^\gamma_\cdot)\right) \leq \lim_{\gamma\to 0}\sum_{l\in S}\frac{1}{|\Gamma^\gamma|}\sum_{x\in\Gamma^\gamma}|g(\gamma x,l)| = \int_{\mathbf{\Gamma}\times S}|g\left(u,l\right)|\,dm(u,i)$$

Therefore, for all $t\in[0,T]$, for all $g\in C(\Gamma\times S)$,

$$\left|\int_{\mathbf{\Gamma}\times S} g(u,l)d\pi^*_t\right| \leq \int_{\mathbf{\Gamma}\times S}|g\left(u,l\right)|\,dm(u,i)$$

so for all $t\in[0,T]$ π^*_t is absolutely continuous with respect to $dm(u,i)$. ■

We also see that all limit points of the sequence $\{\mathbf{Q}^\gamma\}$ are concentrated on the trajectories that equal to $f^0 m$ at time 0, since

$$\begin{aligned}&\mathbf{Q}^*\left\{\pi_\cdot : \left|\int g(u,i)d\pi_0 - \frac{1}{|\Gamma|}\int g(u,i)f^0(u,i)dm(u,i)\right| > \epsilon\right\}\\ \leq\ &\liminf_{k\to\infty}\mathbf{Q}^{\gamma_k}\left\{\pi_\cdot : \left|\int g(u,i)d\pi_0 - \frac{1}{|\Gamma|}\int g(u,i)f^0(u,i)dm(u,i)\right| > \epsilon\right\} = 0,\end{aligned}$$

where the definition of sequence of product measures with a slowly varying parameter implies the last equality by Proposition 0.4 Kipnis and Landim (1999, p.44).

So far we have shown that $\mathbf{Q}^*$'s are concentrated on the trajectories that are the weak solutions of the integro-differential equations. Next we show the uniqueness of weak solutions defined in the following way. Let $\mathcal{A}(f)(u,i) :=$

$\sum_{k\in S}\mathbf{c}(u,k,i,f)f_\Lambda(t,u,k)-f_\Lambda(t,u,i)\sum_{k\in S}\mathbf{c}(u,i,k,f)$. For an initial profile $f^0\in\mathcal{M}$, $f\in\mathcal{M}$ is a weak solution of the Cauchy problem:

$$\frac{\partial f_t}{\partial t}=\mathcal{A}(f_t),\ \ f_0=f^0 \tag{2.30}$$

if for every function $g\in C(\Gamma\times S)$, for all $t<T$, $\langle f_t,g\rangle=\int_0^t\langle\mathcal{A}(f_s),g\rangle\,ds$. Observe that from **C3** $\mathcal{A}$ satisfies the Lipschitz condition: there exists C such that for all $f,\tilde{f}\in L^\infty([0,T];\ L^\infty(\Gamma\times S))$, $\left\|\mathcal{A}(f)-\mathcal{A}(\tilde{f})\right\|_{L^2(\Gamma\times S)}\le C\left\|f-\tilde{f}\right\|_{L^2(\Gamma\times S)}$.

Lemma 2.4.9 (Uniqueness of Weak Solutions) *Weak solutions of the Cauchy problem (2.30) which belong to $L^\infty([0,T];L^2(\Gamma\times S))$ are unique.*

Proof. Let f_t, $\tilde{f}_t$ be two weak solutions and $\bar{f}_t:=f_t-\tilde{f}_t$. Then, we have

$$\left\langle\bar{f}_t,g\right\rangle=\int_0^t\left\langle\mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s),g\right\rangle ds \text{ for all } g\in C(\Gamma\times S)$$

We show that $t\mapsto\left\|\bar{f}_t\right\|^2_{L^2(\Gamma\times S)}$ is differentiable. Define a mollifier $\eta(x):=C\exp\left(\frac{1}{|x|-1}\right)$ if $|x|<1$, $:=0$ if $|x|\ge 1$, $C>0$ is a constant such that $\int_{\mathbb{R}^d}\eta(x)dx=1$. For $\epsilon>0$, set $\eta_\epsilon(x):=\epsilon^{-d}\eta(\epsilon^{-1}x)$. For each $u\in\Gamma$, $i\in S$, define $h^\epsilon_{u,i}(v,k)=\eta_\epsilon(u-v)\,\mathbf{1}_{\{i=k\}}$ and

$$\bar{f}^\epsilon_t(u,i):=\int_{\Gamma\times S}\left(f_t(v,k)-\tilde{f}_t(v,k)\right)h^\epsilon_{u,i}(v,k)dm(v,k)$$

Then,

$$\begin{aligned}\left|\left\langle\mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s),h^\epsilon_{u,i}\right\rangle\right| &\le \left\|\mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s)\right\|_{L^2}\left\|h^\epsilon_{u,i}\right\|_{L^2}\\ &\le C\left\|f_s-\tilde{f}_s\right\|_{L^2}\left\|h^\epsilon_{u,i}\right\|_{L^2}\le C\sup_{s\in[0,T]}\left\|f_s-\tilde{f}_s\right\|_{L^2}\left\|h^\epsilon_{u,i}\right\|_{L^2}.\end{aligned}$$

Since $f_s-\tilde{f}_s\in L^\infty([0,T];L^2(\Gamma\times S))$ and $h^\epsilon_{u,i}\in C(\Gamma\times S)$ for each u,i, $t\mapsto\bar{f}^\epsilon_t(u,i)$ is differentiable and its derivative is $\bar{f}^{\epsilon\prime}_t(u,i)=\left\langle\mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s),h^\epsilon_{u,i}\right\rangle$. Also, it follows that $\left\|\bar{f}_t^\epsilon\right\|_{L^2}^2$ is differentiable with respect to t and

$$\begin{aligned}
\frac{d}{dt}\left\|\bar{f}_t^\epsilon\right\|_{L^2}^2 &= \int_{\Gamma\times S} 2\left\langle \mathcal{A}(f_t)-\mathcal{A}(\tilde{f}_t), h_{u,i}^\epsilon\right\rangle \bar{f}_t^\epsilon(u,i)dm(u,i), \\
\text{so } \left\|\bar{f}_t^\epsilon\right\|_{L^2}^2 &= \int_0^t \left[\int_{\Gamma\times S} 2\left\langle \mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s), h_{u,i}^\epsilon\right\rangle \bar{f}_t^\epsilon(u,i)dm(u,i)\right] ds
\end{aligned}$$

Then since $f_t^\epsilon \to f_t$ in $\| \ \|_{L^2}$ and $\bar{f}_t \in L^\infty\left([0,T];L^2(\Gamma\times S)\right)$ for a given t, we have $|\ \left\|\bar{f}_t^\epsilon\right\|_{L^2}^2 - \left\|\bar{f}_t\right\|_{L^2}^2\ | \to 0$. Also because $\left\langle \mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s), h_{u,i}^\epsilon\right\rangle \to \mathcal{A}(f_t)(u,i) - \mathcal{A}(\tilde{f}_t)(u,i)$ for $a.e.u$, and all i,t, by the dominant convergence theorem we have

$$\left\|\bar{f}_t\right\|_{L^2}^2 = \int_0^t 2\left\langle \mathcal{A}(f_s)-\mathcal{A}(\tilde{f}_s), \bar{f}_s\right\rangle ds,$$

so $\left\|\bar{f}_t\right\|_{L^2}^2$ is differentiable and

$$\frac{d}{dt}\left\|\bar{f}_t\right\|_{L^2}^2 = \left\langle \mathcal{A}(f_t)-\mathcal{A}(\tilde{f}_t), \bar{f}_t\right\rangle \leq 2\left\|\mathcal{A}(f_t)-\mathcal{A}(\tilde{f}_t)\right\|_{L^2}\left\|\bar{f}_t\right\|_{L^2} \leq C\left\|\bar{f}_t\right\|_{L^2}^2$$

Hence from Gronwell lemma, the uniqueness of the solutions follows. ■

Lemma 2.4.10 (Convergence in Probability) *We have*

$$\Pi_t^\gamma \longrightarrow \frac{1}{|\Gamma|} f_t\, m \ \textit{in probability} \ .$$

Proof. So far we established $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^*$ (converge weakly) and equivalently $\Pi_\cdot^\gamma \to \pi_\cdot^*$ in Skorohod topology (topology on $D([0,T],\mathcal{P}(\mathbf{T}^d\times S))$). If we show that $\Pi_t^\gamma \to \pi_t^*$ weakly in $\mathcal{P}\left(\Gamma^d\times S\right)$ or equivalently $\Pi_t^\gamma \xrightarrow{\mathcal{D}} \pi_t^*$ in distribution for fixed time $t<T$, then we have

$$\Pi_t^\gamma \xrightarrow{\mathbf{P}} \pi_t^* \text{ in probability.} \tag{2.31}$$

Since $\Pi_\cdot^\gamma \to \pi_\cdot^*$ in Skorohod topology implies $\Pi_t^\gamma \to \pi_t^*$ weakly for continuity points of $\pi_\cdot^*$ (p.112 Billingsley, 1968), it is enough to show that $\pi_\cdot^* : t \mapsto \pi_t^*$ is continuous for all $t\in[0,T]$ to obtain (2.31). Let $t_0 < T$ and $\{g_k\}$ a dense family in $C(\Gamma\times S)$. Since

$$\left|\int_{t_0}^t \langle A(\pi_s^*), g_k\rangle\, ds\right| \leq (t-t_0) \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle$$

we choose $\delta \leq \min\{1, \epsilon\}$. Then for $|t - t_0| \leq \delta$,

$$\frac{\left|\int_{t_0}^{t} \langle A(\pi_s^*), g_k\rangle \, ds\right|}{1 + \left|\int_{t_0}^{t} \langle A(\pi_s^*), g_k\rangle \, ds\right|} \leq \frac{\delta \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle}{1 + \delta \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle} \leq \delta \frac{\sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle}{1 + \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle} \leq \delta$$

so $\|\pi_t - \pi_{t_0}\|_{\mathcal{P}(\Gamma\times S)} \leq \epsilon$ and $\pi^*_\cdot : t \mapsto \pi_t^*$ is continuous, all $t \in [0, T]$, thus all $t \in [0, T]$ are continuity point of $\pi^*_\cdot$ ■

From Lemma 2.4.10 we have, for $t < T$

$$\Pi_t^{\Lambda\gamma} \xrightarrow{\mathbf{P}} \frac{1}{|\Lambda|} f_{\Lambda,t}\, m$$

So, from (2.20) we obtain

$$\langle f_t, g\rangle = \langle f_0, g\rangle + \int_0^t ds \sum_{k\in S} \int_{\Lambda\times S} c(u, i, k, \frac{1}{|\Gamma|} f_s\, m(u, i)\, (g(u, k) - g(u, i))\, f_{\Lambda,s} dm(u, i)$$

Since $|\Gamma^\gamma|\, \Pi_t^{\Gamma\gamma} = |\Lambda^\gamma|\, \Pi_t^{\Lambda\gamma} + |\Lambda^{\gamma c}|\, \Pi_0^{\Lambda\gamma c}$, $|\Gamma|\, \Pi_t^{\Gamma\gamma} \xrightarrow{\mathbf{P}} f_t\, m$, $|\Lambda|\, \Pi_t^{\Lambda\gamma} \xrightarrow{\mathbf{P}} f_{\Lambda,t}\, m$, and $|\Lambda^c|\, \Pi_0^{\Lambda\gamma c} \xrightarrow{\mathbf{P}} f_{\Lambda^c}\, m$, we have $f_t = f_{\Lambda,t} + f_{\Lambda^c}$ for all t.

2.5 Spatially uniform interactions: Mean-field Dynamics

The goal of this section is to show that under the assumption of uniform interactions the spatially aggregated process is still a Markov chain (such process is called lumpable). Furthermore our IDEs reduce then to the usual ODEs of evolutionary game theory, as it should be. The relationships between the various processes and differential equations are illustrated in Figure 2. Let us take periodic boundary conditions and uniform interactions, i.e., $\mathcal{J} \equiv 1$ on $\mathbf{T}^d$. Let us further define the aggregate variables

$$\eta^\gamma(i) := \frac{1}{|\mathbf{T}^{d,\gamma}|} \sum_{x\in\mathbf{T}^{d,\gamma}} \delta(\sigma(x), i)$$

which counts the proportion of agents with strategy i in the entire domain $\mathbf{T}^{d,\gamma}$. Note that this is obtained, equivalently, by integrating the empirical measure $\pi^\gamma(\sigma)$

$$
\begin{array}{ccc}
\text{Macroscopic Proc. } \eta_t & \xrightarrow[Corollary 4]{approximate} & \text{Mean ODE } \rho_t(i) \\
{\scriptstyle aggregate}\Big\uparrow{\scriptstyle Theorem 2.5.1} & & \Big\uparrow{\scriptstyle aggregate} \\
\text{Microscopic Proc. } \sigma_t & \xrightarrow[Theorem 2.2.4, 2.2.5]{approximate} & \text{Mesoscopic Equation } f_t(u,i)
\end{array}
$$

Figure 2. The relationships between the stochastic processes and the deterministic approximation.

over the spatial domain $\mathbf{T}^d$. We observe that η^γ depends on γ only through the size of the domain n^d i.e., $n^d = \frac{1}{\gamma^d}$ and $n^d \to \infty$ as $\gamma \to 0$. Furthermore since $\mathcal{J} \equiv 1$, the payoff $u(x, \sigma, k)$ depends on σ only through the aggregated variable $\eta^n(i)$. Indeed, we have

$$
u(x, \sigma, k) := \frac{1}{n^d} \sum_{y \in \mathbf{T}^{d,n}} \sum_{l \in S} \delta(\sigma(y), l) a(k, l) = \sum_{i \in S} a(k, i) \eta^n(i)
$$

Thus for the strategy revision rates, if $\sigma(x) = j$ we define

$$
c^M(j, k, \eta^n) := c^\gamma(x, \sigma, k)\,,
$$

since the right hand side is independent of x and depends only on σ through the corresponding aggregate variable η^n. Therefore η^n_t itself is a Markov process as we will show in Theorem 2.5.1 below, and the state space for η^n_t is the discrete simplex

$$
\Delta^n = \left\{ \{\eta(i)\}_{i \in S}\, ;\, \sum_{i \in S} \eta(i) = 1\ ,\, n^d \eta(i) \in \mathbb{N}_+ \right\}
$$

To capture the transition induced by an agent's strategy switching, we write

$$
\eta^{j,k}(i) = \begin{cases} \eta(i) & \text{if } i \neq k, j \\ \eta(i) - \frac{1}{n^d} & \text{if } i = j \\ \eta(i) + \frac{1}{n^d} & \text{if } i = k \end{cases}
$$

Thus $\eta^{j,k}$ is the state obtained from η if one agent switches his strategy from j to k.

Theorem 2.5.1 *Suppose the interaction is uniform, then η^n is a Markov chain with state space Δ^n and generator*

$$L^{M,n} g\,(\eta) = \sum_{k\in S}\sum_{j\in S} n^d \eta^n(j) c(j,k,\eta)(g(\eta^{n,j,k}) - g(\eta^n))\,. \tag{2.32}$$

Proof. First, we define a reduction mapping, $\phi : S^{\Lambda^n} \to \Delta^n$,

$$\sigma \mapsto \phi(\sigma), \quad \phi(\sigma)(i) := \frac{1}{|\Lambda^n|}\sum_{y\in\Lambda^n} \delta_{\sigma(y)}(\{i\})$$

For $g \in L^\infty(\Delta^n;\mathbb{R})$ we let $f := g\circ\phi \in L^\infty(S^{\Lambda^n};\mathbb{R})$, where $f(\sigma) = g(\eta)$. Then for $\eta = \phi(\sigma)$, we have $f(\sigma^{x,k}) - f(\sigma) = g(\eta^{\sigma(x),k}) - g(\eta)$ since

$$\phi(\sigma^{x,k})(i) = \frac{1}{n^d}\sum_{y\in\Lambda_n}\delta_{\sigma(y)}(\{i\}) + \frac{1}{n^d}\delta_k(\{i\}) - \frac{1}{n^d}\delta_{\sigma(x)}(\{i\}) = \eta^{\sigma(x),k}(i)$$

We check the case of imitative and comparing rates. Other cases can be treated as a special case. By writing $m^n(k) := \sum_l a(k,l)\eta^n(l)$, we find

$$\begin{aligned} L_n f(\sigma) &= \sum_{k\in S}\sum_{x\in\Lambda_n} \eta(k)F(m^n(k) - m^n(\sigma(x)))(g(\eta^{\sigma(x),k}) - g(\eta)) \\ &= \sum_{k\in S}\sum_{j\in S}\left[\sum_{x\in\Lambda_n}\delta_{\sigma(x)}(\{j\})\right]\eta(k)F(m^n(k) - m^n(j))(g(\eta^{j,k}) - g(\eta)) \\ &= \sum_{k\in S}\sum_{j\in S} n^d\eta(j)\eta(k)F(m^n(k) - m^n(j))(g(\eta^{j,k}) - g(\eta)) \\ &:= \sum_{k\in S}\sum_{j\in S} n^d c^M(\eta,j,k)(g(\eta^{j,k}) - g(\eta)) \end{aligned}$$

Thus we obtain

$$L^n g(\eta) = \sum_{k\in S}\sum_{j\in S} n^d c^M(\eta,j,k)(g(\eta^{j,k}) - g(\eta))$$

and this makes $\{\eta_t\}$ a Markov chain and the rate is given by $c^M(\eta,j,k)$. ■

The factor n^d in (2.32) comes from the fact that in a time interval of size 1, on average n^d strategy switches take place, and among those, $n^d\eta^n(j)$ are switches from agents with type j. Theorem 2.5.1 shows that the stochastic process with uniform interactions coincides with multi-type birth and death process in population dynamics (Blume, 1998; Benaim and Weibull, 2003). In addition, following Kurtz (1970), Benaim and Weibull (2003), and Darling and Norris (2008), or as a special case of our result (Corollary 2.5.2 below) we can obtain mean field ODEs. Furthermore, at the meso-scopic level, the IDEs reduce to the usual ODEs of evolutionary game theory as follows (See Figure 2). We note that when $\mathcal{J} \equiv 1$, we can define

$$\rho(i) := \int f(u,i)du = \mathcal{J} * f(i)$$

so $\mathbf{c}(u,k,i,f)$ is independent of u and this again allows to define

$$\mathbf{c}^M(k,i,\rho) := \mathbf{c}(u,k,i,f) \tag{2.33}$$

Thus, from the IDE (2.16) we obtain

$$\frac{d\rho_t(i)}{dt} = \sum_{k \in S} \mathbf{c}^M(k,i,\rho)\rho_t(k) - \rho_t(i) \sum_{k \in S} \mathbf{c}^M(i,k,\rho). \tag{2.34}$$

For example, in the case of the comparing and imitative rate we have

$$\mathbf{c}^M(k,i,\rho) = \rho(i) F\left(\sum_{l \in S} a(i,l)\rho(l) - \sum_{l \in S} a(k,l)\rho(l) \right).$$

If $F(s) = \frac{1}{\kappa}\log\left(\exp\left(\kappa s\right) + 1\right)$, then $F(s) - F(-s) = s$ and (2.34) becomes the (imitative) replicator dynamics. Other well-known mean field ODEs, such as logit dynamics and Smith dynamics, are similarly derived by choosing appropriate F. Finally, as a consequence of Theorem 2.2.4 we have the following corollary which is the *continuous-time* version of Benaim and Weibull (2003)'s result. To state the result, we write $\|\eta^n\|_u := \sup_{i \in S} |\eta^n(i)|$.

Corollary 2.5.2 (Uniform Interaction; Benaim and Weibull, 2003) *Suppose that the interaction is uniform and that the strategy revision rate satisfies* $\mathbf{C1} - \mathbf{C3}$.

Suppose there exists $\rho \in \Delta$ *such that the initial condition* η_0^n *satisfies*

$$\lim_{n\to\infty} \eta_0^n \ = \rho \textit{ in probability}$$

Then for every $T > 0$

$$\lim_{n\to\infty} \eta_t^n(i) \longrightarrow \rho_t(i) \textit{ in probability} \tag{2.35}$$

uniformly for $t \in [0, T]$ *and* $\rho_t(i)$ *satisfies the following differential equation: for* $i \in S$

$$\frac{d\rho_t(i)}{dt} = \sum_{k\in S} \mathbf{c}^M(k, i, \rho)\rho_t(k) - \rho_t(i) \sum_{k\in S} \mathbf{c}^M(i, k, \rho) \tag{2.36}$$

$$\rho_0(i) = \rho(i) \tag{2.37}$$

where $\mathbf{c}^M$ *is given by (2.33). Moreover, there exist* C *and* ϵ_0 *such that for all* $\epsilon \le \epsilon_0$, *there exists* n_0 *such that for all* $n \ge n_0$

$$P\left\{\sup_{t\le T} \|\eta_t^n - \rho_t\|_u \ge \epsilon\right\} \le 2\,|S|\,e^{-\frac{n^d \epsilon^2}{TC}}\,. \tag{2.38}$$

Proof. It is enough to prove the exponential estimate. From (2.20) we recall that

$$\langle \Pi_t^\gamma, g\rangle = \langle \Pi_0^\gamma, g\rangle + \int_0^t \sum_{k\in S} \int_{\mathbf{T}^d\times S} c(u, i, k, \Pi_s^\gamma)\,(g(u, k) - g(u, i))\, d\Pi_s^\gamma(u, i) ds + M_t^{g,\gamma}$$

for $g \in C(\mathbf{T}^d \times S)$. By taking $g(u, i) = 1$ if $i = l$, $g(u, i) = 0$ otherwise, we find

$$\eta_{t,l}^n = \eta_{0,l}^n + n^d \int_0^t \left[\sum_{i\in S} c^M(i, l, \eta_s^n)\eta_{s,l}^n - \sum_{k\in S} c^M(l, k, \eta_s^n)\eta_{s,l}^n\right] ds + M_t^{l,n}$$

We define $\beta_l(x) := \sum_{i\in S} \mathbf{c}^M(i, l, x)x_l - \sum_{k\in S} \mathbf{c}^M(l, k, x)x_l$. Thus we have

$$\eta_{t,l}^n = \eta_{0,l}^n + n^d \int_0^t \beta_l(\eta_s^n) ds + M_t^{l,n}, \quad \rho_{t,l} = \rho_{0,l} + \int_0^t \beta_l(\rho_s) ds$$

From Lemma 2.4.5, we have $\mathbf{P}\left\{\sup_{t\le T}\left|M_t^{l,n}\right| \ge \delta\right\} \le 2e^{-n^d \frac{\delta^2}{TC_0}}$ for each l and for $\delta \le \delta_0$, where we note that the choices of C_0 and δ_0 does not depend on g since

$|g(u,i)| \leq 1$ for all u, i. Thus, $\mathbf{P}\left\{\sup_{t\leq T}\|M_t^n\|_u \geq \delta\right\} \leq 2|S|e^{-\frac{n^d\delta^2}{TC_0}}$. Therefore for $t \leq T$, using the Lipschitz condition of β we obtain

$$\sup_{\tau\leq t}\|\eta_\tau^n - \rho_\tau\|_u \leq \|\eta_0^n - \rho_0\|_u + L\int_0^t \sup_{\tau\leq s}\|\eta_\tau^n - \rho_\tau\|_u\, ds + \sup_{t\leq T}\|M_t^n\|_u$$

For ϵ_0 in Lemma 2.4.5, we let $\delta = \frac{1}{3}e^{-LT}\epsilon$ for $\epsilon < \epsilon_0$ and define

$$\Omega_0 = \{\omega : \|\eta_0^n - \rho_0\|_u \leq \delta\},\ \Omega_1 = \left\{\omega : \sup_{t\leq T}\|M_t^n\|_u \leq \delta\right\}$$

Then when $\omega \in \Omega_0 \cap \Omega_1$, we have $\sup_{\tau\leq T}\|\eta_\tau^n - \rho_\tau\|_u \leq 2\delta e^{LT}$ by Gronwell lemma. Choose n_0 such that $\|\eta_0^n - \rho_0\|_u \leq \delta$ for *a.e.* ω for $n \geq n_0$. Then for $\epsilon \leq \epsilon_0$ and $n \geq n_0$,

$$\begin{aligned} P\left\{\sup_{\tau\leq T}\|\eta_\tau^n - \rho_n\| \geq \epsilon\right\} &\leq P(\Omega_0^c) + P(\Omega_1^c) \leq P\{\omega : \|\eta_0^n - \rho_0\|_u \geq \delta\} \\ &\quad + P\left\{\omega : \sup_{t\leq T}\|M_t^n\|_u \geq \delta\right\} \\ &\leq 2|S|e^{-\frac{n^d\delta^2}{TC_0}} = 2|S|e^{-\frac{n^d\epsilon^2}{TC}} \end{aligned}$$

where $C := 9C_0e^{2LT}$. ■

Estimates such as (2.38) describe the validity regimes of the approximation by mean field models (2.36) in terms both of agent number n and the time window $[0, T]$.

2.6 Equilibrium Selection and Pattern Formations

In this section we illustrate the usefulness and the versatility of the IDE's derived in Section 2.2.2 by using a combination of linear analysis and numerical simulations.

(a) Logit/Glauber dynamics: If the rate is given by (2.5) we obtain the IDE

$$\frac{\partial}{\partial t}f_t(u,i) = \frac{\exp\left(\beta\sum_{l\in S}a(i,l)\mathcal{J}*f_t(u,l)\right)}{\sum_{k\in S}\exp\left(\beta\sum_{l\in S}a(k,l)\mathcal{J}*f_t(u,l)\right)} - f_t(u,i)$$

which generalizes the well-known logit ODE of game theory.

(b) Imitative replicator equation: Let us suppose that the rates are given by equation (2.6). Then we obtain

$$\begin{aligned}\frac{\partial}{\partial t} f_t(u,i) \quad = \quad & \sum_{k \in S} \left[f(u,k)\mathcal{J} * f(u,i) F\left(\sum_{l \in S} (a(i,l) - a(k,l))\mathcal{J} * f_t(u,l) \right) \right. \qquad (2.39) \\ & \left. - f(u,i)\mathcal{J} * f(u,k) F\left(\sum_{l \in S} (a(k,l) - a(i,l))\mathcal{J} * f_t(u,l) \right) \right]\end{aligned}$$

Note that the equation depends explicitly on F. This is to be contrasted with the replicator ODE which is independent of F whenever F satisfies the relation $F(t) - F(-t) = t$. This is a purely spatial effect: indeed if we take $f(u,i)$ independent of u for all i then equation (2.39) reduces to the replicator ODE.

2.6.1 Linear stability analysis

As in ODE's, the linearization around stationary solutions captures the local behavior of solutions. For example if all eigenvalues for the linearized system have negative real part then one can show that the stationary solution is a stable stationary solution for the nonlinear equations. Furthermore if the linearization around the stationary solution is hyperbolic, i.e., no eigenvalues has 0 real part then one can analyze the local behavior of the nonlinear equation around the stationary solution by constructing stable and unstable mainfolds.

At a deeper level the linear stability analysis is a first step to understand the generation and propagation of spatial structures. For example traveling wave solutions are constructed by joining two stable spatially homogeneous stationary solutions. Linearization also allows to study bifurcations in the systems, i.e., to identify value of the parameters when the nature of eigenvalues of the linearization changes. The appearance of one or general eigenvalues with positive real part is the sign of an instability in the system. Such instability often makes the appearance of complex spatial structure such as patterns. Several such examples are demonstrated at length in Murray (1989) using mostly numerical tools. We will perform such analysis below and indeed observe the formation of patterns for two strategy coordination games. A rigorous proof of existence of patterns is a challenging problem and will be considered elsewhere.

Let us consider the following general type of integro-differential equations:

$$\begin{cases} \frac{\partial f}{\partial t} = \Phi(\mathcal{J} * f, f) & \text{in } \Lambda \times (0, T] \\ f(0, x) = f^0(x) & \text{on } \Lambda \times \{0\} \end{cases} \tag{2.40}$$

where $\Lambda = \mathbb{R}^d$ or a periodic torus $\mathbf{T}^d$, $f \in L^\infty(\Lambda; \Delta(\mathbb{R}^n))$.$\mathcal{J}*f := (\mathcal{J}*f_1, \mathcal{J}*f_2, \cdots, \mathcal{J} * f_{|S|})^T$, and $\Phi : \mathbb{R}^{|S|} \times \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$, $\Phi(r, s)$ is smooth in both argument, where r and s are variables representing $\mathcal{J} * p$ and p, respectively. First, observe that if f is spatially homogeneous, i.e., $f(u, t) = f(t)$, then $\mathcal{J} * f = f(\mathcal{J} * 1) = f$, where 1 denotes the constant function 1 on Λ. Thus the IDE (2.40) reduces to the ODE

$$\frac{\partial f}{\partial t} = \Phi(f, f) \,.$$

This ODE, in turn, is exactly the ODE obtained if the interactions are uniform $\mathcal{J} \equiv$ const. This shows that the spatially homogenous solutions of (2.40) are exactly the stationary solutions of the corresponding mean-field ODE. In particular every spatially homogenous stationary solution f_0, satisfies $\Phi(f_0, f_0) = 0$. We record this observation in Lemma 2.6.1.

Lemma 2.6.1 (Space Independent Stationary Solutions) *f_0 is a spatially independent stationary solution to (2.40) if and only if $\Phi(f_0, f_0) = 0$.*

Next we study perturbations of such constant states by linearizing around a spatially homogeneous stationary solution, f_0: let $f = f_0 + \epsilon Z$ where $Z = Z(u, i)$

and substituting into (2.40), we obtain

$$\epsilon \frac{\partial D}{\partial t} = \Phi(f_0 + \epsilon \mathcal{J} * Z, f_0 + \epsilon Z). \tag{2.41}$$

For small ϵ we expand the right hand side of equation (2.41) around $\epsilon = 0$, we find

$$\epsilon \frac{\partial D}{\partial t} = \Phi(f_0, f_0) + (M_0 \mathcal{J} * D + N_0 D)\,\epsilon + O(\epsilon^2)$$

where

$$M_0 = \begin{pmatrix} \frac{\partial \Phi_1}{\partial r_1} & \frac{\partial \Phi_1}{\partial r_2} & \cdots & \frac{\partial \Phi_1}{\partial r_n} \\ \frac{\partial \Phi_2}{\partial r_1} & \frac{\partial \Phi_2}{\partial r_2} & \cdots & \frac{\partial \Phi_2}{\partial r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial r_1} & \frac{\partial \Phi_n}{\partial r_2} & \cdots & \frac{\partial \Phi_n}{\partial r_n} \end{pmatrix}, \; N_0 = \begin{pmatrix} \frac{\partial \Phi_1}{\partial s_1} & \frac{\partial \Phi_1}{\partial s_2} & \cdots & \frac{\partial \Phi_1}{\partial s_n} \\ \frac{\partial \Phi_2}{\partial s_1} & \frac{\partial \Phi_2}{\partial s_2} & \cdots & \frac{\partial \Phi_2}{\partial s_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial s_1} & \frac{\partial \Phi_n}{\partial s_2} & \cdots & \frac{\partial \Phi_n}{\partial s_n} \end{pmatrix}$$

and each derivative is evaluated at (f_0, f_0). Therefore we obtain the variational equation system:

$$\frac{\partial D}{\partial t} = M \mathcal{J} * D + ND. \tag{2.42}$$

We solve (2.42) explicitly using Fourier transform. We suppose that $\Lambda = \mathbf{T}^d$ and consider the following initial value problem with periodic boundary condition.

$$\begin{cases} \frac{\partial D}{\partial t} = M \mathcal{J} * D + ND & \text{in } \mathbf{T}^d \times [0, \infty) \\ D = g & \text{on } \mathbf{T}^d \times \{0\} \end{cases} \tag{2.43}$$

for $g \in L^\infty(\mathbf{T}^d; \Delta(\mathbb{R}^n))$. Applying Fourier transform to (2.43) elements by element, we obtain

$$\frac{\partial \hat{D}(k)}{\partial t} = (M \hat{\mathcal{J}}(k) + N) \hat{D}(k)$$

for each $k \in \mathbb{Z}^d$ and $\hat{D}(k) \in \mathbb{C}^n$. By solving the resulting ODE for each k, we obtain

$$\hat{D}(k) = e^{(M \hat{\mathcal{J}}(k) + N)t} \hat{g}(k)$$

where $\hat{g}(k) \in \mathbb{C}^n$. So $D = (\hat{g} e^{(M \hat{\mathcal{J}} + N)t})^\vee$ and we find

$$D(x,t) = \sum_{k \in \mathbb{Z}} e^{(M \hat{\mathcal{J}}(k) + N)t} \hat{g}(k) e^{2\pi i x \cdot k}$$

where $e^{(M\hat{\mathcal{J}}(k)+N)t}$ is $n \times n$ matrix, $\hat{g}(k)$ is $n \times 1$ vector, and $e^{2\pi i x \cdot k}$ is a scalar. Therefore we obtain the dispersion relation:

$$\lambda(k) = \text{eigenvalue}(M\hat{\mathcal{J}}(k) + N) \tag{2.44}$$

We also expect that one of eigenvalues of $M\hat{\mathcal{J}}(k) + N$ is 0 by the invariance of dynamics in the simplex.

2.6.2 Example: Two-strategy symmetric games

We consider two-strategy symmetric games with payoffs (2.2). We call a game the coordination game if $a > c$ and $d > b$ and a game the Hawk and Dove game if $a < c$ and $d < b$. If $p(u) := f(u, 2)$, using that $f(u, 1) + f(u, 2) = 1$ we can write a single equation for $p(u)$ and obtain an equation of the form (2.40) with

$$\textbf{Replicator IDE} \quad \Phi_R(r, s) \ : \ = (1-s)rF_\kappa\left(\alpha(r-\zeta)\right) - s(1-r)F_\kappa\left(\alpha(\zeta-r)\right) \tag{2.45}$$

$$\textbf{Logit IDE} \quad \Phi_L(r, s) \ : \ = l_\beta(\alpha(r-\zeta)) - s \tag{2.46}$$

where $l_\beta\left(t\right) := \frac{1}{1+\exp(-\beta t)}$, and $F_\kappa(t) := \frac{1}{\kappa}\log\left(\exp(\kappa t) + 1\right)$ (recall equation (2.10)).

We refer to (2.45) at $\kappa = \infty$ as a replicator IDE, while we also consider the regularized replicator IDE (2.45) for $\kappa < \infty$, and refer to (2.46) as a logit IDE. We will consider $[-\pi, \pi]^d$ for $d = 1, 2$ as a domain with the periodic boundary condition and $[-1, 1]^d$ for $d = 1, 2$ as a domain with the fixed boundary condition. In addition to the conditions for $\mathcal{J}$ stated in Section 2.3, we assume that $\mathcal{J}$ is symmetric: $\mathcal{J}(x) = \mathcal{J}(-x)$ for $x \in \Lambda$. Frequently, in examples and simulations, we consider localized Gaussian-like kernels $\mathcal{J}(x) \propto \exp(-b\left\|x\right\|^2)$ for some $b > 0$. More specifically, for the case of the periodic boundary, we will use

$$\mathcal{J}(x) = \frac{exp(-b\left\|x\right\|^2)}{\int_{[-\pi,\pi]^d} exp(-b\left\|z\right\|^2)dz} \text{ for } x \in [-\pi, \pi]^d$$

and for the case of the fixed boundary, we will use

$$\mathcal{J}(x) = \begin{cases} \frac{exp(-b\|x\|^2)}{\int_{[-1,1]^d} exp(-b\|z\|^2)dz} & \text{for } x \in [-1,1]^d \\ 0 & \text{otherwise} \end{cases} .$$

Stationary solutions and their linear stability

To find spatially homogenous stationary solutions, we need to set $\Phi_R(p,p) = 0$ and $\Phi_L(p,p) = 0$. Then, for the replicator case $p = 0, 1$, and ζ are three stationary solutions. In the case of the logit dynamics, using $l_\beta(z) = \frac{1}{2} + \frac{1}{2}\tanh(\beta\frac{z}{2})$ and changing the variable, $p \mapsto 2p - 1 := u$, the differential equation becomes

$$\frac{\partial u}{\partial t} = -u + \tanh(\beta\frac{\alpha}{4}(\mathcal{J} * u + 1 - 2\zeta)) \tag{2.47}$$

which is the well-known Glauber meso-scopic equation (DeMasi, Orlandi, Presutti, and Triolo, 1994; Katsoulakis and Souganidis, 1997; Presutti, 2009) with β being the *inverse temperature*. All known results for (2.47), such as the existence of traveling wave solutions in one space dimension and the geometric evolution of interfaces between homogeneous stationary states in higher dimensions, are directly applicable to the logit dynamics. Because of this connection, we have the following characterization of stationary solutions to the logit dynamics; the proof is the consequence of (2.47) and the analysis of Glauber dynamics (Presutti, 2009) or it can easily be done directly.

Lemma 2.6.2 *(1) Suppose that the game is the coordination game. Then, there exists β_C such that for $\beta < \beta_C$ there exists one spatially homogenous stationary solution, p_1, and for $\beta > \beta_C$ there exist three spatially homogenous stationary solutions, p_1, p_2, and p_3.*
(2) Suppose that the game is the Hawk and Dove game. Then there exist a unique spatially homogenous stationary solution.

Next we examine the linear stability of these stationary solutions. By differentiating Φ_R, Φ_L, we find similarly to (2.44) the dispersion relations for the replicator IDE:

$$\begin{array}{ll} p=0 & \lambda_R(k) = F_\kappa\left(-\alpha\zeta\right)\hat{\mathcal{J}}(k) - F_\kappa\left(\alpha\zeta\right) \\ p=1 & \lambda_R(k) = F_\kappa\left(\alpha\left(\zeta-1\right)\right)\hat{\mathcal{J}}(k) - F_\kappa\left(\alpha\left(1-\zeta\right)\right) \\ p=\zeta & \lambda_R\left(k\right) = \left(\frac{\log(2)}{\kappa} + \alpha\zeta\left(1-\zeta\right)\right)\hat{\mathcal{J}}(k) - \frac{\log(2)}{\kappa} \end{array}$$

Table 1. Dispersion relations for the replicator IDE

Note that by our assumptions of $\mathcal{J}$, $\hat{\mathcal{J}}(k)$ is real-valued and $|\hat{\mathcal{J}}(k)| < 1$ for all k. Using this fact, we obtain the first part of Proposition 2.6.3. Since $a-c+b-d<0$ in the Hawk and Dove game, when $\hat{\mathcal{J}}\left(k\right) \geq 0$, $\lambda_R\left(k\right)$ is negative for a sufficiently large κ.

Proposition 2.6.3 (Linear Stability for the Replicator IDE) *(1) $p=0,1$ are linearly stable for the replicator dynamics for coordination games. (2) $p=\zeta$ is linearly stable for the replicator dynamics for Hawk and Dove games when $\hat{\mathcal{J}}\left(k\right) \geq 0$.*

Figure 3 shows one example of the dispersion relations for $p=\zeta$. Observe that $\lambda\left(k\right) > 0$ for $k=0$, ± 1, ± 2 and the solutions to linear equation (2.42) is of the form, $e^{2\pi i k\cdot x}$ (see appendix). So, when $k=0$, the corresponding solution is constant along space and the eigenvalue $\lambda(0)$ is the eigenvalue for the linearized equation of the mean-field ODE (2.36). Thus $\lambda(0) > 0$ merely shows that ζ is unstable in mean-field ODE, and when $k=0$ we do not expect to observe any non-trivial spatial morphologies. At $k=\pm 1$, the corresponding solution has a period 1, involving $\cos(x)$, $\sin(x)$ or both and this solution may grow fast, dominating other solutions with different frequencies. Note that the nonlinearity of the replicator IDE implies a bound on the solutions, so that they remain in the simplex, at each spatial location. An initially fast growing solution may be bounded due to the

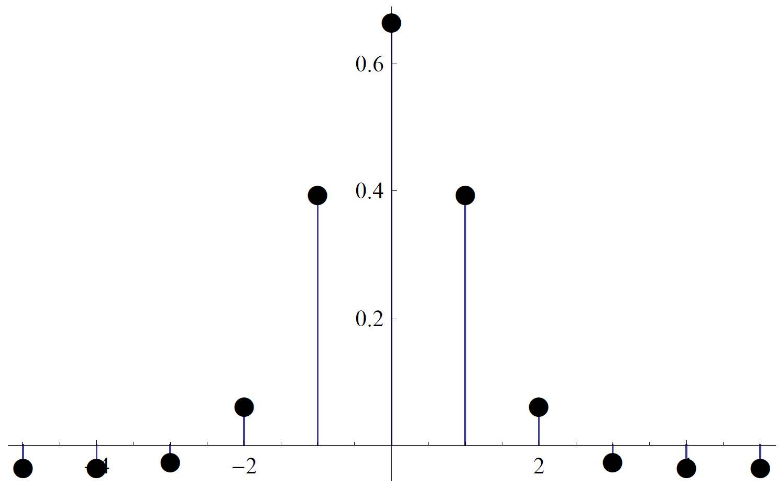


Figure 3. Dispersion relation. **The figure shows the dispersion relation** $\lambda_R(k)$ **at** $p = \zeta$**.** $\mathcal{J}(x) = \exp\left(-bx^2\right) / \int \exp\left(-bx^2\right) dx$**,** $b = 20,\ \kappa = 20,\ \beta = 3, \zeta = \frac{1}{3}$.

nonlinearity effects and, hence, may develop to a spatially heterogeneous solution. This is how we obtain the pattern formation in Figure 1 (upper panels). For $k = \pm 2$, we expect a similar spatial phenomenon, but now the solution involves $\cos(2x)$ or $\sin(2x)$. Hence, we anticipate a finer pattern and, indeed, observe this in the numerical simulation of Figure 1 (lower panels).

In the case of the logit dynamics, we note that $l'_\beta(t) = \beta l_\beta(t)\,(1 - l_\beta(t))$, hence we easily obtain the dispersion relation for any stationary solution, p:

$$\lambda_L(k) = \beta\alpha(1-p)p\hat{\mathcal{J}}(k) - 1, \ \ k \in \mathbb{Z}^d \tag{2.48}$$

Proposition 2.6.4 (Linear Stability for the logit IDEs) *Suppose that* $0 < \hat{\mathcal{J}}(k)$ *for all* k.

(1) Suppose that the game is the coordination game. When $\beta < \beta_C$*, the unique stationary solution* p_0 *is linearly stable. When* $\beta > \beta_C$*, two stationary solutions,* p_1*,* p_3 *are linearly stable where* $p_3 < p_2 < p_1$*.*

(2) If the game is a hawk-dove game, then the unique p_0 *is linearly stable.*

Proof. First we note that $p_1 > \zeta$, $p_2, p_3 < \zeta$,

$$\begin{aligned} \beta(a-c+d-b)(1-l_\beta((a-c+d-b)(p_i-\zeta)))l_\beta((a-c+d-b)(p_i-\zeta)) &< 1 \text{ for } i=1,3, \\ \beta(a-c+d-b)(1-l_\beta((a-c+d-b)(p_i-\zeta)))l_\beta((a-c+d-b)(p_i-\zeta)) &> 1 \text{ for } i=2. \end{aligned}$$

Suppose that $\beta > \beta_C$ and consider p_1. Since $l_\beta((a-c+d-b)(p_1-\zeta)) = p_1$, we have $\beta(a-c+d-b)(1-p_1)p_1 < 1$. Then since $\hat{\mathcal{J}}(k) \leq 1$ for all k, we have

$$\lambda_L(k) = \beta(a-c+d-b)(1-p_1)p_1\hat{\mathcal{J}}(k) - 1 < \beta(a-c+d-b)(1-p_1)p_1 - 1 < 0$$

Thus p_1 is linearly stable. Similar argument shows that p_3 is linearly stable. The case for Hawk-dove games follows from $a-c+d-b<0$ ■

We note that the Gaussian kernel satisfies the hypothesis, $0 < \hat{\mathcal{J}}(k)$ for all k.

2.6.3 Traveling front solutions and equilibrium selection: Imitation versus Perturbed Best Responses

Suppose that the domain is a subset of $\mathbb{R}$ with the fixed boundary conditions or the whole real line $\mathbb{R}$. Then, this provides a natural setting to study traveling front solutions. A solution is called a traveling front or wave solution if it moves at a constant speed: i.e., a traveling front solution $p(x,t)$ can be written as $P(x-ct)$ for some constant c and some function P. The existence of traveling front solutions for the logit dynamics is the direct consequences of known results for the Glauber equations. When $\zeta = \frac{1}{2}$, the existence of a unique standing wave (i.e. $c=0$) was proved and when there are three equilibrium states, the existence of traveling waves was established (DalPasso and DeMottoni, 1991; DeMasi, T.Gobron, and Presutti, 1995; Orlandi and Triolo, 1997). Particularly, if $\zeta < \frac{1}{2}$ one can find a solution that satisfies $P(-\infty) = 0$ and $P(\infty) = 1$, and travels at a negative speed. Thus the value of $P(\infty)$ propagates to the whole real line and as $t \to \infty$, the solution

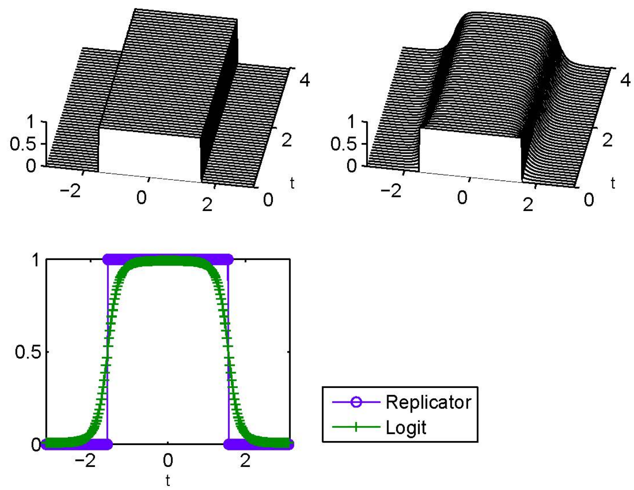


Figure 4. Comparison of standing waves. ($a_{11} = a_{22}$, **Periodic BC). The upper left panel shows the time evolution of the population density of strategy 1 in the replicator dynamic. The upper right panel describes the case of the logit dynamic. The bottom panel shows the shapes of standing waves in both cases at time** 4. **We consider the replicator with** $\kappa = \infty$. $N = 256$. $\Lambda = [-\pi, \pi]$. $dt = 0.001/(0.25N^2)$, $a_{11} = 5$, $a_{22} = 5$, $a_{12} = a_{21} = 0$. $b = 2$. **The initial condition is** $\mathbf{1}_{[-\frac{1}{2}\pi, \frac{1}{2}\pi]}$

becomes 1 everywhere; coordination to a state with the higher payoffs becomes a dominating behavior. However, there is no existing rigorous result, so far, on the replicator IDE, though we have observed this solution in numerical simulations.

To compare the traveling wave solutions for each meso-scopic dynamic, we first study the shapes of the standing waves. This is because the shapes of the standing waves may depend on how "diffusive" the system is and the diffusiveness of the system may, in turn, determine the speed of traveling waves. As in the usual analysis of Allen-Cahn type PDE and Glauber IDE, we believe that the sharpness of the standing wave varies with the diffusion effect of the equations and the more "diffusive" the system is, the faster interfaces move (Carr and Pego, 1989; Katsoulakis and Souganidis, 1997).

As Figure 4 shows, the shape of the standing wave in the replicator dynamics with $\kappa = \infty$ is much sharper than that of the logit dynamics. In other numerical simulations, we have observed that the shape of the regularized replicator dynamics depend on κ; as κ become larger, the shape is getting sharper. Since $F_\kappa(t) \to [t]_+$ as $\kappa \to \infty$, as κ increases marginal gains from switching to a different strategy become higher in response to increases in the payoff of that strategy; in particular, at $\kappa = \infty$, this marginal gain becomes infinity. Thus in the replicator IDEs of high payoffs, there is a zero probability for actions against the optimal choice, hence the interface is very sharp. However, the players in the logit dynamics do not have zero probabilities for doing such an action when an agent is right on the "interface"; i.e., there is a nonzero probability to select something not optimal. That creates the "mushy" mixed region of a transition. From this observation we infer that the logit dynamic is more "diffusive" than the replicator dynamic with $\kappa = \infty$; hence the interfaces in the logit IDEs would move faster than those in the replicator IDEs. This is numerically exhibited in Figure 5.

We note that in the coordination game used for Figure 5, the equilibrium of coordination to strategy 1 is the one predicted by the existing equilibrium selection theories (Harsanyi and Selton, 1988; Young, 1998; Hofbauer, Hutson, and Vickers, 1997; Hofbauer, 1997). Particularly Hofbauer (1997) shows, under the best response dynamics, the existence of a traveling wave solution which drives out the equilibrium of strategy 2, and at the same time propagates the equilibrium of strategy 1. Although we observe the existence of similar traveling wave solutions under various dynamics, the speed of traveling varies dramatically. As Figure 9 shows the transition is extremely slow in the replicator equation with $\kappa = \infty$. So, when the society is characterized by imitative behaviors and marginal gains from switching are high, our model predicts that the transition to a "better equilibrium" is very slow and it takes a long time for equilibrium selection to occur.

Finally we present another comparison between the imitative behavior with the

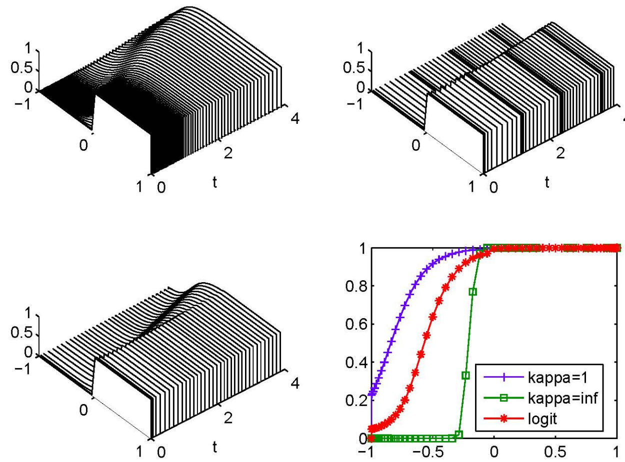


Figure 5. Comparison of traveling waves. **(Fixed BC) The upper panels show the time paths of the population densities for strategy 1 in the replicator with** $\kappa = 1$ **(left) and the one with** $\kappa = \infty$ **(right). The lower left panel shows the case of the logit dynamic. In the bottom right panel we show the shapes of traveling waves at time 4. The initial condition is** $\mathbf{1}_{[0,1]}$. $N = 256$. $\Lambda = [-1, 1]$**,** $dt = 0.001/(0.05N^2)$, $a_{11} = 20/3, a_{12} = a_{21} = 0$. $b = 2$ **for the Gaussian kernel.** $\partial\Lambda = [-3, , 3]$ $\cup[-1, 1]$ **with the fixed boundary condition.**

perturbed best response rule using unequal payoff coordination games ($a_{11} > a_{22}$) with the periodic boundary condition (Figure 6). Observe that the time evolution of the replicator dynamic IDE in the left panel of Figure 6 corresponds to the 1-dimensional snap shot of the pattern formation in two dimensional replicator systems in Figure 1. In Figure 6, the replicator system developed a spatial pattern; in the logit dynamic all population coordinate to an equilibrium of strategy 1 exponentially fast. Thus, in a society where agents adopt strategies by imitating their neighbors, the significant proportion of the population may spend a long time in an inefficient equilibrium, whereas agents with perturbed best response rules coordinate "better" to an efficient outcome.

Throughout numerical simulations, we frequently observed the development of patterns in the replicator IDEs, while this is not the case for the logit IDEs, except

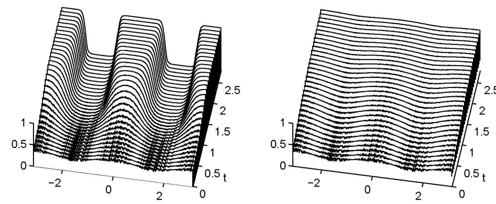


Figure 6. Replicator versus Logit. (Periodic BC) The left panel shows population density of strategy 1 for the replicator IDE with $\kappa = \infty$, and the right panel depicts the population density in the logit dynamics. $N = 512$. $\Lambda = [-\pi, \pi]$ **with the periodic condition**. $dt = 0.001/(0.05N^2)$, $a_{11} = 20/3$, $a_{22} = 10/3$, $a_{12} = a_{21} = 0$. $b = 10$. **for the Gaussian kernel. The initial datum is** $\frac{1}{2} + \frac{1}{10}\,\mathrm{rand}\cos(2x)$,**where rand denotes a realization of the uniform random variable at each node.**

for the equal payoff coordination games. We have also observed the similar pattern formations in the regularized replicator IDEs for a reasonable range of κ; the regularized replicator IDEs with $\kappa = 10$ showed similar patterns to the replicator IDEs.

2.6.4 PDE Approximations

If the interaction kernel $\mathcal{J}$ is highly concentrated at the origin, or equivalently, the density f varies slowly with respect to space, we can consider $\mathcal{J}_\epsilon(x) = \epsilon^{-d}\mathcal{J}(x/\epsilon)$ as an interaction kernel for small ϵ. Then by a change of variables and a Taylor expansion, we find

$$\mathcal{J}_\epsilon * f \approx f + \frac{\epsilon^2}{2} J_2 \Delta f$$

where we ignore smaller order terms like ϵ^3 and $\Delta f = (\Delta f_1, \Delta f_2, \cdots, \Delta f_n)^T$, $\Delta f_1 = \frac{\partial^2 f_1}{\partial r_1^2} + \cdots + \frac{\partial^2 f_1}{\partial r_d^2}$, and $J_2 = \int_\Lambda |w|^2 \mathcal{J}(w) dw$. Thus, by expanding $F(f + \frac{\epsilon^2}{2} J_2 \Delta f, f)$ in equation 2.40 around $\epsilon \approx 0$ again, we find the PDE approximations of IDEs:

$$\frac{\partial f}{\partial t} = F(f, f) + \frac{1}{2}\epsilon^2 J_2 M \Delta f \tag{2.49}$$

where $(M)_{i,j} := \frac{\partial F_i}{\partial r_j}$, and the derivatives are evaluated at (f, f). Intuitively, the coordinating behaviors imply that agents try to choose the same strategy as their neighbors, and this, in turn, means that the density of a given strategy tends to diffuse toward locations where the coordination of that strategy is more likely. This is how our original IDEs are related to the reaction diffusion equations in (2.49). For specific PDE expressions, we find

$$\textbf{Replicator}\quad \frac{\partial f}{\partial t} = \beta f(1-f)(f-\zeta) + \left[\beta f(1-f) + (1-f)F_{\kappa}\left(\beta\left(f-\zeta\right)\right) + fF_{\kappa}\left(\beta\left(\zeta-f\right)\right)\right]\frac{\epsilon^2}{2}J_2\Delta f$$

$$\textbf{Logit}\quad \frac{\partial f}{\partial t} = l(\beta\left(f-\zeta\right)) - f + \beta l(\beta(f-\zeta))(1-l(\beta(f-\zeta)))\frac{\epsilon^2}{2}J_2\Delta f \tag{2.50}$$

Both PDEs in (2.50) are reaction diffusion equations, whose reaction terms are of the same functional form as the mean field reactions (term $\beta f(1-f)(f-\zeta)$ in the replicator and $l(\beta\left(f-\zeta\right))-f$ in the logit). The diffusion terms are *nonlinear* as the coefficients of the terms Δf depend on the strategy density f. In PDE that Hutson and Vickers (1992), Vickers, Hutson, and Budd (1993), Hofbauer, Hutson, and Vickers (1997), and Hofbauer (1997) have studied for the existence of traveling wave solutions and pattern formation, the diffusion coefficients are constant, implicitly modeling 'fast' diffusion of strategies between players at different lattice sites in space at the microscopic level, in contrast to the 'slow' strategy updating dynamics; such derivations of reaction-diffusion PDE from interacting particle systems with combined fast/slow mechanisms are discussed in Durrett (1999) and references therein. However, in our long-range interaction models the diffusion terms are concentration-dependent, induced by the nonlinearities in the logit and replicator microscopic stochastic dynamics. In biology models, when the population pressure tends to enhance dispersal as the population density increases, the density dependent reaction diffusion models have been used (Murray, 1989; Morishita, 1971; Shigesada, 1980).

Overall the PDEs in (2.50) provide additional insights for the IDEs in (2.45) and (2.46), and their corresponding microscopic stochastic dynamics. For example, in the case of (2.45), when p is close to either 0 or 1 the diffusion term is weakest and when p lies in the intermediate range, the effect becomes strong. This means that the individuals playing strategy 1 diffuse fast, as p reaches $\frac{1}{2}$, because it is more likely for them to play with 2-strategists, so more likely to be uncoordinated. When it is highly likely to be coordinated, as in $p = 0$ or 1, the individuals with the corresponding strategy do not diffuse at all.

Remark. Here we derive the numerical scheme that is used in simulations for the equations with fixed boundary conditions (Figure 5). We suppose that $\Lambda = [-1, 1], \Lambda^c = [-3, -1] \cup [1, 3]$ and would like to solve

$$\begin{cases} \frac{\partial p}{\partial t} = F_1(\mathcal{L}(p + p_{\Lambda^c}), \mathcal{L}(q + q_{\Lambda^c}), p, q) & \text{in } \Lambda \times (0, T] \\ \frac{\partial q}{\partial t} = F_1(\mathcal{L}(p + p_{\Lambda^c}), \mathcal{L}(q + q_{\Lambda^c}), p, q) & \\ p(0, x) = p^0(x) & \text{on } \Lambda \times \{0\} \\ q(0, x) = q^0(x) & \\ p(t, x) = p_{\Lambda^c}(x),\ q(t, x) = q_{\Lambda^c}(x) & \text{on } \Lambda^c \times (0, T] \end{cases} \tag{2.51}$$

and we suppose that $p_{\Lambda_C}(x) = \beta_R$ for $x \in [1, 3]$, $p_{\Lambda_C}(x) = \alpha_R$ for $x \in [-3, -1]$, $q_{\Lambda_C}(x) = \beta_C$ for $x \in [1, 3]$ and $q_{\Lambda^c}(x) = \alpha_R$ for $x \in [-3, -1]$ and

$$F_1(r_1, r_2, s_1, s_2) = (1 - s_1)r_1\beta_C(1 - \zeta_C)r_2 - s_1(1 - r_1)\beta_C\zeta_C(1 - r_2)$$
$$F_2(r_1, r_2, s_1, s_2) = (1 - s_2)r_2\beta_R(1 - \zeta_R)r_1 - s_2(1 - r_2)\beta_R\zeta_R(1 - r_1)$$
$$\mathcal{L}p(t, x) := \int_{-1}^{1} \mathcal{J}(x - y)p(t, y)dx,\ \ \mathcal{L}q(t, x) := \int_{-1}^{1} \mathcal{J}(x - y)q(t, y)dx$$

Let $h = \frac{1}{N}$. We approximate solutions for (2.51) by

$$p_h(t, x) = \sum_{j=0}^{N} a_j(t)\phi_j(x), \quad q_h(t, x) = \sum_{j=0}^{N} b_j(t)\phi_j(x) \tag{2.52}$$

where $\phi_j(x)$ is jth order Chebyshev polynomial. So we would like to find $a_j(t)$, $b_j(t)$ such that when we substitute (2.52) into (2.51), the resulting equation is satisfied at $N+1$ collocation points:

$$x_k = \cos(\frac{\pi k}{N}) \text{ for } k = 0, 1, 2, \cdots, N$$

First from the boundary condition we have

$$\begin{aligned} p_h(t,1) &= \beta_R \rightarrow \sum_{j=0}^{N} a_j(t)\phi_j(1) = \beta_R \\ p_h(t,-1) &= \alpha_R \rightarrow \sum_{j=0}^{N} a_j(t)\phi_j(-1) = \alpha_R \end{aligned}$$

Since $\phi_j(1) = 1$ and $\phi_j(-1) = (-1)^j$ for all j, we obtain

$$\begin{aligned} a_0(t) + a_N(t) + \sum_{j=1}^{N-1} a_j(t)\phi_j(1) &= \beta_R \\ a_0(t) - a_N(t) + \sum_{j=1}^{N-1} a_j(t)\phi_j(-1) &= \alpha_R \end{aligned}$$

where we assume that N is odd. Thus we find

$$\begin{aligned} a_0(t) &= \frac{1}{2}(\beta_R + \alpha_R) - \frac{1}{2}\sum_{j=1}^{N-1}(\phi_j(1) + \phi_j(-1))a_j(t) \\ a_N(t) &= \frac{1}{2}(\beta_R - \alpha_R) - \frac{1}{2}\sum_{j=1}^{N-1}(\phi_j(1) - \phi_j(-1))a_j(t) \end{aligned} \tag{2.53}$$

Then we have for $k = 1, \cdots, N-1$

$$\begin{aligned}
p_h(x_k,t) &= \sum_{j=0}^{N} a_j(t)\phi_j(x_k) \\
&= \left[\frac{1}{2}\left(\beta_R + \alpha_R\right) - \frac{1}{2}\sum_{j=1}^{N-1}(\phi_j(1) + \phi_j(-1))a_j(t)\right]\phi_0(x_k) \\
&\quad + \left[\frac{1}{2}\left(\beta_R - \alpha_R\right) - \frac{1}{2}\sum_{j=1}^{N-1}(\phi_j(1) - \phi_j(-1))a_j(t)\right]\phi_N(x_k) \\
&\quad + \sum_{j=1}^{N-1} a_j(t)\phi_j(x_k) \\
&= \frac{1}{2}\left[\left(\beta_R + \alpha_R\right)\phi_0(x_k) + \left(\beta_R - \alpha_R\right)\phi_N(x_k)\right] \\
&\quad + \sum_{j=1}^{N-1} a_j(t)\left(\phi_j(x_k) - \frac{1}{2}\left[(1 + (-1)^j)\phi_0(x_k) + (1 - (-1)^j)\phi_N(x_k)\right]\right)
\end{aligned}$$

So we can write in the vector notation

$$\vec{p}_h(t) = \frac{1}{2}\vec{c}_R + M\vec{a}(t)$$

where $(\vec{p}_h)_k := p_h(x_k,t)$, $(\vec{c}_R)_k := (\beta_R + \alpha_R)\,\phi_0(x_k) + (\beta_R - \alpha_R)\,\phi_N(x_k)$, $(M)_{k,j} = \phi_j(x_k) - \frac{1}{2}\left[(1 + (-1)^j)\phi_0(x_k) + (1 - (-1)^j)\phi_N(x_k)\right]$, $(\vec{a})_k := a_k$. From this we have

$$\frac{d\vec{p}_h}{dt}(t) = M\vec{a}'(t)$$

Similarly we have

$$\vec{q}_h(t) = \frac{1}{2}\vec{c}_C + M\vec{b}(t),\ \ \frac{d\vec{q}_h}{dt}(t) = M\vec{b}'(t)$$

where $(\vec{c}_C)_k := (\beta_C + \alpha_C)\,\phi_0(x_k) + (\beta_C - \alpha_C)\,\phi_N(x_k)$ Also for $k = 1, \cdots, N-1$,

$$\begin{aligned}
(\mathcal{L}(p_h + p_{\Lambda^c}))\,(t, x_k) &= \int_\Lambda \mathcal{J}(x_k - y) p_h(t, y) dy + \int_{\Lambda^c} \mathcal{J}(x_k - y) p_{\Lambda^c}(y) dy \\
&= \sum_{j=0}^{N} a_j(t) \int_\Lambda \mathcal{J}(x_k - y)\phi_j(y) dy \\
&= \left(\frac{1}{2}(\beta_R + \alpha_R) - \frac{1}{2}\sum_{j=1}^{N-1}(\phi_j(1) + \phi_j(-1)) a_j(t) \right) a_{k,0} \\
&\quad + \left(\frac{1}{2}(\beta_R - \alpha_R) - \frac{1}{2}\sum_{j=1}^{N-1}(\phi_j(1) - \phi_j(-1)) a_j(t) \right) a_{k,N} + \sum_{j=1}^{N-1} a_j(t) a_{k,j} \\
&\quad + \int_{\Lambda^c} \mathcal{J}(x_k - y) p_{\Lambda^c}(y) dy \\
&= \frac{1}{2}\left[(\beta_R + \alpha_R)\, a_{k,0} + (\beta_R - \alpha_R)\, a_{k,N} \right] \\
&\quad + \sum_{j=1}^{N-1} a_j(t) \left(a_{k,j} - \frac{1}{2}\left[(1 + (-1)^j) a_{k,0} + (1 - (-1)^j) a_{k,N} \right] \right) \\
&\quad + \int_{\Lambda^c} \mathcal{J}(x_k - y) p_{\Lambda^c}(y) dy
\end{aligned}$$

where $a_{k,j} := \int_\Lambda \mathcal{J}(x_k - y)\phi_j(y) dy$. Therefore we can write

$$\overrightarrow{\mathcal{L}p_h} = \frac{1}{2}\vec{d}_R + N\vec{a}(t)$$

where $\left(\overrightarrow{\mathcal{L}p_h}\right)_k := (\mathcal{L}p_h)\,(x_k)$, $(\vec{d}_R)_k = (\beta_R + \alpha_R)\, a_{k,0} + (\beta_R - \alpha_R)\, a_{k,N} + 2\int_{\Lambda^c} \mathcal{J}(x_k - y) p_{\Lambda^c}(y) dy$, $(N)_{k,j} := a_{k,j} - \frac{1}{2}\left[(1 + (-1)^j) a_{k,0} + (1 - (-1)^j) a_{k,N} \right]$. Also from the initial condition we have $p_h(x_k, 0) = p^0\,(x_k)\,,\ k = 0, \cdots, N$. So $\frac{1}{2}\vec{c}_R + M\vec{a}(0) = \vec{p}^0$. Again the similar computation shows

$$\overrightarrow{\mathcal{L}q_h} = \frac{1}{2}\vec{d}_C + N\vec{b}(t), \quad \frac{1}{2}\vec{c}_C + M\vec{b}(0) = \vec{q}^0$$

where $(\vec{d}_C)_k = \left[(\beta_C + \alpha_C)\, a_{k,0} + (\beta_C - \alpha_C)\, a_{k,N} + 2\int_{\Lambda^c} \mathcal{J}(x_k - y) q_{\Lambda^c}(y) dy \right]$. Therefore our numerical scheme for (2.51) yields ODE:

$$\begin{cases} M\vec{a}'(t) = \vec{F}_1(\frac{1}{2}\vec{d}_R + N\vec{a}(t), \frac{1}{2}\vec{d}_C + N\vec{b}(t), \frac{1}{2}\vec{c}_R + M\vec{a}(t), \frac{1}{2}\vec{c}_C + M\vec{b}(t)) \\ M\vec{b}'(t) = \vec{F}_1(\frac{1}{2}\vec{d}_R + N\vec{a}(t), \frac{1}{2}\vec{d}_C + N\vec{b}(t), \frac{1}{2}\vec{c}_R + M\vec{a}(t), \frac{1}{2}\vec{c}_C + M\vec{b}(t)) \\ M\vec{a}(0) = \vec{p}^0 - \frac{1}{2}\vec{c}_R \\ M\vec{b}(0) = \vec{q}^0 - \frac{1}{2}\vec{c}_C \end{cases} \tag{2.54}$$

where $\left(\vec{F}_i(\vec{r}_1, \vec{r}_2, \vec{s}_1, \vec{s}_2)\right)_k := F_i(r_{1,k}, r_{2,k}, s_{1,k}, s_{2,k})$. We solve (2.54) and obtain $(a_1(t), \cdots, a_{N-1}(t))$ and $(b_1(t), \cdots, b_{N-1}(t))$. Then we can find $a_0(t)$, $a_N(t), b_0(t)$, $b_N(t)$ by using (2.53). Finally from (2.52) we obtain the numerical solution of (2.51).

CHAPTER 3

DECOMPOSITION OF NORMAL FORM GAMES: POTENTIAL, ZERO-SUM, AND STABLE GAMES

3.1 Decompositions of the Space of Games into Orthogonal Subspaces

3.1.1 Reduction of Games Modulo Payoff Transformation.

To illustrate the idea of our first decomposition, we decompose the well-known generalized Rock-paper-scissors game by performing a simple calculation.

$$\begin{pmatrix} \gamma_1 & -a+\gamma_2 & b+\gamma_3 \\ b+\gamma_1 & 0+\gamma_2 & -a+\gamma_3 \\ -a+\gamma_1 & b+\gamma_2 & \gamma_3 \end{pmatrix} \tag{3.1}$$

$$= \underbrace{\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}}_{\text{Passive Game}} + \frac{1}{2}(b-a)\underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{\text{Potential Part}} + \frac{1}{2}(b+a)\underbrace{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}}_{\text{Anti-potential Part}} \tag{3.2}$$

It is easy to see that the game (3.1) is a potential game if and only if $a = -b$ and is equivalent to the Rock-paper-scissors game if and only if $a = b$. In this section we show that such a decomposition as in (3.2) holds for any game.

We start with symmetric games: let us denote the space of all $l \times l$ matrices by $\mathcal{L}$. Let us endow $\mathcal{L}$ with the inner product, $\langle A, B \rangle_{\mathcal{L}} = \mathrm{tr}(A^T B)$. A *passive game* (in the terminology of Sandholm (2010b)) is a game in which players' payoffs do not depend on the choice of strategies. Let $E_\gamma^{(j)} \in \mathcal{L}$ be the matrix given by

$$E_\gamma^{(j)}(k,l) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

; i.e., $E_\gamma^{(j)}$ is a matrix which has 1's in its jth column and 0's at all other entries. Then the set of all symmetric passive games is given by $\mathcal{I} := \mathrm{span}\{E_\gamma^{(i)}\}_{j.}$ It is well-known that the set of Nash equilibria for a symmetric game is left invariant under the addition of a passive game to the payoff matrix.

3.1.2 Potential games and zero-sum games decompositions

To characterize the spaces of all potential games and all zero-sum games, we define the following special matrices:

$$K^{(ij)} = \begin{array}{r|ccc|} & i\text{-th} & & j\text{-th} \\ \hline i\text{-th} \rightarrow & -1 & \cdots & 1 \\ & \vdots & & \vdots \\ j\text{-th} \rightarrow & 1 & \cdots & -1 \\ \hline \end{array}\,, \quad N^{(ij)} = \begin{array}{r|ccccc|} & 1\text{st} & & i\text{-th} & & j\text{-th} \\ \hline 1\text{st} & 0 & \cdots & -1 & \cdots & 1 \\ & \vdots & & \vdots & & \vdots \\ i\text{-th} \rightarrow & 1 & \cdots & 0 & \cdots & -1 \\ & \vdots & & \vdots & & \vdots \\ j\text{-th} \rightarrow & -1 & \cdots & 1 & \cdots & 0 \\ \hline \end{array}\,,$$

where all other elements in the matrices are zeros. Similarly, $N^{(ij)}$ is a game whose restriction on the strategy set $\{1, i, j\} \times \{1, i, j\}$ is the Rock-Paper-Scissors game. Monderer and Shapley (1996)

Recall that a symmetric game A is a *potential game* (Monderer and Shapley (1996)) if there exist a symmetric matrix S and a passive game $\sum_j \gamma_j E_\gamma^{(j)} \in \mathcal{I}$ such that

$$A = S + \sum_j \gamma_j E_\gamma^{(j)} \,. \tag{3.3}$$

We will use the word "exact " to indicate that a game is a potential game with no passive part, i.e., all $\gamma_j = 0$, so exact potential games refers to full potential games in Sandholm (2010b). We denote by $\mathcal{M}$ the linear subspace of all potential games and we have the orthogonal decomposition $\mathcal{L} = \mathcal{M} \oplus \mathcal{M}^\perp$ with respect to the inner product $<,>_\mathcal{L}$. We call a game in $\mathcal{M}^\perp$ an *anti-potential game.*

Note that the dimension of the subspace of $\mathcal{L}$ consisting of all symmetric matrices is $\frac{1}{2}l(l+1)$ and the dimension of the subspace of passive games is l. Since the sum of all E_γ is an exact potential game, namely the game whose payoffs are all 1's, the dimension of the intersection between the subspace of all symmetric matrices and $\mathcal{I}$ is at least 1. Conversely if a matrix belongs to this intersection, then the entries of this matrices should be all the same (see also the discussion in Sandholm (2010a, p.15)) and so the dimension of the intersection is exactly 1. Hence the dimension of $\mathcal{M}$ is given by

$$\dim(\mathcal{M}) = \frac{l(l+1)}{2} + l - 1 = l^2 - \frac{(l-1)(l-2)}{2}. \tag{3.4}$$

Note that the extended Rock-Paper-Scissors $N^{(ij)}$ is an anti-symmetric matrix whose column sums and row sums are all 0's. Thus, we have

$$\left\langle A, N^{(ij)} \right\rangle_\mathcal{L} = 0 \ ,$$

for all $A \in \mathcal{M}$, because $\left\langle S, N^{(ij)} \right\rangle_\mathcal{L} = 0$ and $\left\langle P, N^{(ij)} \right\rangle_\mathcal{L} = 0$ for all symmetric matrix S and all passive game P (See the appendix for the properties of $\langle \rangle_\mathcal{L}$). In other words, $N^{(ij)} \in \mathcal{M}^\perp$ for all i, j. The set $\{N^{(ij)} : j > i, \ i = 2, \cdots, l-1\}$

has $\frac{(l-1)(l-2)}{2}$ elements and they are linearly independent since since each $N^{(ij)}$ is uniquely determined by the property of having 1 in its (i, j) th position. This set forms a basis for $\mathcal{M}^{\perp}$. If a matrix B is antisymmetric and the sums of elements in each column in B are all zeros, $\langle S, B\rangle_{\mathcal{L}} = 0$ for a symmetric matrix and $\langle P, B\rangle_{\mathcal{L}} = 0$ for a passive game P. Therefore $B \in \mathcal{M}^{\perp}$. On the other hand, if $B \in \mathcal{M}^{\perp}$, B can be written as a linear combination of $N^{(ij)}$, and hence B is antisymmetric and the sums of elements in each column in B are all zeros.

Proposition 3.1.1 (Anti-potential games) *We have*

$$B \in \mathcal{M}^{\perp} \text{ if and only if } B^T = -B \text{ and } \sum_j B(i,j) = \sum_i B(i,j) = 0\,.$$

Moreover the set $\{N^{(ij)} : j > i,\, i = 2, \cdots, l\}$ *forms a basis for* $\mathcal{M}^{\perp}$.

Proposition 3.1.1 shows that a basis for $\mathcal{M}^{\perp}$ can be obtained from the extended Rock-Paper-Scissors. As a corollary of Proposition 3.1.1 we obtain immediately the criterion for potential games given by Hofbauer and Sigmund (1998).

Corollary 3.1.2 (Potential games) *A is a potential game if and only if*

$$a(l,m) - a(k,m) + a(k,l) - a(m,l) + a(m,k) - a(l,k) = 0 \quad \text{for all } l, m, k \in S \tag{3.5}$$

Proof. First note from Proposition 3.1.1 that we have A is a potential game if and only if $\langle A, N^{(ij)}\rangle_{\mathcal{L}} = 0$ for all i, j. Then notice that

$$a(l,m) - a(k,m) + a(k,l) - a(m,l) + a(m,k) - a(l,k) = \langle A, E\rangle_{\mathcal{L}}$$

where

$$E = \begin{array}{c|ccc|} & k & l & m \\ \hline k & 0 & 1 & -1 \\ l & -1 & 0 & 1 \\ m & 1 & -1 & 0 \\ \hline \end{array}$$

and all other entries in E are 0's.

Then clearly (3.5) implies $\left\langle A, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ for all i, j. Conversely, the matrix E is anti-symmetric and its row sums and column sums are zero, so $E \in \mathcal{M}^{\perp}$. Therefore E can be uniquely written as $N^{(ij)}$ and thus $\left\langle A, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ for all i, j implies (3.5).

■

We provide a similar decomposition starting with zero-sum games. We call an anti-symmetric matrix A an *exact zero-sum game* and call a game *zero-sum* if it can be written as the sum of a antisymmetric matrix and a passive game. Let us denote by $\mathcal{N}$ the subspace of all zero-sum games. The dimension of the subspace all anti-symmetric matrices is $\frac{(l-1)l}{2}$ and the dimension of the intersection between the subspace of anti-symmetric matrices and $\mathcal{I}$ is 0 (the diagonal elements of anti-symmetric matrices are all zeros and hence all off diagonal elements are again all zeros if this game is also a passive game). Thus

$$\dim(\mathcal{N}) = \frac{(l-1)l}{2} + l = l^2 - \frac{(l-1)l}{2}. \tag{3.6}$$

We decompose the space of game as $\mathcal{L} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ and we call a game in $\mathcal{N}^{\perp}$ an *anti-zero-sum* game. Note that $K^{(ij)}$ is a symmetric matrix whose row sums and column sums are zeros, so $K^{(ij)} \in \mathcal{N}^{\perp}$. The set $\{K^{(ij)} : j > i, \ i = 1, \cdots, l\}$ has $\frac{(l-1)l}{2}$ elements which are linearly independent since early independent since each $K^{(ij)}$ is uniquely determined by having 1 in its (i, j)th entry. Thus we obtain

Proposition 3.1.3 (Anti-zero-sum games) *We have*

$$B \in \mathcal{N}^{\perp} \quad \textit{if and only if} \quad B^T = B \text{ and } \sum_j B(i,j) = \sum_i B(i,j) = 0\,.$$

Moreover the set $\{K^{(ij)} : j > i, \, i = 1, \cdots, l-1\}$ *forms a basis for* $\mathcal{N}^{\perp}$.

Using this orthogonal decomposition we obtain a new criterion to identify a zero-sum game similar to the criterion in Corollary 3.1.2.

Corollary 3.1.4 (Zero-sum games) *A is a zero-sum game if and only if*

$$a(j,i) - a(i,i) + a(i,j) - a(j,j) = 0 \text{ for all } i,j \in S\,. \tag{3.7}$$

Proof. If $A \in \mathcal{N}$ then $\left\langle A, K^{(ij)} \right\rangle_{\mathcal{L}} = 0$ which yields (3.7). ■

3.1.3 Decompositions using the projection mapping Γ

The subspaces of potential games and zero-sum games have a non-trivial intersection $\mathcal{M} \cap \mathcal{N}$. In order to understand this set let $P = I - \frac{1}{l}\mathbf{1}\mathbf{1}^T$ where I is the identity matrix and $\mathbf{1}$ the constant vector with entries equal to 1. It is easy to see that P is the orthogonal projection onto the subspace $\{x \in \mathbb{R}^l\,; \sum_i x_i = 0\}$, i.e., onto the tangent space to the unit simplex $\{x \in \mathbb{R}^l\,; x_i \geq 0\,, \sum_i x_i = 1\}$. Let us define a linear transformation Γ on $\mathcal{L}$ by

$$\Gamma : \mathcal{L} \to \mathcal{L},\ A \mapsto PAP.$$

To characterize the kernel and the range of the map Γ, let us say that a game is *constant game* if the player's payoff does not depend on his opponent's strategy, that is the payoff matrix is constant on each row. The matrices $E_\eta^{(i)} := (E_\gamma^{(i)})^T$ form an orthonormal basis of the subspace of constant games. Note that $E_\eta^{(i)}$ has a strictly dominant strategy.

Furthermore let us define for each $i \in \{2, \cdots, l_r\}$, $j \in \{2, \cdots, l_c\}$

$$E_\kappa^{(ij)} = \begin{array}{r} \\ \\ i\text{-th} \\ i+1\text{-th} \\ \\ \end{array} \begin{array}{cccccc} & & j\text{-th} & j+1\text{-th} & & \\ \hline & & \vdots & \vdots & & \\ & \cdots & -1 & 1 & \cdots & \\ & \cdots & 1 & -1 & \cdots & \\ & & \vdots & \vdots & & \end{array}$$

where all other entries are 0's.

It is easy to see that

$$\text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\} \subset \ker\Gamma. \tag{3.8}$$

Conversely, one can show that the left and right actions of the projection matrices makes only this class belongs to $\ker\Gamma$. Then note that

$$\sum_i E_\gamma^{(l)} = \sum_i E_\eta^{(l)},$$

so by throwing away one element from the spanning set (3.8), we may obtain the independent spanning set, hence a basis for the kernel of Γ. Concerning the range of Γ, by counting the basis elements, we have $\dim(\ker\Gamma) = 2l-1$ and, thus, $\dim(\text{range}\Gamma) = l^2 - (2l-1) = (l-1)^2$. Since $\mathbf{1}E_\kappa^{(ij)} = \mathbf{0}$ and $E_\kappa^{(ij)}\mathbf{1} = \mathbf{0}$,

$$\{E_\kappa^{(ij)} : i = 1, \cdots, l-1, j = 1, \cdots l-1\}$$

provides a natural candidate for the basis of the range. These observations lead to Proposition 3.1.5 whose formal proof is elementary but tedious.

Proposition 3.1.5 (Characterizations of $\ker(\Gamma)$ and range(Γ)) *We have*
(1) $\{E_\eta^{(i)}\}_{i\neq 1} \cup \{E_\gamma^{(j)}\}_j$ *form a basis for* $\ker\Gamma$.
(2) $\{E_\kappa^{(ij)} : i = 1, \cdots, l-1, j = 1, \cdots l-1\}$ *form a basis for range*(Γ).

Proof. (1) We first show that

$$\ker\Gamma = \text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$$

Note that $PE_\gamma^{(j)} = O$ for all j. Then $E_\eta^{(i)}P = (P(E_\eta^{(i)})^T)^T = O$ for all i. Thus we have span $\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\} \subset \ker\Gamma$. Conversely, let A such that $\Gamma(A) = O$. Since

$$PAP = A - \frac{1}{l}\mathbf{1}\mathbf{1}^T A - \frac{1}{l}A\mathbf{1}\mathbf{1}^T + \frac{1}{l^2}\mathbf{1}\mathbf{1}^T A\mathbf{1}\mathbf{1}^T,$$

we have

$$A = \frac{1}{l}\mathbf{1}\mathbf{1}^T A + \frac{1}{l} A\mathbf{1}\mathbf{1}^T - \frac{1}{l^2}\mathbf{1}\mathbf{1}^T A\mathbf{1}\mathbf{1}^T$$

Then note the following properties of $\mathbf{1}\mathbf{1}^T$:

$$\mathbf{1}\mathbf{1}^T A = (\sum_k a_{k1}\mathbf{1} : \sum_k a_{k2}\mathbf{1} : \cdots : \sum_k a_{kl}\mathbf{1})$$

i.e., the left action of $\mathbf{1}\mathbf{1}^T$ on A turns A into a matrix with the same elements in each column. Since $A\mathbf{1}\mathbf{1}^T = (\mathbf{1}\mathbf{1}^T A^T)^T$, the right action of $\mathbf{1}\mathbf{1}^T$ on A turns A into a matrix with same elements in each column. Also it is easy to see that $\mathbf{1}\mathbf{1}^T A\mathbf{1}\mathbf{1}^T = \sum_k \sum_m a_{km}\mathbf{1}\mathbf{1}^T$. Thus A can be written as

$$A = \sum_i (\sum_k a_{ki}) E_\gamma^{(i)} + \sum_j (\sum_k a_{jk}) E_\eta^{(j)} + (\sum_k \sum_m a_{km}) \sum_j E_\gamma^{(j)}$$

So $A \in \text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$. Thus $\ker \Gamma = \text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$. Next note that

$$\sum_i E_\eta^{(i)} = \sum_j E_\gamma^{(j)}, \quad \text{so } E_\gamma^{(1)} = -\sum_{j \neq 1} E_\gamma^{(j)} - \sum_i E_\eta^{(i)},$$

thus

$$\text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\} = \text{span}\{E_\eta^{(2)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}.$$

To show linear independency among $\{E_\eta^{(2)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$, consider the linear combination of these matrices:

$$O = \sum_{i \neq 2} \eta_i E_\eta^{(2)} + \sum_j \gamma_j E_j^{(2)}.$$

Then since $E_\eta^{(1)}$ does not appear in the linear combination, we have $\gamma_j = 0$ for all j and this implies $\eta_i = 0$ for $i \neq 2$.

(2) Note because of $\mathbf{1}E_\kappa^{(ij)} = \mathbf{0}$ and $E_\kappa^{(ij)}\mathbf{1} = \mathbf{0}, \Gamma(E_\kappa^{(ij)}) = PE_\kappa^{(ij)}P = E_\kappa^{(ij)}$. So,

$$E_\kappa^{(ij)} \subset \text{range}(\Gamma) \quad \text{for all } i, j \geq 2$$

and it is easy to see that $E_\kappa^{(ij)}$ are linearly independent. Finally by $\left|\{E_\kappa^{(ij)}\}_{ij}\right| = (l-1)^2$ and since $\dim(\text{range}(\Gamma)) = (l-1)^2$, $\{E_\kappa^{(ij)}\}_{ij}$ is a basis for range(Γ) ■

Next, we study the relationship among these subspaces. First every game in the subspace $\mathcal{N}^\perp$ is a symmetric matrix and thus a potential game. Similarly every anti-potential game is a zero-sum game, so we have $\mathcal{N}^\perp \subset \mathcal{M}$ and $\mathcal{M}^\perp \subset \mathcal{N}$. To understand the relationship among these spaces further, note the following facts:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

From this clearly any game in ker(Γ) which is not a passive game is both a potential game and zero-sum game; i.e., every constant game can be transformed into potential games and zero-sum games. As Proposition 3.1.6 illustrates, the direction of implication goes the other as well: the games which are both anti-potential and anti-zero-sum games are equivalent to a constant game.

Proposition 3.1.6 $\ker(\Gamma) = \mathcal{M} \cap \mathcal{N}$ *and range*$(\Gamma) = \mathcal{M}^\perp \oplus \mathcal{N}^\perp$.

Proof. First we show that $\ker(\Gamma) = \mathcal{M} \cap \mathcal{N}$. Observe that for $i \geq 2$

$$(E_\eta^{(i)} + E_\gamma^{(i)})^T = (E_\eta^{(i)})^T + (E_\gamma^{(i)})^T = E_\gamma^{(i)} + E_\eta^{(i)}$$

Thus $(E_\eta^{(i)} + E_\gamma^{(i)})$ is symmetric and $(E_\eta^{(i)} + E_\gamma^{(i)})_{11} = 0$, so $E_\eta^{(i)} = (E_\eta^{(i)} + E_\gamma^{(i)})^T - E_\gamma^{(i)} \in \mathcal{M}$. Also

$$(E_\eta^{(i)} - E_\gamma^{(i)})^T = (E_\eta^{(i)})^T - (E_\gamma^{(i)})^T = -(E_\eta^{(i)} - E_\gamma^{(i)}),$$

so $(E_\eta^{(i)} - E_\gamma^{(i)})$ is anti-symmetric and $E_\eta^{(i)} = -(E_\eta^{(i)} - E_\gamma^{(i)})^T + E_\gamma^{(i)} \in \mathcal{N}$. Therefore $\ker\Gamma \subset \mathcal{M}\cap\mathcal{N}$. Conversely let $A \in \mathcal{M}\cap\mathcal{N}$. Then

$$A = S + \mathbf{1}\mathbf{c}_1^T \text{ and } A = B + \mathbf{1}\mathbf{c}_2^T \text{ for a symmetric } S \text{ and anti-symmetric } B.$$

Thus $B + \mathbf{1}\mathbf{c}_2^T - \mathbf{1}\mathbf{c}_1^T = B^T + \mathbf{c}_2\mathbf{1}^T - \mathbf{c}_1\mathbf{1}^T$ and using the anti-symmetry of B, we obtain

$$B = \frac{1}{2}(\mathbf{c}_2\mathbf{1}^T - \mathbf{c}_1\mathbf{1}^T + \mathbf{1}\mathbf{c}_2^T - \mathbf{1}\mathbf{c}_1^T)$$

and so

$$A = \frac{1}{2}(\mathbf{c}_2\mathbf{1}^T - \mathbf{c}_1\mathbf{1}^T + \mathbf{1}\mathbf{c}_2^T - \mathbf{1}\mathbf{c}_1^T) + \mathbf{1}\mathbf{c}_2^T \in \ker\Gamma.$$

Next we show that $\text{range}(\Gamma) = \text{span}(\mathcal{M}^\perp \cup \mathcal{N}^\perp)$. Then, we have

$$\begin{aligned} \text{span}(\mathcal{M}^\perp \cup \mathcal{N}^\perp) &= \text{span}(\{N^{(ij)}\}_{j>i\geq 2} \cup \{H^{(ij)}\}_{j>i\geq 2} \cup \{K^{(ii)}\}_{i\geq 2}\} \\ &= \text{span}(\{K^{(ij)}\}_{j>i\geq 2} \cup \{K^{(ij)}\}_{i>j\geq 2} \cup \{K^{(ii)}\}_{i\geq 2}\} \\ &= \text{span}(\{K^{(ij)}\}_{i\geq 2, j\geq 2}\} = \text{range}(\Gamma) \end{aligned}$$

■

Proposition 3.1.6 provides the essential characterization of the relationship among spaces. Since $\mathcal{L} = \ker(\Gamma) \oplus \text{range}(\Gamma)$, from Proposition 3.1.6, we obtain the decomposition of a given game into three parts; $\mathcal{L} = \mathcal{M}^\perp \oplus \mathcal{N}^\perp \oplus \ker(\Gamma)$. Also since $\mathcal{N} \cap (\mathcal{M}^\perp \cup \mathcal{N}^\perp) = \mathcal{M}^\perp$, we will have $\mathcal{N}\cap$ range$(\Gamma) = \mathcal{M}^\perp$ and this provide another characterization of $\mathcal{M}^\perp$ as follows. From Proposition 3.1.1, we know that a game is anti-potential if and only if it is an antisymmetric matrix whose row sums and column sums are zeros. We know that all row sums and column sums of games belonging to range(Γ) are zeros and the zero sum game is the sum of an antisymmetric matrix and a passive game; thus we can show that $\mathcal{M}^\perp = \mathcal{N}\cap$ range$(\mathbf{\Gamma})$. In this way we obtain the following key result in the paper.

Theorem 3.1.7 *We have*

(1) $\mathcal{M} = \mathcal{N}^{\perp} \oplus \ker(\Gamma)$ *and* $\mathcal{M}^{\perp} = \mathcal{N} \cap$ *range*$(\mathbf{\Gamma})$

(2) $\mathcal{N} = \mathcal{M}^{\perp} \oplus \ker(\Gamma)$ *and* $\mathcal{N}^{\perp} = \mathcal{M} \cap$ *range*$(\mathbf{\Gamma})$

(3) $\mathcal{L} = \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp} \oplus \ker(\Gamma)$

Proof. (1) From Proposition 3.1.6, we have $\mathcal{N}^{\perp} + \ker(\Gamma) = \text{span}(\mathcal{N}^{\perp} \cup \ker(\Gamma)) = \text{span}((\mathcal{N}^{\perp} \cup \mathcal{M}) \cap (\mathcal{N}^{\perp} \cup \mathcal{N})) = \mathcal{M}$. Since $\mathcal{N}^{\perp} \perp \ker(\Gamma)$, we have $\mathcal{M} = \mathcal{N}^{\perp} \oplus \ker(\Gamma)$. From proposition 3.1.6, we have $\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp} = \text{range}(\Gamma)$ and see that $\mathcal{M}^{\perp} \subset \mathcal{N} \cap \text{range}(\Gamma)$. Conversely again from proposition 3.1.6, we have

$$\mathcal{N} \cap \text{range}(\Gamma) = \mathcal{N} \cap (\text{span}(\mathcal{M}^{\perp} \cup \mathcal{N}^{\perp})) \supset \text{span}(\mathcal{N} \cap (\mathcal{M}^{\perp} \cup \mathcal{N}^{\perp})) = \mathcal{M}^{\perp}.$$

By changing the roles of $\mathcal{M}$ and $\mathcal{N}$, we obtain (2). (3) follows from $\mathcal{L} = \mathcal{M}^{\perp} \oplus \mathcal{M} = \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp} \oplus \ker(\Gamma)$. ■

Sandholm (2010a) provides a method of decomposing normal form games by using the orthogonal projection P : for a given A

$$A = \underbrace{PAP}_{\in \text{ range}(\Gamma)} + \underbrace{(I-P)AP + PA(I-P) + (I-P)A(I-P)}_{\in \ker(\Gamma)}. \tag{3.9}$$

The first term in (3.9) belongs to the range of Γ and the remaining three terms belong to the kernel of Γ. Our decompositions (Proposition 3.1.6) show that PAP can be further decomposed into games having nice properties − potential games and zero-sum games − and every game in $\ker(\Gamma)$ is a game which is both a potential and a zero-sum game and possesses (generically) a dominant strategy.

Theorem 3.1.7 also provides a convenient way to compute an anti-zero-sum part (an anti-potential part, resp.) of the game when an anti-potential part (anti-zero-sum part, resp.) is known. Suppose that A is a symmetric game and its anti-potential part is Z. Then the part of A that belongs to $\ker(\Gamma)$ is $A - PAP$. Hence from (3) of Theorem 3.1.7, its anti-zero-sum part is given by $A-Z-(A-PAP) = PAP-Z$; in fact Theorem 3.1.7 shows that $PAP-Z$ is a symmetric matrix in $\mathcal{L}$ and its all row sums and column sums are zeros.

3.1.4 Decompositions of bimatrix games

In this section we prove a decomposition theorem for a general bimatrix game and elucidate the relationship between the decompositions of symmetric games and the bimatrix games. Here most of propositions are the bimatrix extension of the corresponding proposition in section 2.2-2.3. We denote (with a slight abuse of notation) by $\mathcal{L}$ the space of all $l_r \times l_c$ matrices and endow $\mathcal{L}$ with the inner product $\langle A, B\rangle_{\mathcal{L}} := tr(A^T B)$. Without loss of generality we assume $l_r \leq l_c$. The set of all bimatrix games is $\mathcal{L}^2 := \mathcal{L} \times \mathcal{L}$ and sometimes we will view a bimatrix game (A, B) as a $(l_r + l_c) \times (l_r + l_c)$ matrix as follows:

$$(A, B) := \begin{pmatrix} O_r & A \\ B^T & O_c \end{pmatrix}$$

where O_r and O_c are $l_r \times l_r$ and $l_c \times l_c$ zero matrices, respectively. The space $\mathcal{L}^2$ is a subspace of the set of all $(l_r+l_c)\times(l_r+l_c)$ matrices of dimension $2l_r l_c$. We endow $\mathcal{L}^2$ with the inner product $< \cdot, \cdot >_{\mathcal{L}^2}$, where $\langle (A, B), (C, D)\rangle_{\mathcal{L}^2} := tr((A, B)^T (C, D))$. The elementary properties of this scalar product are summarized as follows. First we observe that

1. $(A, B)^T = (B, A)$

2. (A, B) is symmetric in $\mathcal{L}^2$ if $A = B$

3. (A, B) is anti-symmetric in $\mathcal{L}^2$ if $A = -B$

4. (A, B) is a symmetric game if $l_r = l_c$ and $A = B^T$

So, a bimatrix symmetric game is not necessarily a symmetric matrix in $\mathcal{L}^2$. We endow $\mathcal{L}^2$ with an inner product $<,>_{\mathcal{L}^2}$ defined by:

$$\langle (A,B),(C,D)\rangle_{\mathcal{L}^2} := tr((A,B)^T(C,D))$$

We provide some properties of $\langle\ \rangle_{\mathcal{L}^2}$ and its relationship with $\langle\ \rangle_{\mathcal{L}}$.

Lemma 3.1.8 *For $(l_r \times l_c)$ matrices A, B, C, D , we have*

(1) $\langle (A,B),(C,D)\rangle_{\mathcal{L}^2} = \langle A, C\rangle_{\mathcal{L}} + \langle B, D\rangle_{\mathcal{L}}$

(2) $\langle SA, B\rangle_{\mathcal{L}} = \langle A, SB\rangle_{\mathcal{L}}$ *for a symmetric $(l_r \times l_r)$ matrix S*

(3) $\langle (A,A),(B,-B)\rangle_{\mathcal{L}^2} = 0$

(4) For $\mathbf{c} \in \mathbb{R}^{l_r}$ and A such that $A\mathbf{1}_{l_c} = \mathbf{0}$, $\left\langle A, \mathbf{c}\mathbf{1}_{l_c}^T\right\rangle_{\mathcal{L}} = 0$.

(5) For $\mathbf{c} \in \mathbb{R}^{l_c}$ and A such that $\mathbf{1}_{l_r}^T A = \mathbf{0}$, $\left\langle A, \mathbf{1}_{l_r}\mathbf{c}^T\right\rangle_{\mathcal{L}} = 0$.

Proof. (1) and (2) are obvious. (3) follows from

$$\langle (A,A),(B,-B)\rangle_{\mathcal{L}^2} = \langle A, B\rangle_{\mathcal{L}} - \langle A, B\rangle_{\mathcal{L}} = 0.$$

(4) follows from

$$\left\langle A, \mathbf{c}\mathbf{1}_{l_c}^T\right\rangle = tr(\mathbf{1}_{l_c}\mathbf{c}^T A) = tr(\mathbf{c}^T A\mathbf{1}_{l_c}) = 0$$

by the commutativity of trace and (5) follows from

$$\left\langle A, \mathbf{1}_{l_r}\mathbf{c}^T\right\rangle_{\mathcal{L}} = tr(\mathbf{c}\mathbf{1}_{l_r}^T A) = 0$$

■

The set of all bimatrix *passive games* $\bar{\mathcal{I}}$ is given by

$$\bar{\mathcal{I}} := \mathrm{span}(\{(E_\gamma^{(j)}, O)\}_j \cup \{(O, E_\gamma^{(i)})\}_i).$$

We say that a game (A, B) and a game (C, D) are equivalent if $(A, B) - (C, D) \in \bar{\mathcal{I}}$ In this case we write $(A, B) \sim (C, D)$. The set of Nash equilibria for a bimatrix game is invariant under this equivalence relation.

Notice that $(E_\kappa^{(ij)}, -E_\kappa^{(ij)})$ is a game whose restriction on the strategy set $\{i, i+1\} \times \{j, j+1\}$ is the Matching Pennies game, and we call $(E_\kappa^{(ij)}, -E_\kappa^{(ij)})$ is an extended Matching Pennies game.

From Monderer and Shapley (1996) we recall that (A, B) is a *potential game* if there exist a matrix S and $\{\gamma_j\}_j$, $\{\eta_i\}_i$ such that

$$(A, B) = (S, S) + \sum_j \gamma_j (E_\gamma^{(j)}, O) + \sum_i \eta_i (O, E_\eta^{(i)}) \,.$$

Letting $\bar{\mathcal{M}}$ be the subspace of all potential game, we similarly have the orthogonal decomposition $\mathcal{L}^2 = \bar{\mathcal{M}} \oplus \bar{\mathcal{M}}^\perp$. Note that the dimension of the subspace of all exact potential games is $l_r \times l_c$ and the dimension of the subspace of all passive games is $l_r + l_c$. Arguing as for symmetric games, one finds that the dimension of $\bar{\mathcal{M}}$ is given by

$$\dim(\bar{\mathcal{M}}) = l_r l_c + l_r + l_c - 1 = 2 l_r l_c - (l_r - 1)(l_c - 1). \tag{3.10}$$

Also note that $({E_\kappa}^{(ij)}, -{E_\kappa}^{(ij)})$ is an anti-symmetric matrix as an element in $\mathcal{L}^2$ whose column sum and row sum are all 0's, thus we have $\left\langle (A, B), ({E_\kappa}^{(i,j)}, -{E_\kappa}^{(i,j)}) \right\rangle_{\mathcal{L}^2} = 0$ for all $(A, B) \in \bar{\mathcal{M}}$. In other words, $({E_\kappa}^{(ij)}, -{E_\kappa}^{(ij)}) \in \bar{\mathcal{M}}^\perp$ for all i, j and the number of such $({E_\kappa}^{(i,j)}, -{E_\kappa}^{(i,j)})$ is $(l_r - 1)(l_c - 1)$. Hence

Proposition 3.1.9 (Anti-potential games) $\{({E_\kappa}^{(i,j)}, -{E_\kappa}^{(i,j)})\}_{i,j}$ *form an orthonormal basis for* $\bar{\mathcal{M}}^\perp$.

Proof. From the discussion above, it is enough to show the linear independency among $(E_\kappa^{(ij)}, -E_\kappa^{(ij)})$. To do this, we consider the following linear combination:

$$\sum_{i,j} \kappa^{(ij)} E_\kappa^{(ij)} = 0.$$

Then, it is easy to see that $\kappa^{(11)} = 0$. This implies $\kappa^{(1,j)} = 0$ for all j which, in turn, implies $\kappa^{(i,j)} = 0$ for all i. ■

Proposition 3.1.9 shows that a basis for $\mathcal{M}^{\perp}$ can be obtained from the Matching Pennies games and its extensions. From this, we say that (A, B) is an bimatrix *anti-potential* game whenever $(A, B) \in \mathcal{M}^{\perp}$. Proposition 3.1.9 provide an alternative and simple proof for the well-known criterion for the potential game by Monderer and Shapley (1996):

Corollary 3.1.10 (Potential games) *(A, B) is a potential game if and only if for all $i, i' \in S_r$, $j, j' \in S_c$,*

$$a(i',j) - a(i,j) + b(i',j') - b(i',j) + a(i,j') - a(i',j') + b(i,j) - b(i,j') = 0$$

Proof. It is enough to notice that

$$\begin{aligned} & a(i',j) - a(i,j) + b(i',j') - b(i',j) + a(i,j') - a(i',j') + b(i,j) - b(i,j') \\ = & \left\langle (A, B), (K^{(i,i')(j,j')}, -K^{(i,i')(j,j')}) \right\rangle_{\mathcal{L}^2} \end{aligned}$$

where $(K^{(i,i')(j,j')}, -K^{(i,i')(j,j')})$ is an extended Matching Pennies game whose restriction on $\{i, i'\} \times \{j, j'\}$ is a Matching Pennies game. ■

Next we consider a decomposition using zero-sum games as in Section 2.2. We call a game of the form $(A, -A)$ an exact zero-sum game and say that a game is a *zero-sum* game if it can be written as the sum of an exact zero-sum game and a passive game. We denote by $\bar{\mathcal{N}}$ the subspace of all bimatrix zero-sum games and we have $\dim(\bar{\mathcal{N}}) = 2l_r l_c - (l_r - 1)(l_c - 1)$. The similar argument as in Section 2.2 yields

Proposition 3.1.11 (Anti-zero-sum games) *$\{(E_\kappa^{(ij)}, E_\kappa^{(ij)})\}_{i \geq 2, j \geq 2}$ form an orthonormal basis for $\bar{\mathcal{N}}^{\perp}$.*

Notice that in this case *the extended Hawk-Dove games* form a basis for anti-zero-sum games. Again the following corollary is an immediate consequence of orthogonality (See Exercise 11.2.9 in Hofbauer and Sigmund, 1998).

Corollary 3.1.12 *(A, B) is a zero-sum game if and only if for all $i, i' \in S_r$, $j, j' \in S_c$,*

$$a(i',j) - a(i,j) - b(i',j') + b(i',j) + a(i,j') - a(i',j') - b(i,j) + b(i,j') = 0.$$

To consider the decomposition in terms of the projection mapping onto the tangent space as in Section 2.3, we first modify the definition of Γ :

$$\Gamma : \mathcal{L} \to \mathcal{L},\ A \mapsto P_r A P_c,\ P_l = I_l - \frac{1}{l}\mathbf{1}_r\mathbf{1}_r^T,\ P_c = I_c - \frac{1}{l}\mathbf{1}_c\mathbf{1}_c^T.$$

and define $\mathbf{\Gamma} : \mathcal{L}^2 \to \mathcal{L}^2$ by

$$(A,B) \mapsto \mathbb{P}(A,B)\mathbb{P} := \begin{pmatrix} P_r & O \\ O & P_c \end{pmatrix} \begin{pmatrix} O & A \\ B^T & O \end{pmatrix} \begin{pmatrix} P_r & O \\ O & P_c \end{pmatrix}.$$

Then analyzing similarly as in the symmetric games (Proposition 3.1.5), we obtain the following characterizations for ker($\mathbf{\Gamma}$) and range ($\mathbf{\Gamma}$):

Proposition 3.1.13 *We have*

$$\ker(\mathbf{\Gamma}) = \operatorname{span}(\{(E_\eta^{(i)}, O)\}_{i \neq 1} \cup \{(E_\gamma^{(i)}, O)\}_i \cup \{(O, E_\eta^{(i)})\}_i \cup \{(O, E_\gamma^{(i)})\}_{i \neq 1})$$

$$\operatorname{range}(\mathbf{\Gamma}) = \operatorname{span}(\{(E_\kappa^{(ij)}, O)\}_{i \geq 2, j \geq 2} \cup \{O, E_\kappa^{(ij)}\}_{i \geq 2, j \geq 2})$$

Clearly results similar to Proposition 3.1.6, and Theorem 3.1.7 hold for $\mathcal{L}^2$ and the subspaces $\bar{\mathcal{M}}$, $\bar{\mathcal{M}}^\perp$, $\bar{\mathcal{N}}, \bar{\mathcal{N}}^\perp$, ker($\mathbf{\Gamma}$), and range($\mathbf{\Gamma}$). To understand the relationship between the decompositions of the symmetric games and the bimatrix games, note that the set of two player symmetric games is a special class of bimatrix games when $l = l_r = l_c$ namely,

$$(A, B) \text{ is a symmetric game if } A = B^T.$$

To avoid confusion, we denote by $\mathcal{L}_{sym}$ the set of all symmetric games which is the subspace of $\mathcal{L}^2$ and write $[A] = (A, A^T)$ for a symmetric game A. Consider the following example:

$$[E_\kappa^{(12)} - E_\kappa^{(21)}] = (E_\kappa^{(12)}, -E_\kappa^{(12)}) - (E_\kappa^{(21)}, -E_\kappa^{(21)})$$

$$= \begin{array}{|c|c|c|}\hline 0,0 & -1,1 & 1,-1 \\ \hline 0,0 & 1,-1 & -1,1 \\ \hline 0,0 & 0,0 & 0,0 \\ \hline \end{array} - \begin{array}{|c|c|c|}\hline 0,0 & 0,0 & 0,0 \\ \hline -1,1 & 1,-1 & 0,0 \\ \hline 1,-1 & -1,1 & 0,0 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0,0 & -1,1 & 1,-1 \\ \hline 1,-1 & 0,0 & -1,1 \\ \hline -1,1 & 1,-1 & 0,0 \\ \hline \end{array}.$$

Thus $[E_\kappa^{(12)} - E_\kappa^{(21)}]$ is the Rock-paper-scissors game; these examples show how one can "symmetrize" the bimatrix games to obtain the symmetric version of them. More generally, we obtain the orthonormal bases of anti-potential games and anti-zero-sum symmetric games in symmetric games by restricting the bases of subspaces of bimatrix games using the following lemma.

Lemma 3.1.14 *Suppose that* $\{(A^{(ij)}, A^{(ij)})\}_{i,j\in\mathcal{I}_1} \cup \{(B^{(ij)}, -B^{(ij)})\}_{i,j\in\mathcal{I}_2} \cup \{(C^{(i)}, O)\}_{i\in\mathcal{I}_3} \cup \{(O, (C^{(i)})^T)\}_{i\in\mathcal{I}_3}$ *form a basis for* K*, a subspace of* $\mathcal{L}^2$ *and* $\{A^{(ij)}\}_{i,j} \cup \{B^{(ij)}\}_{i,j} \cup \{C^{(i)}\}_i$ *are linearly independent. Then* $\{[A^{(ij)}+A^{(ji)}]\}_{i,j\in\mathcal{I}_1\cap\{j\geq i\}} \cup \{[B^{(ij)}-B^{(ji)}]\}_{i,j\in\mathcal{I}_2\cap\{j>i\}} \cup \{[C^{(i)}]\}_{i\in\mathcal{I}_3}$ *form a basis for* $K \cap \mathcal{L}_{sym}$.

Proof. First we show that $\text{span}(\{[A^{(ij)}+A^{(ji)}]\}_{i,j\in\mathcal{I}_1\cap\{j\geq i\}} \cup \{[B^{(ij)}-B^{(ji)}]\}_{i,j\in\mathcal{I}_2\cap\{j>i\}} \cup \{[C^{(i)}]\}_{i\in\mathcal{I}_3}) = \mathcal{K} \cap \mathcal{L}_{sym}$. Obviously,

$$\begin{aligned} [A^{(ij)} + A^{(ji)}] &= (A^{(ij)} + A^{(ji)}, (A^{(ij)})^T + (A^{(ji)})^T) = (A^{(ij)}, A^{(ji)}) + (A^{(ji)}, A^{(ij)}) \\ &= (A^{(ij)}, A^{(ij)}) + (A^{(ji)}, A^{(ji)}) \in \mathcal{K} \cap \mathcal{L}_{sym} \end{aligned}$$

Similarly we have $\{(B^{(ij)}, -B^{(ij)})\}_{i,j} \in \mathcal{K} \cap \mathcal{L}_{sym}$. Also $[C^{(i)}] = (C^{(i)}, (C^{(i)})^T) = (C^{(i)}, O) + (O, (C^{(i)})^T) \in \mathcal{K} \cap \mathcal{L}_{sym}$. Conversely, let $(E, F) \in \mathcal{K} \cap \mathcal{L}_{sym}$. Then

$$\begin{aligned}
& (E,F) \\
= \quad & \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)}(B^{(ij)}, -B^{(ij)}) + \sum_{i\in\mathcal{I}_3} \kappa_{(3)}^{(i)}(C^{(i)}, O) + \sum_{i\in\mathcal{I}_3} \kappa_{(4)}^{(i)}(O, (C^{(i)})^T) \\
= \quad & (\sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ij)} + \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ij)} + \sum_{i\in\mathcal{I}_3} \kappa_{(3)}^{(i)} C^{(i)}, \\
& \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ij)} - \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ij)} + \sum_{i\in\mathcal{I}_3} \kappa_{(4)}^{(i)} (C^{(i)})^T)
\end{aligned}$$

Since $E = F^T$, we have

$$\sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ij)} + \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ij)} + \sum_{i\in\mathcal{I}_3} \kappa_{(3)}^{(i)} C^{(i)} = \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ji)} - \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ji)} + \sum_{i\in\mathcal{I}_3} \kappa_{(4)}^{(i)} C^{(i)}$$

Thus we obtain

$$\sum_{i,j\in\mathcal{I}_1} (\kappa_{(1)}^{(ij)} - \kappa_{(1)}^{(ji)}) A^{(ij)} + \sum_{i,j\in\mathcal{I}_2} (\kappa_{(2)}^{(ij)} + \kappa_{(2)}^{(ji)}) B^{(ij)} + \sum_{i\in\mathcal{I}_3} (\kappa_{(3)}^{(i)} - \kappa_{(4)}^{(i)}) C^{(i)} = O \qquad (3.11)$$

Then from the linear independency of $\{A^{(ij)}\}_{ij} \cup \{B^{(ij)}\}_{ij} \cup \{C^{(i)}\}_i$ in $\mathcal{L}$, we conclude that

$$\kappa_{(1)}^{(ij)} = \kappa_{(1)}^{(ji)}, \ \kappa_{(2)}^{(ij)} = -\kappa_{(2)}^{(ji)}, \text{ and } \kappa_{(3)}^{(i)} = \kappa_{(4)}^{(i)} \text{ for all } i, j$$

Note that $\kappa_{(2)}^{(ii)} = 0$ for all i. Thus we have

$$\begin{aligned}
& \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) \\
= \quad & \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{\{j<i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{\{i=j\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) \\
= \quad & \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ji)}(A^{(ji)}, A^{(ji)}) + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ii)}(A^{(ii)}, A^{(ii)}) \\
= \quad & \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}((A^{(ij)} + A^{(ji)}, A^{(ij)} + A^{(ji)}) + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \frac{1}{2}\kappa_{(1)}^{(ii)}(A^{(ii)} + A^{(ii)}, A^{(ii)} + A^{(ii)}) \\
= \quad & \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}((A^{(ij)} + A^{(ji)}, A^{(ji)} + A^{(ij)}) + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \frac{1}{2}\kappa_{(1)}^{(ii)}(A^{(ii)} + A^{(ii)}, A^{(ii)} + A^{(ii)}) \\
= \quad & \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)} \left[A^{(ij)} + A^{(ji)}\right] + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \frac{1}{2}\kappa_{(1)}^{(ii)}[A^{(ii)} + A^{(ii)}]
\end{aligned}$$

Similar manipulation yields

$$\begin{aligned}
&\sum_{i,j\in\mathcal{I}_2}\kappa^{(ij)}_{(2)}(B^{(ij)},-B^{(ij)})\\
=&\sum_{\{j>i\}\cap\mathcal{I}_2}\kappa^{(ij)}_{(2)}(B^{(ij)},-B^{(ij)})+\sum_{\{j<i\}\cap\mathcal{I}_2}\kappa^{(ij)}_{(2)}(B^{(ij)},-B^{(ij)})\\
=&\sum_{\{j>i\}\cap\mathcal{I}_2}\kappa^{(ij)}_{(2)}(B^{(ij)},-B^{(ij)})+\sum_{\{j>i\}\cap\mathcal{I}_2}\kappa^{(ji)}_{(2)}(B^{(ji)},-B^{(ji)})\\
=&\sum_{\{j>i\}\cap\mathcal{I}_2}\kappa^{(ij)}_{(2)}((B^{(ij)}-B^{(ji)},-B^{(ij)}+B^{(ji)})=\sum_{\{j>i\}\cap\mathcal{I}_2}\kappa^{(ij)}_{(2)}((B^{(ij)}-B^{(ji)},B^{(ji)}-B^{(ij)})\\
=&\sum_{\{j>i\}\cap\mathcal{I}_2}\kappa^{(ij)}_{(2)}\left[B^{(ij)}-B^{(ji)}\right]
\end{aligned}$$

and finally

$$\sum_i\kappa^{(i)}_{(3)}(C^{(i)},O)+\sum_i\kappa^{(i)}_{(4)}(O,(C^{(i)})^T)=\sum_i\kappa^{(i)}_{(3)}(C^{(i)},(C^{(i)})^T)=\sum_i\kappa^{(i)}_{(3)}[C^{(i)}].$$

Therefore, we have $\mathrm{span}(\{[A^{(ij)}+A^{(ji)}]\}_{i,j}\cup\{[B^{(ij)}-B^{(ji)}]\}_{i,j}\cup\{[C^{(i)}]\}_i)=\mathcal{K}\cap\mathcal{L}_{sym}$. Next we show that $\{[A^{(ij)}+A^{(ji)}]\}_{i,j}\cup\{[B^{(ij)}-B^{(ji)}]\}_{i,j}\cup\{[C^{(i)}]\}_i$ are linearly independent in $\mathcal{L}^2$. Suppose that

$$\sum_{\{j\geq i\}\cap\mathcal{I}_1}\alpha_{ij}[A^{(ij)}+A^{(ji)}]+\sum_{\{j>i\}\cap\mathcal{I}_2}\beta_{ij}[B^{(ij)}-B^{(ji)}]+\sum_{\mathcal{I}_3}\gamma_i[C^{(i)}]=O\text{ in }\mathcal{L}^2$$

Then we have

$$\sum_{\{j\geq i\}\cap\mathcal{I}_1}\alpha_{ij}(A^{(ij)}+A^{(ji)})+\sum_{\{j>i\}\cap\mathcal{I}_2}\beta_{ij}(B^{(ij)}-B^{(ji)})+\sum_{\mathcal{I}_3}\gamma_i(C^{(i)})=O\text{ in }\mathcal{L}$$

and note that we have

$$\begin{aligned}
\sum_{\{j\geq i\}\cap\mathcal{I}_1}\alpha_{ij}(A^{(ij)}+A^{(ji)}) &= \sum_{\{j>i\}\cap\mathcal{I}_1}\alpha_{ij}A^{(ij)}+\sum_{\{i>j\}\cap\mathcal{I}_1}\alpha_{ji}A^{(ij)}+\sum_{\{i=j\}\cap\mathcal{I}_3}\alpha_{ii}A^{(ii)}\\
\sum_{\{j>i\}\cap\mathcal{I}_2}\beta_{ij}(B^{(ij)}-B^{(ji)}) &= \sum_{\{j>i\}\cap\mathcal{I}_2}\beta_{ij}B^{(ij)}-\sum_{\{i>j\}\cap\mathcal{I}_2}\beta_{ji}B^{(ij)}
\end{aligned}$$

and since $\{A^{(ij)}\}_{i,j}\cup\{B^{(ij)}\}_{i,j}\cup\{C^{(i)}\}_i$ are linearly independent in $\mathcal{L}$, we conclude that $\alpha_{ij}=0,\beta_{ij}=0$, and $\gamma_i=0$ for all i,j. ■

As an immediate consequence of the decomposition we obtain the alternative proof for the following well-known characterization for potential games (Hofbauer and Sigmund, 1998; Sandholm, 2010b). Notice that similar characterization for the symmetric potential and zero-sum games are also readily available.

Proposition 3.1.15 *The following conditions are equivalent:*

(1) (A, B) *is a potential game (a zero-sum game, respectively)*

(2) $\mathbb{P}(A, B)\mathbb{P}$ *is a symmetric* $(l_r + l_c) \times (l_r + l_c)$ *matrix (an antisymmetric* $(l_r + l_c) \times (l_r + l_c)$ *matrix, respectively)*

(3) $(A, B) - (A, B)^T \in \ker(\mathbf{\Gamma})$ $((A, B) + (A, B)^T \in \ker(\mathbf{\Gamma})$, *respectively.)*

Proof. For a given (A, B), using range$(\mathbf{\Gamma}) = \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp}$ (Proposition 3.1.6) we have

$$\mathbb{P}(A, B)\mathbb{P} = (V, V) + (N, -N) \text{ for some } V \text{ and } N \in \mathcal{L}.$$

Since (V, V) is a $(l_r+l_c)\times(l_r+l_c)$ symmetric matrix and $(N, -N)$ is a $(l_r+l_c)\times(l_r+l_c)$ anti-symmetric, so (1) $\Leftrightarrow$ (2). For (2) $\Leftrightarrow$ (3), we first note that $(A, B)^T = (B, A)$. Thus $(A \pm B, B \pm A) \in \ker(\mathbf{\Gamma})$, if and only if $\mathbb{P}(A \pm B, B \pm A)\mathbb{P} = O$, if and only if $\mathbb{P}(A, B)\mathbb{P} = \pm\mathbb{P}(B, A)\mathbb{P}$, if and only if $\mathbb{P}(A, B)\mathbb{P} = \pm(\mathbb{P}(A, B)\mathbb{P})^T$. ■

3.1.5 Decompositions of n-player normal form games.

In this section we will briefly discuss how one can generalize the decompositions of previous sections into the case of $n-$player normal form games. For the simplicity of exposition, we suppose that all $n-$players have the same strategy set S. We denote by $\mathcal{L}_n$ the set of all n player games, by $\mathcal{S}$ the set of all strategy profiles and by $\mathcal{P}$ the set of all players. First note that we have $\dim(\mathcal{L}_n) = nl^n$. We use a l^n dimensional tensor A to denote a player's payoffs and thus a normal form game is given by $(A_{p_1}, A_{p_2}, \cdots, A_{p_n})$ for $p_l \in \mathcal{P}$. We introduce a similar inner product $\langle\rangle_{\mathcal{L}_n}$ in $\mathcal{L}_n$:

$$\langle (A_{p_1}, \cdots, A_{p_n}), (B_{p_1}, \cdots, B_{p_n}) \rangle_{\mathcal{L}_n} = \sum_{i=1,\cdots n} \langle A_{p_i}, B_{p_i} \rangle_{\mathcal{L}} \ ,$$

where

$$\langle A, B \rangle_{\mathcal{L}} = \sum_{(i_{p_1},\cdots,i_{p_n}) \in S} a_{i_{p_1},\cdots,i_{p_n}} b_{i_{p_1},\cdots,i_{p_n}}.$$

Similarly we let $\mathcal{M}_n$ be the subspace of all potential games. Then we have the following recursive formula for the dimension of $\mathcal{M}_n$.

Proposition 3.1.16 *We have* $\dim(\mathcal{M}_{n+1})^{\perp} = (l-1)^2 n l^{n-1} + \dim(\mathcal{M}_n)^{\perp}$.

Proof. First note that $\dim(\mathcal{M}_n) = l^n - 1 + nl^{n-1}$.Using this, we factorize as follows:

$$\begin{aligned} \dim(\mathcal{M}_{n+1})^{\perp} &= (n+1)l^{n+1} - l^{n+1} - (n+1)l^n + 1 \\ &= (l-1)^2(nl^{n-1} + (n-1)l^{n-2} + \cdots + 2l + 1) \\ &= (l-1)^2 n l^{n-1} + \dim(\mathcal{M}_n)^{\perp}. \end{aligned}$$

■

In particular this recursive relation in Proposition 3.1.16 shows that a basis for $(\mathcal{M}_{n+1})^{\perp}$ can be obtained from the existing basis of $(\mathcal{M}_n)^{\perp}$ by adding $(l-1)^2 n l^{n-1}$ additional elements. To illustrate this, we consider two strategy three player games. From

$\mathcal{M}_2 = \text{span}($

-1,1	1,-1
1,-1	-1,1

$)$,

we expand this basis to obtain an element of the basis set for $\mathcal{M}_3$ by making player 3 as a null player (See the first cubic in Figure 7) . That is,

$M_1 =$

-1,1,0	1,-1,0	0,0,0	0,0,0
1,-1,0	-1,1,0	0,0,0	0,0,0

.

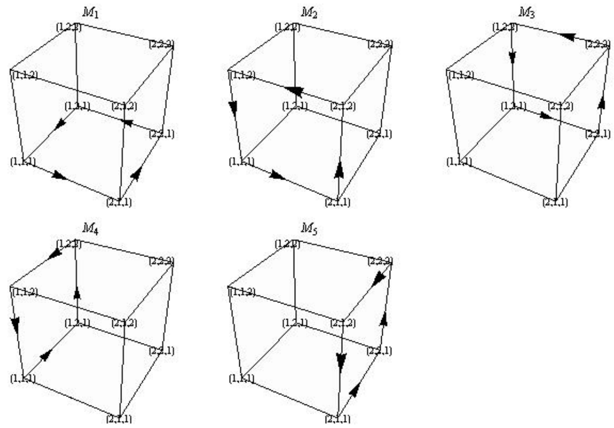


Figure 7. A basis set for three-player anti-potential games. Each vertex in each cube represents the strategy profile and the arrows show the deviation motivations based on the payoffs from the game.

Now we imagine that one of existing players, player 1 and player 2, is matched with player 3 to play the Matching Pennies game. Then since the null player, either player 1or player 2, can choose one strategy from the two strategies, there are four possible situation in which two players play the Matching Pennies game and one player plays the null player (See Figure 7). Thus we obtain the following basis games.

M_2 =

-1,0,1	0,0,0	1,0,-1	0,0,0
1,0,-1	0,0,0	-1,0,1	0,0,0

, M_3 =

0,0,0	-1,0,1	0,0,0	1,0,-1
0,0,0	1,0,-1	0,0,0	-1,0,1

M_4 =

0,-1,1	0,1,-1	0,1,-1	0,-1,1
0,0,0	0,0,0	0,0,0	0,0,0

, M_5 =

0,0,0	0,0,0	0,0,0	0,0,0
0,-1,1	0,1,-1	0,1,-1	0,-1,1

It is easy to see that $M_1, \cdots, M_5$ are independent and belong to $\mathcal{M}_3$. Thus $\{M_1, \cdots, M_5\}$ form a basis for $\mathcal{M}_3$. Here we verify Proposition 3.1.16 as follows:

$$\dim(\mathcal{M}_3)^{\perp} = (2-1)^2 2 \times 2^{2-1} + \dim(\mathcal{M}_2)^{\perp}.$$

Note that

$M_6 =$

0,0,0	0,0,0	-1,1,0	1,-1,0
0,0,0	0,0,0	1,-1,0	-1,1,0

can be obtained by taking $M_1 - (M_2 - M_3 - M_4 + M_5)$. Next we characterize the subspace of all zero-sum games. We call a game $(A_{p_1}, A_{p_2}, \cdots, A_{p_n})$ is an exact zero-sum game if

$$(A_{p_1})_{(i_{p_1},\cdots,i_{p_n})} + \cdots + (A_{p_n})_{(i_{p_1},\cdots,i_{p_n})} = 0 \text{ for all } (i_{p_1}, \cdots, i_{p_n}) \in \mathcal{S}.$$

The following lemma reveals the structure of the subspace of all zero-sum games.

Lemma 3.1.17 $A = (A_{p_1}, A_{p_2}, \cdots, A_{p_n})$ *is an exact zero-sum game if and only if* A *can be written as a finite sum of tensors* Z*'s of the form:*

$$Z = (O, \cdots, O, Z_{p_i}, O, \cdots, O, -Z_{p_i}, O, \cdots).$$

Proof. "If part" is trivial. For "only if part", we decompose A first into l^n tensors whose $(i_{p_1}, \cdots, i_{p_n})$th element is the same as $((A_{p_1})_{(i_{p_1},\cdots,i_{p_n})}, \cdots, (A_{p_n})_{(i_{p_1},\cdots,i_{p_n})})$ and other elements are all 0's. Then since $((A_{p_1})_{(i_{p_1},\cdots,i_{p_n})}, \cdots, (A_{p_n})_{(i_{p_1},\cdots,i_{p_n})}) \in T\Delta_n$ and $\{(1, -1, 0, \cdots, 0), (1, 0, -1, \cdots, 0), \cdots, (1, 0, 0, \cdots, -1)\}$ form a basis for $T\Delta_n$, we have the desired representation. ■

Now we extend our treatment a bit further. We will denote by S, S_{-p}, and $S_{-p\cup q}$ the set of all strategy profiles, the set of all strategy profiles except player p,and the set of all strategy profiles except player p and q ; i.e.,

$$\begin{aligned}
\mathcal{S} &: = \{(i_{p_1}, \cdots, i_{p_n}) : i_{p_1}, \cdots, i_{p_n} \in S\} \\
\mathcal{S}_{-q} &: = \{(i_{p_1}, \cdots, \hat{i}_q, \cdots, i_{p_n}) : i_{p_1}, \cdots, i_{p_n} \in S\} \\
\mathcal{S}_{-q\cup r} &: = \{(i_{p_1}, \cdots, \hat{i}_q, \cdots, \hat{i}_r \cdots, i_{p_n}) : i_{p_1}, \cdots, i_{p_n} \in S\},
\end{aligned}$$

where $\hat{i}_q$ means that we omit the qth element. Then it is easy to see that $|\mathcal{S}_{-p}| = l^{n-1}$. Also for $\vec{i}_{-q\cup r} \in S_{-q\cup\gamma}$,

$$(A_{p_1})_{\vec{i}_{-q\cup r}} := (A_{p_1})_{(i_{p_1},\cdots,\hat{i}_q,\cdots,\hat{i}_r,\cdots,i_{pn})}$$

can be written as an $l \times l$ matrix and for $\vec{i}_{-q} \in S_{-q}, (A_{p_1})_{\vec{i}_{-q}}$ can be written as a $l \times 1$ vector. We also write

$$\vec{i}_{-q} \subset \vec{j} \;\text{ if } (i_{p_1}, \cdots, k, \cdots, i_{p_n}) = (j_{p_1}, \cdots, j_q, \cdots, j_{p_n}) \;\text{ for some } k \in S$$

for $\vec{i}_{-q} \in S_{-q}$ and $\vec{j} \in S$. To define passive games, we define a tensor $E_\gamma^{\vec{i}_{-q}}$ for $\vec{i}_{-q} \in S_{-q}$ as follows:

$$(E_\gamma^{\vec{i}_{-q}})_{\vec{i}_{-q}} = \mathbf{1} \text{ and 0's in other positions} \tag{3.12}$$

where 1 denotes a $l \times 1$ vector consisting of 1's. Then $E_\gamma^{\vec{i}_{-q}}$ in (3.12) is an tensor that describes the payoffs of player q and under this payoffs, given other players' strategy profile $(i_1, \cdots, \hat{i}_q, \cdots, i_n)$ for any choice of q player's strategy, q obtains payoff 1. Then similarly we set

$$\mathcal{I} = \text{span}(\{(E_\gamma^{\vec{i}_{-p_1}}, O, \cdots, O)\}_{\vec{i}_{-p_1}\in\mathcal{S}_{-p_1}}, \cdots, \{(O, \cdots, E_\gamma^{\vec{i}_{-p_n}})\}_{\vec{i}_{-p_n}\in\mathcal{S}_{-p_n}})$$

where O denotes a l^n- dimensional zero tensor. Then I is the set of all passive games. We also define the following tensors: for $\vec{i} \in S$,

$$(E_\beta^{\vec{i}})_{\vec{j}} = 1 \text{ if } \vec{i} = \vec{j}\ .$$

Then $E_\beta^{\vec{i}}$ is a tensor which has 1 at the position $\vec{i}$ and 0's at others. Then similarly we set $M := span(\{(E_\beta^{\vec{i}}, \cdots, E_\beta^{\vec{i}})\}_{\vec{i}\in\mathcal{S}}, I\}$. Then we obtain Proposition 3.1.16. Next, using Lemma 3.1.17, we define the subspace of all zero-sum games:

$$\mathcal{N} := \text{span}(\{(O, \cdots, \underbrace{E_\beta^{\vec{i}}}_{i\text{th}}, \cdots, \underbrace{-E_\beta^{\vec{i}}}_{j\text{th}}, \cdots, O)\}_{(i_{p_1},\cdots,i_{p_n})\in\mathcal{S}, p_i, p_j\in\mathcal{P}} \cup \mathcal{I}).$$

Then we have the following characterization for anti-zero-sum games. For the strategy profile $\vec{i} = (i_{p_1}, i_{p_2}, \cdots, i_{p_n})$ such that $i_p \geq 2$ for all p,we define

$$\begin{aligned}(E_\kappa^{\vec{i}})_{(\hat{\imath}_{p_1},\hat{\imath}_{p_2},1,\cdots 1)} &= E_\kappa^{(i_1,i_2)}, (E_\kappa^{\vec{i}})_{(\hat{\imath}_{p_1},\hat{\imath}_{p_2},\hat{\imath}_{p_3},\cdots 1)} = -E_\kappa^{(i_1,i_2)}, \cdots ,\\ (E_\kappa^{\vec{i}})_{(\hat{\imath}_{p_1},\hat{\imath}_{p_2},\hat{\imath}_{p_3},\cdots,\hat{\imath}_{p_n})} &= (-1)^{2n-1} E_\kappa^{(i_1,i_2)}\end{aligned}$$

and all other entries are zeros. An example of such tensors for 4 player 2 strategy is given by

$$E_\kappa^{(2,2,2,2)} = \frac{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Bigg\| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Bigg\| \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}.$$

Then we have the following proposition.

Proposition 3.1.18 $\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i}\in\mathcal{S},\ i_p\geq 2 \text{ for all } p}$ *form a basis for* $N^\perp$. *Thus* $\dim(N^\perp) = (l-1)^p$

Proof. First since $(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})$ is a symmetric tensor, $\left\langle (E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}}), N \right\rangle_{\mathcal{L}^n} = 0$ for every exact zero-sum game N. Also

$$\left\langle (E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}}), (O, \cdots, E_\gamma^{\vec{i}_{-q}}, \cdots O) \right\rangle_{\mathcal{L}^n} = \left\langle E_\kappa^{\vec{i}}, E_\gamma^{\vec{i}_{-q}} \right\rangle_{\mathcal{L}} = 0.$$

Thus $span(\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i}\in\mathcal{S},\ i_p\geq 2 \text{ for all } p}) \subset N^\perp$. Now we show $N^\perp \subset span\,(\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i}\in\mathcal{S},\ i_p\geq 2 \text{ for all } p})$. If $(A_{p_1}, \cdots, A_{p_n}) \in N^\perp$, then since all $(O, \cdots, Z, \cdots, -Z, \cdots, O) \in N$, $(A_{p_1}, \cdots, A_{p_n}) = (V, \cdots, V)$. We now show how to express $(V, \cdots, V)$ in terms of $\{(E_\kappa^{\vec{i}}, \cdots, E_\kappa^{\vec{i}})\}_{\vec{i}\in\mathcal{S},\ i_p\geq 2 \text{ for all } p}$. To do this we use an induction. We suppose that $\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i}\in\mathcal{S},\ i_p\geq 2 \text{ for all } p}$ form a basis for the subspace of anti-zero-sum games for $n-1$ player games. Then for each $i_{p_n} \in S$ such that $i_{p_n} \geq 2$ (the strategy of n th player), $\{(V)_{(i_{p_1}, i_{p_2}, \cdots, i_{p_n})}\}_{i_{p_1},\cdots,i_{p_{n-1}}\in S}$ can be viewed as l^{n-1} dimensional tensor and hence can be decomposed in terms of a basis of $\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i}\in\mathcal{S},\ i_p\geq 2 \text{ for all } p}$ of $n-1$ player games by the induction hypothesis. In this way we obtain $(l-1)^{n-1}$ coefficients of the basis elements for each $i_{p_n} \geq 2$ and, thus, in total $(l\!\!/-1)^n$ coefficients. We write this linear combination as follows:

$$B = \sum_{\vec{i}} \kappa^{\vec{i}} E_\kappa^{\vec{i}}$$

Then we have

$$(V)_{(i_{p_1},i_{p_2},\cdots,i_{p_n})} = (B)_{(i_{p_1},i_{p_2},\cdots,i_{p_n})} \text{ for } i_{p_n} \geq 2$$

by construction. Then it follows that $(V)_{(i_{p_1},i_{p_2},\cdots,1)} = (B)_{(i_{p_1},i_{p_2},\cdots,1)}$ since

$$(V)_{(i_{p_1},i_{p_2},\cdots,1)} = -\sum_{j\geq 2}(V)_{(i_{p_1},i_{p_2},\cdots,j)} = -\sum_{j\geq 2}(B)_{(i_{p_1},i_{p_2},\cdots,j)} = (B)_{(i_{p_1},i_{p_2},\cdots,1)}.$$

■

We illustrate the above proof by the following example. Suppose that $p = 2$ and $l = 3$. Suppose that a symmetric bimatrix game (A, A) is given; $A = [a_1 : a_2 : a_3]$. Then we know that the basis for $N^\perp$ is given by

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

If $A \in N^\perp$, A can be uniquely written as a linear combination of the above basis. On the other hand, if $A \in N^\perp$, then $a_2, a_3 \in T\Delta$, so a_2, a_3 can be uniquely written as a linear combination of $(1, -1, 0)^T, (1, 0, -1)^T$. Clearly, the four coefficients that we obtain in the second way also are the same as the coefficients of the basis elements of $N^\perp$.

3.1.6 Examples of Decompositions

From the previous Sections, we see that a game $(A, B) \in \mathcal{L}^2$ can be *uniquely* decomposed into (i) a representative of equivalent classes of a potential game and

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \Leftrightarrow$$

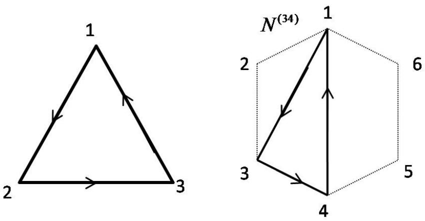


Figure 8. Representation of the Rock-paper-scissors games

an anti-potential game, (ii) a representative of equivalent classes of a zero-sum game and an anti-zero-sum game, or (iii) an anti-potential part, an anti-zero-sum part, and an part belonging to $\ker(\mathbf{\Gamma})$. Because of the simple structure of basis games in the anti-potential subspace, we can associate a class of anti-potential games with a set of graphs. To explain this we focus on the set of the symmetric games. First observe that all basis elements in $\mathcal{M}^{\perp}$, $N^{(ij)}$ have payoffs consisting 0, 1, and -1. Thus we can assign a binary relation to (i, j): for given A, $i \succ j$ if $a(i,j) = 1$ (i is better than j), $i \prec j$ if $a(i,j) = -1$ (i is worse than j), and $i \sim j$ if $a(i,j) = 0$ (i is as good as j). Since every anti-potential game is anti-symmetric, the relation is symmetric; i.e., $i \succ j$ if and only if $j \prec i$. Therefore we can represent a given basis element of anti-potential games in a diagram as in Figure 8.

For games with cyclic symmetry (Hofbauer and Sigmund, 1998, p.173) we have the following decomposition.

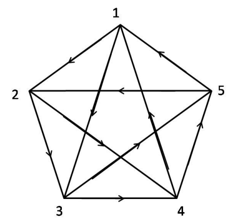


Figure 9. Games with cyclic symmetry.

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ a_4 & 0 & a_1 & a_2 & a_3 \\ a_3 & a_4 & 0 & a_1 & a_2 \\ a_2 & a_3 & a_4 & 0 & a_1 \\ a_1 & a_2 & a_3 & a_4 & 0 \end{pmatrix} \sim \frac{1}{2}\underbrace{\begin{pmatrix} 0 & a_1+a_4 & a_2+a_3 & a_2+a_3 & a_1+a_4 \\ a_1+a_4 & 0 & a_1+a_4 & a_2+a_3 & a_2+a_3 \\ a_2+a_3 & a_1+a_4 & 0 & a_1+a_4 & a_2+a_3 \\ a_2+a_3 & a_2+a_3 & a_1+a_4 & 0 & a_1+a_4 \\ a_1+a_4 & a_2+a_3 & a_2+a_3 & a_1+a_4 & 0 \end{pmatrix}}_{\mathcal{M}}$$

$$+ \quad \frac{1}{2}\underbrace{\begin{pmatrix} 0 & a_1-a_4 & a_2-a_3 & -a_2+a_3 & -a_1+a_4 \\ -a_1+a_4 & 0 & a_1-a_4 & a_2-a_3 & -a_2+a_3 \\ -a_2+a_3 & -a_1+a_4 & 0 & a_1-a_4 & a_2-a_3 \\ a_2-a_3 & -a_2+a_3 & -a_1+a_4 & 0 & a_1-a_4 \\ a_1-a_4 & a_2-a_3 & -a_2+a_3 & -a_1+a_4 & 0 \end{pmatrix}}_{\mathcal{M}^{\perp}}$$

If $a_1 - a_4 = a_2 - a_3 - 1$, then the anti-potential part of game can be represented in Figure 9.

In case of 2-strategy bimatrix coordination games, we have the following decomposition of a 2-strategy.

$$\begin{array}{|c|c|}\hline a,b & 0,0 \\ \hline 0,0 & c,d \\ \hline \end{array} \sim \frac{1}{2}\underbrace{\begin{array}{|c|c|}\hline 0,0 & 0,-b+d \\ \hline -a+c,0 & -a+c,-b+d \\ \hline \end{array}}_{\ker(\mathbf{\Gamma})} + \frac{1}{8}(a+b+c+d)\underbrace{\begin{array}{|c|c|}\hline 1,1 & -1,-1 \\ \hline -1,-1 & 1,1 \\ \hline \end{array}}_{\bar{\mathcal{N}}^{\perp}}$$

$$+\frac{1}{8}(-a+b-c+d)\underbrace{\begin{array}{|c|c|}\hline -1,1 & 1,-1 \\ \hline 1,-1 & -1,1 \\ \hline \end{array}}_{\bar{\mathcal{M}}^{\perp}}$$

Therefore, a 2 strategy coordination game is a potential game if and only if $-a+b-c+d=0$ and a zero-sum game if and only if $a+b+c+d=0$. In other words, the coefficients of the anti-potential game and the anti-zero-sum game corresponds to the condition for payoffs in four-cycle criteria as in Corollary 3.1.10 and 3.1.12.

3.2 Applications of Decompositions

3.2.1 Decompositions and Stable Games

In this section, we provide a characterization of stable games using decompositions (For properties of stable games see Hofbauer and Sandholm, 2009). We recall the definition of stable games in terms of matrix notations in order to facilitate the applications of our decompositions. A symmetric game $[A]$ is a stable game if $\langle y-x, A(y-x)\rangle_{\mathbb{R}_l} \leq 0$ for all $x, y \in \Delta_l$. A bimatrix game (A,B) is a stable game if $\langle y-x, (A,B)(y-x)\rangle_{\mathbb{R}_{l_r+l_c}} \leq 0$ for all $x, y \in \Delta_{l_r} \times \Delta_{l_c}$. A stable game which satisfies the inequality by the equality is called a null-stable game.

Note that since $[A] = (A, A^T)$, the condition for stable games can be written as

$$\left\langle y-x, (A, A^T)(y-x)\right\rangle_{\mathbb{R}_{l+l}} \leq 0 \;\text{ for all } x, y \in \{(p,q) \in \Delta_l \times \Delta_l : p = q\} \quad (3.13)$$

So by comparing this to the condition for stable bimatrix games in the case of $l_r = l_c$, we see that the definition for a symmetric stable game requires that the condition (3.13) holds for a smaller subset of $\mathbb{R}^{2l}$. This opens the possibility that more stable games arise in symmetric games, even though symmetric games belong to the special class of bimatrix games. Note that using the projection operator in Section 2 a symmetric game A is a stable game if and only if $\langle x, PAPx \rangle \leq 0$ for all $x \in \mathbb{R}_l$ and a bimatrix game (A, B) is a stable game if and only if $\langle x, \mathbb{P}(A, B)\mathbb{P}x \rangle \leq 0$ for all $x \in \mathbb{R}_{l_r + l_c}$ (Hofbauer and Sandholm, 2009, Theorem 2.1).

We first characterize stable symmetric matrix games. To do this we define a function V_A for a given symmetric game A, which will play an important role in characterizing stable games: $V_A(x) := \frac{1}{2} \langle x, Ax \rangle$. Then using the decomposition, we obtain the following representation of V_A.

Proposition 3.2.1 *Suppose that $A \in \mathcal{L}$. Then for $x \in \Delta$ and $z \in T\Delta$, there exists a symmetric matrix S and a column vector c such that $S\mathbf{1} = 0$ and*

$$V_A(x) = \frac{1}{2} \langle x, Sx \rangle + \langle x, c \rangle, \quad V_A(z) = \frac{1}{2} \langle z, Sz \rangle .$$

Moreover all eigenvectors v_i for S such that $v_i \neq \mathbf{1}$ belongs to $T\Delta$ and $S = \sum_{i=1}^{n-1} \lambda_i S_i$ where S_i is an orthogonal projection of $\mathbb{R}^n$ onto eigenspace of v_i such that $v_i \neq \mathbf{1}$.

Proof. Let $A \in \mathcal{L}$. Then since $A \in \mathcal{N}^{\perp} \oplus \ker(\Gamma) \oplus \mathcal{M}^{\perp}$, we can write

$$A = S + c_1 \mathbf{1}^T + \mathbf{1} c_2^T + N$$

where S is symmetric, $S\mathbf{1} = 0$, N is anti-symmetric and $N\mathbf{1} = 0$.Thus

$$V_A = \frac{1}{2} \langle x, Sx \rangle + \frac{1}{2} \left\langle x, (c_1 \mathbf{1}^T + \mathbf{1} c_2^T) x \right\rangle$$

Then since $\left\langle x, c_1 \mathbf{1}^T x \right\rangle = \sum_i x_i \langle x, c_1 \rangle = \langle x, c_1 \rangle$ and $\left\langle x, \mathbf{1} c_2^T x \right\rangle = \sum_i x_i \langle x, c_2 \rangle = \langle x, c_2 \rangle$, we have

$$\frac{1}{2} \left\langle x, (c_1 \mathbf{1}^T + \mathbf{1} c_2^T) x \right\rangle = \langle x, c \rangle \qquad \text{where } c = c_1 + c_2.$$

The second representation, we note that $S = PSP$ and $\left\langle z, c_1 \mathbf{1}^T z \right\rangle = \left\langle z, \mathbf{1} c_2^T z \right\rangle = 0$. Since S is symmetric, all eigenvectors are orthogonal and since $\mathbf{1}$ is an eigenvector with corresponding eigenvalue $\lambda = 0$, all other eigenvectors belongs to $T\Delta$ and the representation of S follows from the spectral theorem. ■

To characterize the stable games using Proposition 3.2.1, we let $A \in \mathcal{L}$ and $z \in T\Delta$. Then

$$V_A(z) = \frac{1}{2} \langle z, Sz \rangle = \frac{1}{2} \left\langle \sum_i \xi_i v_i, S \sum_i \xi_i v_i \right\rangle = \frac{1}{2} \sum_i \xi_i^2 \lambda_i$$

where v_i is orthonormal basis for $T\Delta$ consisting of eigenvectors of S. Thus A is null-stable if $\lambda_i = 0$ for all i. Then since S is a symmetric matrix, $S = O$ if and only if all its eigenvalues are 0. Therefore A is null-stable game if and only if $A \in \mathcal{N}$. We put this fact as Proposition 3.2.2 of which another direct proof is presented in the Appendix. Similarly note that $V_A(z) < 0$ for all $z \neq 0$ if and only if $\lambda_i < 0$ for all i. Thus a game is a strict stable game if and only if the eigenvalues for S, except the one corresponding to $\mathbf{1}$, are all negative.

Proposition 3.2.2 *$\langle x, PAPx \rangle = 0$ for all $x \in \mathbb{R}^l$ if and only if $A \in \mathcal{N}$.*

Proof. "If part" is obvious, so we let $A \in \mathcal{L}$ such that $\langle x, PAPx \rangle = 0$ for all $x \in \mathbb{R}^l$. From the decomposition we can write A as the following:

$$A = \sum_{j \geq i \geq 2} \kappa^{(ij)} (E_\kappa^{(ij)} + E_\kappa^{(ji)}) + N + C, \;\; N \in (\mathcal{M}_{\mathcal{L}})^\perp, C \in \ker(\Gamma)$$

Since $\langle x, PAPx \rangle = 0$ for all $x \in \mathbb{R}^l$, we have

$$\sum_{j \geq i} \kappa^{(ij)} \left\langle x, (E_\kappa^{(ij)} + E_\kappa^{(ji)}) x \right\rangle = 0 \text{ for all } x \in \mathbb{R}^l.$$

Let $K^{(ij)} := \frac{1}{2}(E_\kappa^{(ij)} + E_\kappa^{(ji)})$. Next by choosing appropriate x, we show that $\kappa^{(ij)} = 0$ for all $j \geq i$. Then it follows that $A \in \mathcal{N}_{\mathcal{L}}$. To do this, observe that

$$\left\langle x, K^{(ij)}x \right\rangle = -x_1^2 + x_1 x_i + x_1 x_j - x_i x_j,$$

so whenever $x_1 = x_i$ or $x_1 = x_j$, $\left\langle x, K^{(ij)}x \right\rangle = 0$.We first show that $\kappa^{(ii)} = 0$ for all i. For a given $\kappa^{(mm)}$, we choose

$$x = (1, 1, \cdots, 1, \overset{m\text{th}}{0}, 1, \cdots, 1)^T$$

i.e., x is a vector that has 0 in nth element and 1 otherwise. Then all $i < m$, $x_1 = x_i = 1$, so $\left\langle x, K^{(ij)}x \right\rangle = 0$. Similarly for all $i > m$, $x_1 = x_i = 1$, so $\left\langle x, K^{(ij)}x \right\rangle = 0$. When $i = m$, since $j > i$, $x_j = x_1 = 1$, thus $\left\langle x, K^{(ij)}x \right\rangle = 0$. For $i = j = m$, $x_i = x_j = 0$ so $\left\langle x, K^{(mm)}x \right\rangle = -1$. Therefore we have $-\kappa^{(mm)} = 0$,which implies $\kappa^{(mm)} = 0$. Next we show that $\kappa^{(ij)} = 0$ for all $i < j \leq l$ using induction. We start from the highest index, i.e., $\kappa^{(l-1,l)}$. For this case we set

$$x = (1, \cdots, 1, \overset{l-1\text{ th}}{0}, 0)^T.$$

where we assign an arbitrary value to x_n. For all $i < l-1$, $x_1 = x_i = 1$, $\left\langle x, K^{(ij)}x \right\rangle = 0$ and $\left\langle x, K^{(l-1,l)}x \right\rangle = -1$, so $\kappa^{(l-1,l)} = 0$. Next, we suppose that $\kappa^{(ij)} = 0$ for all $i > m$ and $j > n$ and show that $\kappa^{(mn)} = 0$. In this case, we set

$$x = (1, \cdots, 1, \overset{m\text{ th}}{0}, 1 \cdots, 1, \overset{n\text{ th}}{0}, x_{n+1}, \cdots, x_l)^T.$$

where we assign arbitrary values to elements over nth position. Since $n < l$, $x \in \mathbb{R}^l$. For all $i < m$, $x_i = x_1$, $\left\langle x, K^{(ij)}x \right\rangle = 0$. When $i = m$ and $j < n$, $x_j = 1$, so again $\left\langle x, K^{(ij)}x \right\rangle = 0$. When $i = m$, $j = n$, $x_i = x_j = 0$. Thus $\left\langle x, K^{(ij)}x \right\rangle = -1$ and we conclude $\kappa^{(ij)} = 0$. ■

As is well-known, *the Hawk-Dove game* provides the simplest possible strictly stable game,

$$(z_1, z_2)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -(z_1 - z_2)^2 < 0, \text{ for } (z_1, z_2) \neq (0, 0).$$

This observation can be generalized via the basis of the subspace of anti-zero-sum games $\mathcal{N}^\perp$.

Corollary 3.2.3 ($l-$strategy strictly stable games) *Suppose that*

$$A \in \{\sum_{j>i} \alpha^{(ij)} K^{(ij)} : \alpha^{(ij)} > 0\} + \ker(\mathbf{\Gamma}) + \mathcal{M}^\perp.$$

Then A is a strict stable game.

Proof. Recall that $[A]$ is a strict stable game if $\langle z, Az\rangle < 0$ for all $z \in T\Delta$ such that $z \neq \mathbf{0}$.Let $A \in S$. Then we have $\langle z, Az\rangle = -\sum_{j>i} \alpha^{(ij)}(z_i - z_j)^2 \leq 0$.Now suppose that $-\sum_{j>i} \alpha^{(ij)}(z_i - z_j)^2 = 0$. Then we have $z_i - z_j = 0$ for all $j > i$. Since $z \in T\Delta$, this implies that $z = \mathbf{0}$. ■

In case of 3-strategy games, we can strengthen Corollary 3.2.3 so as to characterize 3 strategy strict stable games completely, since the computation in 3-strategy case is less demanding.

Corollary 3.2.4 (3-strategy strictly stable games) *A $3-$strategy symmetric game A is strictly stable if and only if*

$$A \in \left\{ \begin{pmatrix} -a-b & a & b \\ a & -a-c & c \\ b & c & -b-c \end{pmatrix} : 4a + b + c > 0,\ ab + bc + ca > 0 \right\} + \ker(\Gamma) + \mathcal{M}^\perp.$$

Proof. From proposition 3.2.1, we see that $[A]$ is strictly stable if and only if S, A's part belonging to $\mathcal{N}_{\mathcal{L}}^{\perp}$, is strictly stable and S has the following parameterization.

$$S = \begin{pmatrix} -a-b & a & b \\ a & -a-c & c \\ b & c & -b-c \end{pmatrix}$$

We recall that $\langle x, Sx \rangle$ satisfying $\sum_i x_i = 0$ is negative if and only if its bordered Hessians, given below, satisfies some sign condition as we will check below. In our case, these conditions are

$$\det \begin{pmatrix} -a-b & a & 1 \\ a & -a-c & 1 \\ 1 & 1 & 0 \end{pmatrix} > 0, \quad \det \begin{pmatrix} -a-b & a & b & 1 \\ a & -a-c & c & 1 \\ b & c & -b-c & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} < 0.$$

Then by computing determinants we find that

$$4a + b + c > 0 \text{ and } ab + bc + ca > 0$$

and obtain the desired result. ■

First we note that when $l = 3$ in Corollary 3.2.3 the condition for strictly stable games is a special case of Corollary 3.2.4 by the choices of $a, b > 0$ and $c = 0$. As another important special case of Corollary 3.2.4, consider game B given by

$$B = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ \beta_{12} & 0 & \beta_{23} \\ \beta_{13} & \beta_{23} & 0 \end{pmatrix}.$$

First note that B is a potential game, so there is no anti-potential part of B. Thus

B can be decomposed into

$$B = \begin{pmatrix} -a-b & a & b \\ a & -a-c & c \\ b & c & -b-c \end{pmatrix} + \underbrace{C}_{\in \ \ker(\Gamma)} \text{ and}$$
$$a = \frac{1}{9}(5\beta_{12} - \beta_{13} - \beta_{23}), b = \frac{1}{9}(-\beta_{12} + 5\beta_{13} - 2\beta_{23}), c = \frac{1}{9}(-\beta_{12} - \beta_{13} + 5\beta_{23}).$$

Then the conditions in Corollary 3.2.4 imply

$$\beta_{12} > 0 \quad \text{and } (\beta_{12} + \beta_{23} + \beta_{13})^2 > 2(\beta_{12}^2 + \beta_{23}^2 + \beta_{13}^2). \tag{3.14}$$

Recall that the generalized Rock-paper-scissors game can be decomposed as follows:

$$\begin{pmatrix} 0 & -l & w \\ w & 0 & -l \\ -l & w & 0 \end{pmatrix} \sim \frac{1}{2}(w-l) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \frac{1}{2}(w+l) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

We see that the case when $\beta_{12} = \beta_{23} = \beta_{13}$, $\beta_{12} > 0$ satisfies conditions in (3.14), so using Corollary 3.2.4 we conclude that the generalized Rock-paper-scissors is strictly stable if and only if $w > l$ (See the discussion in (See the discussion in Hofbauer and Sandholm, 2009)). In the next section we will provide another useful parametrization of 3-strategy anti-zero-sum games.

Next we characterize the bimatrix stable game. First we recall that for J given by

$$J := \begin{pmatrix} O & A \\ B^T & O \end{pmatrix},$$

the characteristic polynomial $p(\lambda) = \det(J - \lambda I)$ satisfies $p(\lambda) = (-1)^{l_r + l_c} p(-\lambda)$. Hence if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue. For a given bimatrix game (A, B), we can write $(A, B) \sim (V, V) + (C, D) + (N, -N)$ where $(C, D) \in \ker(\mathbf{\Gamma})$. Thus, $\mathbb{P}(A, B)\mathbb{P} = \mathbb{P}(V, V)\mathbb{P}$. So if (A, B) is a stable game, all its eigenvalues must have the same sign and, thus, they must be all zeros. Hence every stable bimatrix game is always null-stable (Hofbauer and Sandholm, 2009, Theorem2.1). Then, as the similar argument as Proposition 3.2.1 shows, every null-stable bimatrix game is a zero-sum game. As a result, we provide the complete characterization of the set of all stable bimatrix games; the set of all stable bimatrix games are the set of all zero-sum games. Proposition 3.2.5 can be proved via either the straightforward extension of Proposition 3.2.2 or the direct proof by use of the basis of decompositions. We provide the direct proof in the Appendix.

Proposition 3.2.5 *$\langle w, \mathbb{P}(A,B)\mathbb{P}w\rangle = 0$ for all $w \in \mathbb{R}_{l_r+l_c}$ if and only if $(A,B) \in \bar{\mathcal{N}}$.*

Proof. Again "If part" is obvious, so we let $(A,B) \in \mathcal{L}^2$ such that $\langle w, \mathbb{P}(A,B)\mathbb{P}w\rangle = 0$ for all $w \in \mathbb{R}_{l_r+l_c}$. From corollary 3.1.7 (3), we can write (A,B) as

$$(A,B) = \sum_{i\geq 2, j\geq 2} \kappa^{(ij)}(E_\kappa^{(ij)}, E_\kappa^{(ij)}) + (N,-N) + (C_1, C_2),$$

where $(N,-N) \in \mathcal{M}^\perp, (C_1,C_2) \in \ker(\mathbf{\Gamma})$. Since $\langle w, \mathbb{P}(A,B)\mathbb{P}w\rangle = 0$ for all $w \in \mathbb{R}_{l_r+l_c}$, we have

$$\sum_{i\geq 2, j\geq 2} \kappa^{(ij)}(\langle y, (E_\kappa^{(ij)})^T x\rangle_{\mathcal{L}} + \langle x, E_\kappa^{(ij)} y\rangle_{\mathcal{L}}) = 0 \text{ for all } x \in \mathbb{R}^{l_r},\ y \in \mathbb{R}^{l_c}.$$

Similarly to the previous section, by choosing appropriate x and y we show that $\kappa^{(ij)} = 0$ for all $i \geq 2, j \geq 2$. Then it follows that $(A,B) \in \mathcal{N}$. To do this, observe that

$$\frac{1}{2}(\langle y, E_\kappa^{(ji)} x\rangle_{\mathcal{L}} + \langle x, E_\kappa^{(ij)} y\rangle_{\mathcal{L}}) = -x_1y_1 + x_iy_1 + x_1y_j - x_iy_j, \tag{3.15}$$

so whenever $x_1 = x_i$ or $y_1 = y_j$,(3.15) becomes zero. We choose the following $(x^{(i)}, y^{(j)})$:

$$x^{(i)} = (1,1,\cdots,1,\overset{i\text{th}}{0},1,\cdots,1)^T,\ y^{(j)} = (1,1,\cdots,1,\overset{j\text{th}}{0},1,\cdots,1)^T.$$

Then for (k, m) such that $k \neq i$ or $m \neq j$, we have either $x_1^{(i)} = x_k^{(i)}$ or $y_j^{(j)} = y_1^{(j)}$. Thus for all (k, m) such that $k \neq i$ or $m \neq j$,

$$\left\langle y^{(j)}, E_\kappa^{(mk)} x^{(i)} \right\rangle_{\mathcal{L}} + \left\langle x^{(i)}, E_\kappa^{(km)} y^{(j)} \right\rangle_{\mathcal{L}} = 0$$

and

$$\left\langle y^{(j)}, E_\kappa^{(ji)} x^{(i)} \right\rangle_{\mathcal{L}} + \left\langle x^{(i)}, E_\kappa^{(ij)} y^{(j)} \right\rangle_{\mathcal{L}} = -2.$$

From this we conclude that $\kappa^{(ij)} = 0$. Thus, $(A, B) \in \mathcal{N}$. ■

3.2.2 Decompositions and Deterministic Dynamics

Evolutionary dynamics based on the normal form games have been extensively examined and their important properties are closely related to the underlying games; for example, potential games yield the gradient like replicator dynamics (Hofbauer and Sigmund, 1998). Moreover the replicator dynamics are linear with respect to the underlying game matrix (or matrices), so our decompositions naturally induce decompositions at the level of vector fields. We will consider the replicator dynamics given by

$$\text{One population:} \quad \dot{x}_i = x_i((Ax)_i - x^T Ax) \text{ for all } i \tag{3.16}$$

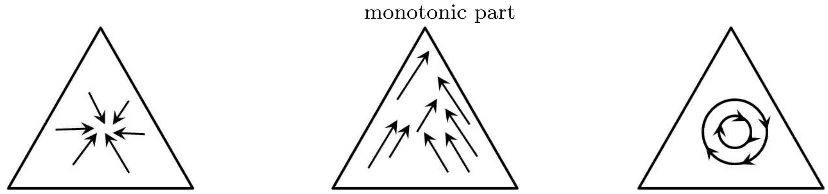
$$\text{Two population: } x_i = x_i((Ay)_i - x^T Ay), \quad \dot{y}_j = y_j((B^T x)_j - y^T B^T x)$$

When we have $A \sim S + G + N$, where $S \in \mathcal{N}^\perp$, $G \in \ker(\Gamma)$, $N \in \mathcal{M}^\perp$, the replicator dynamics can also be decomposed in three parts. First note that if $G = \sum_i \eta_i E_\eta^{(i)}$, then $(Gx)_i = \eta_i$ and $\langle x, Gx \rangle = \sum_{l \neq 1} \eta_l x_l$, so the vector field for the replicator dynamics induced by G is given by

$$x_i(\eta_i - \sum_{l \neq 1} \eta_l x_l)$$

and the system monotonically moves towards the dominating strategy state. Also when $x^T Nx = 0$ for $N \in \mathcal{M}^\perp$. Thus, the replicator ordinary differential equation for the matrix A can be decomposed into

$$f_i(x) \sim \underbrace{x_i((Sx)_i - x^T Sx)}_{\text{potential part}} + \underbrace{x_i(\eta_i - \sum_{l \neq 1} \eta_l x_l)}_{\text{monotonic part}} + \underbrace{x_i Nx}_{\text{conservative part}}$$



This decomposition of the vector field of the replicator ordinary differential equations coincides with the known Hodge decomposition which plays an important role in understanding the underlying dynamics.

We recall that a function $H\colon D \to \mathbb{R}$ is an integral of (3.16) on a region D if H is continuous differentiable and $H(x(t))$ is constant along the solution of (3.16);i.e., $LH(x(t)) := \langle \nabla H(x(t)), f(x(t)) \rangle = 0$ for a solution $x(t)$. The orbits of a conservative system must therefore lie on level curves of the integral H. A system (3.16) is said to be conservative if it has an integral H. We again recall that a function $V\colon D \to \mathbb{R}$ is a strict Lyapunov function for $C \subset D$ if V is continuous that achieves its minimum at C, is non-increasing along the solutions and is decreasing outside of C; i.e., $LV(x) := \langle \nabla V(x), f(x) \rangle \leq 0$ for x in D and $LV(x) < 0$ for $x \notin C$.

It is well-known that the replicator dynamic for the Rock-paper-scissors games is conservative and volume-preserving, the dynamics of the Matching Pennies games can be transformed to Hamiltonian systems by change in velocity of solutions, and all the bimatrix games preserve volume up to change in velocity of solutions (Hofbauer and Sigmund, 1998). As Proposition 3.2.6 shows, the class of anti-potential games provides the dynamics which are volume-preserving without involving the change of time.

Proposition 3.2.6 *(1) Suppose that* $[A]$ *is an anti-potential game. Then (3.16) is conservative and volume-preserving.*

(2) Suppose $(A, B) \in range(\mathbf{\Gamma})$. *Then* (A, B) *is conservative.*

(3) Suppose that (A, B) *is an anti-potential game and* $l_c = l_r$. *Then* (A, B) *is volume preserving.*

Proof. (1) First we note that $x^T Ax = 0$ and $A\mathbf{1} = 0$, so $x_0 = (\frac{1}{n}, \cdots, \frac{1}{n})$ is a rest point for (3.16). We consider $H(x) := \sum_i \log(x_i)$. Then $LH = \sum (Ax)_i = 0$, thus H is an integral of (3.16). Thus (3.16) is conservative. To show the preservation of volume we first write $\hat{x} = (1 - \sum_{i \neq 1} x_i, x_2, \cdots, x_n)$ and when $x \in \Delta$, $Ax = A\hat{x}$ and $\langle x, Ax \rangle = \langle \hat{x}, A\hat{x} \rangle$. Also we note that for $k \geq 2$,

$$\begin{aligned} \frac{\partial}{\partial x_k}(A\hat{x})_k &= -a_{k1} + a_{kk} \\ \frac{\partial}{\partial x_k}\langle \hat{x}, A\hat{x} \rangle &= -(Ax)_1 - (A^T x)_1 + (A^T x)_k + (Ax)_k \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{div}_\Delta f_A &= \sum_{k \neq 1} \frac{\partial f_k}{\partial x_k}(x) = \sum_{k \neq 1} (Ax)_k - (l-1)\langle x, Ax \rangle - \sum_{k \neq 1} x_k a_{k1} + \sum_{k \neq 1} x_k a_{kk} \\ &\quad + (1 - x_1)(Ax)_1 + (1 - x_1)(A^T x)_1 - \sum_{k \neq 1} x_k (A^T x)_k - \sum_{k \neq 1} x_k (Ax)_k \\ &= \sum_k (Ax)_k - l\langle x, Ax \rangle + \sum_k x_k a_{kk} - \left\langle x, A^T x \right\rangle \end{aligned}$$

If A is anti-potential, then $\sum_k (Ax)_k = \langle \mathbf{1}, Ax \rangle = \left\langle A^T \mathbf{1}, x \right\rangle = 0$ and all diagonal elements of A are zero. Thus $\mathrm{div}_\Delta f_A = 0$

(2) Recall that $(A, B) \in range(\mathbf{\Gamma})$ if and only if $(\mathbf{1}_r^T A, \mathbf{1}_r^T B) = 0$ and $(A\mathbf{1}_c, B\mathbf{1}_c) = 0$.Thus (A, B) has an interior rest point, so from Hofbauer and Sigmund (1998) (p.130) the result follows.

(3) Similarly we have, for $l_r =: l \geq 2$

$$\frac{\partial}{\partial x_l} \langle \hat{x}, Ay \rangle = (Ay)_l - (Ay)_1, \text{ and } \frac{\partial}{\partial y_l} \left\langle \hat{y}, B^T x \right\rangle = (B^T x)_l - (B^T x)_1.$$

Thus

$$\begin{aligned} \mathrm{div}_\Delta f_{(A,B)} &= \sum_{i \neq 1} \frac{\partial f_i}{\partial x_i}(x,y) + \sum_{j \neq 1} \frac{\partial f_j}{\partial y_j}(x,y) = \sum_{i \neq 1}((Ay)_i - \langle x, Ay \rangle) - \sum_{i \neq 1} x_i((Ay)_i - (Ay)_1) \\ &\quad + \sum_{j \neq 1}((B^T x)_i - \left\langle y, B^T x \right\rangle) - \sum_{j \neq 1} y_j((B^T x)_j - (B^T x)_1) \\ &= \sum_i (Ay)_i - l_r \langle x, Ay \rangle + \sum_j (B^T x)_j - l_c \left\langle y, B^T x \right\rangle. \end{aligned}$$

Then since (A, B) is anti-symmetric in $\mathcal{L}^2$, which implies $\langle x, Ay \rangle + \left\langle y, B^T x \right\rangle = 0$,and $(A, B) \in range(\mathbf{\Gamma})$, the result follows. ■

In the generalized Rock-paper-scissors game, it is easy to check that when $b > a$, $H(x) := \sum_i \log(x_i)$ is a strict Lyapunov function for $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Our decompositions show that this observation generalizes to the bigger class of games that have the similar structure to the generalized Rock-paper scissors game.

Proposition 3.2.7 *Suppose*

$$A \in \{\sum_{j > i} \alpha^{(ij)} K : \alpha^{(ij)} > 0\} + \mathcal{M}^\perp .$$

Then, $H(x) := \sum_i \log(x_i)$ *is a strict Lyapunov function for* $\frac{1}{n}\mathbf{1}$. *And thus a unique NE* $\frac{1}{n}\mathbf{1}$ *is evolutionarily stable.*

Proof. Let $A = S + N$, where $S \in \{\sum_{j>i} \alpha^{(ij)} K : \alpha^{(ij)} > 0\}$ and $N \in \mathcal{M}^\perp$. Note that for $x \neq \frac{1}{n}\mathbf{1}$, we have

$$\begin{aligned} LH &= \sum_i ((Ax)_i - x^T A x) = \sum_i (Sx)_i - \langle x, Sx \rangle + \sum_i (Nx)_i \\ &= -\langle x, Sx \rangle = -\langle x, PSPx \rangle = -\langle z, Sz \rangle > 0 \end{aligned}$$

■

Next we explain how to obtain a game that has a pure strategy ESS and an interior NE who is an attractor (called the Zeeman game) using the decomposition. We consider a game $A \in \mathcal{N}^\perp \oplus \mathcal{M}^\perp$. Then since $A\mathbf{1} = \mathbf{0}$, $\frac{1}{n}\mathbf{1}$ is NE. From the previous discussion, the anti-zero-sum part S of A is completely determined by its eigenvalues and (orthonormal) eigenvectors. Recall that S always has an eigenvector $\mathbf{1}$ with the corresponding eigenvalue 0 and other eigenvectors lie in the tangent space. Thus when the number of strategies is 3, any two eigenvectors in the tangent space can be obtain by rotating given reference orthogonal eigenvectors around the axis $(1, 1, 1)$. First we denote the matrix for the Rock paper scissors game by N :

$$N := \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Next, to express this parameterization of S we define a rotation matrix R, which rotates a given vector in $\mathbb{R}^3$ around the axis $(1, 1, 1)$, as follows:

$$R = I - P + (\cos\theta I + \sin\theta \frac{1}{\sqrt{3}} N)P. \tag{3.17}$$

To explain the meaning of R, we first recall that the rotation matrix in $\mathbb{R}^2$ acts as follows:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x = \cos\theta I x + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x.$$

Thus the rotation matrix map x to a linear combination of x itself and a vector orthogonal to x, and the coefficients of the combination are parameterized by an angle. Now note that $\langle Nx, x \rangle = 0$ for all x. Thus when $z \in T\Delta$,

$$Rz = \cos\theta I z + \sin\theta \frac{1}{\sqrt{3}} N z$$

and since Nz is orthogonal to z, R acts in the same way as the rotation in two-dimension. Also clearly $R\mathbf{1} = \mathbf{0}$. When $x \in \mathbb{R}^3$, x can be uniquely written as

$x = (I - P)x + Px$ and R rotates the part belonging to $range(P)$. Thus, R have the representation in (3.17). Using the rotation matrix R, we can write a 3-strategy game A as follows:

$$A = R \begin{pmatrix} \alpha + \frac{\beta}{3} & -\frac{2\beta}{3} & -\alpha + \frac{\beta}{3} \\ -\frac{2\beta}{3} & \frac{4\beta}{3} & -\frac{2\beta}{3} \\ -\alpha + \frac{\beta}{3} & -\frac{2\beta}{3} & \alpha + \frac{\beta}{3} \end{pmatrix} R^{-1} + \eta \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$\text{where} \begin{pmatrix} \alpha + \frac{\beta}{3} & -\frac{2\beta}{3} & -\alpha + \frac{\beta}{3} \\ -\frac{2\beta}{3} & \frac{4\beta}{3} & -\frac{2\beta}{3} \\ -\alpha + \frac{\beta}{3} & -\frac{2\beta}{3} & \alpha + \frac{\beta}{3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & 0 \\ 0 & 0 & 2\beta \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}^{-1}$$

Then the matrix A has the characteristic polynomial $\phi(t) = t(t^2 - 2(\alpha + \beta)t + 4\alpha\beta + 3\eta^2)$, so it has eigenvalues 0, $\alpha + \beta \pm \sqrt{(\alpha - \beta)^2 - 3\eta^2}$ and the eigenvector corresponding to 0 is $\mathbf{1}$. Note the eigenvalues for A does not depend on the choice of θ. We can also verify this as follows. From $RN = NR$, we have

$$A = REDE^{-1}R^{-1} + \eta N = REDE^{-1}R^{-1} + \eta RNR^{-1} = R(EDE^{-1} + \eta N)R^{-1},$$

where E denotes the matrix whose columns consist of orthogonal eigenvectors and D denotes the diagonal matrix which has 0, 2α,and 2β on the diagonal. Since $R(EDE^{-1} + \eta N)R^{-1}$ has the same eigenvalues as $EDE^{-1} + \eta N$, eigenvalues of A do not depend on the particular choice of θ.

Since $\frac{\partial}{\partial x_i}(x^T Ax) = (Ax)_i + (A^T x)_i$, by differentiating (3.16), we find that

$$\frac{\partial f_i(x)}{\partial x_j} = x_i(a_{ij} - (Ax)_j - (A^T x)_j) \quad \text{for } j \neq i, \; \frac{\partial f_i(x)}{\partial x_i} = ((Ax)_i - xA^T x) + x_i(a_{ii} - (Ax)_i). \tag{3.18}$$

So if we evaluate the expressions in (3.18) at $x = \frac{1}{n}\mathbf{1}$, from $A\mathbf{1} = \mathbf{0}$, $A^T\mathbf{1} = \mathbf{0}$,and $(Ax)_i - xA^T x = 0$, we find the following Jacobian matrix

$$\left.\frac{\partial f_i(x)}{\partial x_j}\right|_{x=\frac{1}{n}\mathbf{1}} = \frac{1}{n}a_{ij}.$$

Thus the eigenvalues for the linearized system around $\frac{1}{n}\mathbf{1}$ are a nth of the eigenvalues of A with the same corresponding eigenvectors. Also we note that if $(\alpha-\beta)^2 < 3\eta^2$, then two non-zero eigenvalues are complex and in this case real parts of eigenvalues are negative (zero, positive, resp.) if and only if $\alpha+\beta<0$ ($\alpha+\beta=0, \alpha+\beta>0$,resp.). Now we set $\theta = 0$. Then

$$A = \frac{1}{3}\begin{pmatrix} 3\alpha+\beta & -2\beta+3\eta & -3\alpha+\beta-3\eta \\ -2\beta-3\eta & 4\beta & -2\beta+3\eta \\ -3\alpha+\beta+3\eta & -2\beta-3\eta & 3\alpha+\beta \end{pmatrix}$$

so it is easy to see that if $-(\alpha+\beta) < \eta < 2\alpha$, then strategy 1 is a strict Nash equilibrium, hence an evolutionary stable strategy. Thus we obtain the following characterization of Zeeman games.

Proposition 3.2.8 *Suppose that $-(\alpha+\beta) < \eta < 2\alpha$, and $(\alpha-\beta)^2 < 3\eta^2$. Then strategy 1 is an ESS and the interior fixed point is a sink (center, source, resp.) if $\alpha+\beta<0$ ($\alpha+\beta=0, \alpha+\beta>0$,resp.).*

In Figure 10 we show how the vector field of the system changes when θ varies. To find 4-strategy Zeeman game we consider the following matrix using the similar idea:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -3 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -3 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}^{-1} + \eta \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Then, it is easy to see that if $-\gamma < \eta < \gamma$ and $\gamma > 0$, strategy 2 become a strict Nash equilibrium, so an ESS. The characteristic polynomial for A is

$$\phi(t) = t(t^3 - (\alpha+\beta+\gamma)t^2 + (\alpha\beta+\beta\gamma+\gamma\alpha+4\eta^2)t - \alpha\beta\gamma - \frac{1}{3}(6\alpha+2\beta+4\gamma)\eta^2).$$

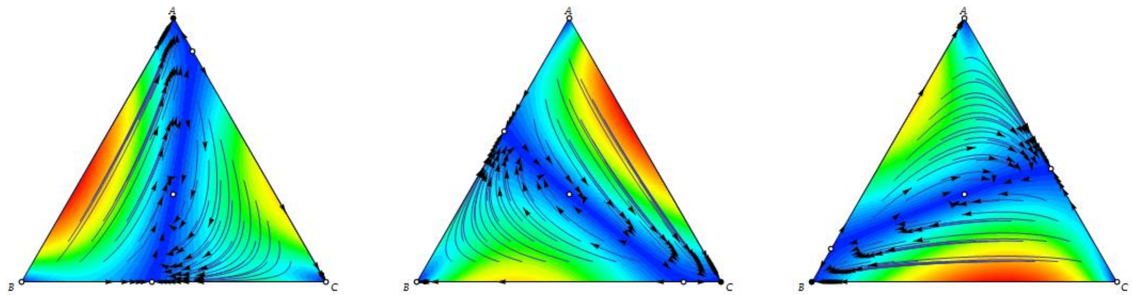


Figure 10. Rotation of eigenvectors in the Zeeman games. Figures are drawn using Dynamo: W. H. Sandholm, E. Dokumaci, and F. Franchetti (2010). Dynamo: Diagrams for Evolutionary Game Dynamics, version 0.2.5.

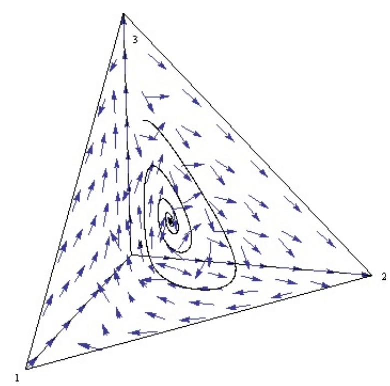


Figure 11. Four-strategy Zeeman game. $\alpha : -2.5,\ \beta : -2.5,\ \gamma : 2,\ \eta : 1.9.$

Thus from the Routh-Hurwitz criterion, we see that eigenvalues λ for A all have negative real parts (except 0 eigenvalue) if and only if

$$\alpha+\beta+\gamma<0,\ \alpha\beta+\beta\gamma+\gamma\alpha+4\eta^2>0,\ 3\alpha\beta\gamma+(6\alpha+2\beta+4\gamma)\eta^2<0,$$
$$\alpha\beta\gamma+\frac{1}{3}(6\alpha+2\beta+4\gamma)\eta^2>(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha+4\eta^2).$$

Using these conditions we exhibit a 4 strategy Zeeman game in Figure 11.

CHAPTER 4

HIERARCHICAL MULTI-SCALE MODELS

4.1 Coarse-Grained Stochastic Processes

4.1.1 Setup

We take $\Lambda \subset \mathbb{Z}^d$ and define a coarse cell $C_k := [k_1, k_1+1) \times [k_2, k_2+1) \times \cdots \times [k_d, k_d+1)$ for $k \in (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d$. We denote a coarse lattice by $\Lambda_C \subset \mathbb{Z}^d$ and identify each cell C_k with a site of Λ_C, so $C_k \sim k$. We suppose that $|\Lambda_c| = M$. There are Q_k sites in each coarse cell C_k, and we set $N := \sum_{k \in \Lambda_C} Q_k$. We assume that an interaction kernel in the microscopic space satisfies

$$\mathcal{W}(x-y) = \bar{\mathcal{W}}(k,l) \text{ for } x \in C_k, y \in C_l \tag{4.1}$$

We will write $\bar{\mathcal{W}}(0) := \bar{\mathcal{W}}(k,k)$ for all $k \in \Lambda_C$, and interpret $\bar{\mathcal{W}}(k,l)$ for $k \neq l$ as a between-group interaction intensity and $\bar{\mathcal{W}}(0)$ as a within-group interaction intensity. Recall that the imitative comparing strategy revision rates: for $s' \in S$

$$c(x, \sigma, s') := \sum_{y \in \Lambda} w(x, y, \sigma, s')\, G(u(x, \sigma^{x,s'}) - u(x, \sigma)).$$

and

$$w(x, y, \sigma, s') = \mathcal{W}_m(x-y)\delta_{\sigma(y)}(\{s'\}).$$

Note that the weight which determines the probability with which agent x imitate y is not necessarily true. Similarly, we assume that $\mathcal{W}_m(x-y)$ satisfies

$$\mathcal{W}_m(x-y) = \bar{\mathcal{W}}_m\,(k,l) \text{ for } x \in C_k, y \in C_l$$

4.1.2 Dynamics

We recall that the state space of the microscopic stochastic process is $\Xi := \{\sigma : \sigma(x) \in S, x \in \Lambda\}$ and

$$\sigma^{x,s}(y) := \begin{cases} \sigma(x) & \text{if } y \neq x \\ s & \text{if } y = x \end{cases}.$$

We define the current payoff of an individual at x when the configuration is σ, $u(x,\sigma)$, and the payoff of an individual at x adopting strategy s, $u(x,\sigma^{x,s})$, as follows:

$$u(x,\sigma) \quad : \quad = \sum_{y\in\Lambda} \mathcal{W}(x-y)a(\sigma(x),\sigma(x)). \tag{4.2}$$

$$u(x,\sigma^{x,s}) \quad : \quad = \sum_{y\in\Lambda} \mathcal{W}(x-y)a(\sigma^{x,s}(x),\sigma^{x,s}(x)). \tag{4.3}$$

Note that equation (4.3) can also be written as

$$u(x,\sigma^{x,s}) = \sum_{\substack{y\in\Lambda \\ y\neq x}} \mathcal{W}(x-y)a(s,\sigma(y)) + \mathcal{W}(0)a(s,s), \text{ for } s \in S$$

and the total utility u includes the self-interaction term (the second term) that gives a payoff $a(s,s)$ when an individual x chooses the strategy s. Note that this term disappears when $\mathcal{W}(0) = 0$ or $a(s,s) = 0$ for all $s \in S$. Later, the addition of the self-interaction term will ensure the detailed balance for the stochastic process of the potential games.

$$Lg(\sigma) := \sum_{s'\in S}\sum_{x\in\Lambda} c(x,\sigma,s')(g(\sigma^{x,s'}) - g(\sigma)) \tag{4.4}$$

for $g \in L^{\infty}(\Xi; \mathbb{R})$. The goal of this section is to aggregate the microscopic generator (4.4) at the coarse cell level and to obtain hierarchical stochastic processes which can capture between group interactions and within group interactions.

4.1.3 Aggregation

First we define a new coarse variable η :

$$\eta_s(k) := \sum_{x \in C_k} \delta_{\sigma(x)}(\{s\})$$

and write $\eta(k) := \left(\eta_1(k), \cdots, \eta_{|S|}(k)\right)^{\mathbf{T}}$ and $\eta = (\eta\left(k^{(1)}\right), \cdots, \eta(k^{(M)}))^{\mathbf{T}}$. Hence the state space is given by

$$\Sigma := \left\{\eta : \sum_{s \in S} \eta_s\left(k^{(l)}\right) = Q_k, k^{(l)} \in \Lambda_C\right\}$$

Sometimes we think elements in Σ as $|S| \times M$ matrices. Next we define a transition coarse variable $\eta^{l,s,s'}$

$$\eta_t^{k,s,s'}(l) := \begin{cases} \eta_t(l) & \text{if } \ l \neq k \text{ or } l = k,\ t \neq s, s' \\ \eta_t(l) - 1 & \text{if } l = k, t = s \\ \eta_t(l) + 1 & \text{if } l = k, t = s' \end{cases} \tag{4.5}$$

:i.e., $\eta^{k,s,s'}$ represent a new state induced by a strategy change of an individual belonging to the coarse cell k from s to s'. Note that we have

$$\begin{aligned} \eta_t^{k,s,s'}(l) &= \eta_t(l) - \delta_s(\{t\}) + \delta_{s'}(\{t\}\} \quad \text{for } l = k \\ \eta_t^{k,s,s'}(l) &= \eta_t(l) \quad \text{for } l \neq k \end{aligned}$$

Next we define a reduction mapping ϕ which will connect the microscopic state space to the coarse state space:

$$\phi : \Xi \to \Sigma,\ \phi(\sigma)(t, l) := \sum_{x \in C_l} \delta_{\sigma(x)}(\{t\}) \quad \text{for } l \in \Lambda_c,\ t \in S$$

Then for $h \in L^{\infty}(\Sigma : \mathbb{R})$, if we define $g := h \circ \phi$, then

$$g \in L^{\infty}(\Xi; \mathbb{R}) \text{ and } g(\sigma) = h(\eta)$$

In addition we have the following lemma.

Lemma 4.1.1 *For $\eta = \phi(\sigma)$, $x \in C_k$ $s' \in S$, we have*

$$g(\sigma^{x,s'}) - g(\sigma) = h(\eta^{k,\sigma(x),s'}) - h(\eta)$$

Proof. Let $x \in C_k$, $s' \in S$ and $\sigma^{x,s'}, \sigma \in \Xi$. Let $s \in S$ fixed. Then for $l = k$,

$$\begin{aligned} \phi(\sigma^{x,s'})(t,l) \quad &: = \sum_{y \in C_l} \delta_{\sigma^{x,s'}(y)}(\{t\}) = \sum_{\substack{y \in C_l \\ y \neq x}} \delta_{\sigma(y)}(\{t\}) + \delta_{s'}(\{t\}) \\ &= \sum_{y \in C_l} \delta_{\sigma(y)}(\{t\}) - \delta_{\sigma(x)}(\{t\}) + \delta_{s'}(\{t\}) = \eta_t^{k,\sigma(x),s'}(l) \end{aligned}$$

and $l \neq k$, $\phi(\sigma^{x,s'})(t,l) = \eta_t(l)$. ■

Next we proceed to find group-level average payoffs.

Lemma 4.1.2 *For $x \in C_k$, we have*

$$u(x, \sigma^{x,s'}) = \sum_{\substack{l \in \Lambda_C \\ l \neq k}} \sum_{s \in S} \bar{\mathcal{W}}(k,l)\, a(s',s)\, \eta_s^{k,\sigma(x),s'}(l) + \sum_{s \in S} \bar{\mathcal{W}}(0)\, a(s',s)\, \eta_s^{k,\sigma(x),s'}(k)$$

Proof. For $x \in C_k$, we have

$$\begin{aligned} u\left(x, \sigma^{x,s'}\right) \quad &: = \sum_{\substack{y \in \Lambda \\ y \neq x}} \mathcal{W}(x-y) a(s', \sigma(y)) + \mathcal{W}(0) a(s', s') \\ &= \sum_{\substack{y \in C_k \\ y \neq x}} \mathcal{W}(x-y)\, a(s', \sigma(y)) + \mathcal{W}(0) a(s', s') + \sum_{\substack{l \in \Lambda_C \\ l \neq k}} \sum_{y \in C_l} \mathcal{W}(x-y)\, a(s', \sigma(y)) \\ &= : I + II. \end{aligned}$$

We compute I as follows:

$$\begin{aligned}
I &= \sum_{y\in C_k} \bar{\mathcal{W}}(0)a(s',\sigma(y)) - \bar{\mathcal{W}}(0)a(s',\sigma(x)) + \bar{\mathcal{W}}(0)a(s',s') \\
&= \bar{\mathcal{W}}(0)\sum_{s\in S} a(s',s)\eta_s(k) - \bar{\mathcal{W}}(0)\sum_{s\in S} a(s',s)\delta_{\sigma(x)}(\{s\}) + \bar{\mathcal{W}}(0)\sum_{s\in S} a(s',s)\delta_{s'}(\{s\}) \\
&= \bar{\mathcal{W}}(0)\sum_{s\in S} a(s',s)\left[\eta_s(k) - \delta_{\sigma(x)}(\{s\}) + \delta_{s'}(\{s\})\right] \\
&= \bar{\mathcal{W}}(0)\sum_{s\in S} a(s',s)\eta_s^{k,\sigma(x),s'}(k).
\end{aligned}$$

Similarly for II, we find

$$\begin{aligned}
II &= \sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{y\in C_l} \bar{\mathcal{W}}(k,l)\, a(s',\sigma(y)) = \sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{y\in C_l}\sum_{s\in S} \bar{\mathcal{W}}(k,l)\, a(s',s)\delta_{\sigma(y)}(\{s\}) \\
&= \sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{s\in S} \bar{\mathcal{W}}(k,l)\, a(s',s)\sum_{y\in C_l}\delta_{\sigma(y)}(\{s\}) = \sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{s\in S} \bar{\mathcal{W}}(k,l)\, a(s',s)\eta_s(l)
\end{aligned}$$

■

From lemma 4.1.2, we define aggregate utilities at the coarse-grained level, a payoff derived from between group interactions $\bar{\mathcal{U}}^B$,a payoff derived from within group interactions $\bar{\mathcal{U}}^W$, a total payoff for a representative agent in a group $\bar{\mathcal{U}}^T$:

$$\begin{aligned}
\bar{\mathcal{U}}^B(k,\eta,s') &:= \sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{s\in S} \bar{\mathcal{W}}(k,l)\, a(s',s)\eta_s(l) \\
\bar{\mathcal{U}}^W(k,\eta,s') &:= \sum_{s\in S} \bar{\mathcal{W}}(0)\, a(s',s)\ \eta_s(k) \\
\bar{\mathcal{U}}^T(k,\eta,s') &:= \bar{\mathcal{U}}^W(k,\eta,s') + \bar{\mathcal{U}}^B(k,\eta,s').
\end{aligned}$$

Then lemma 4.1.2 shows that

$$u(x,\sigma^{x,s'}) = \bar{\mathcal{U}}^T(k,\eta^{k,\sigma(x),s'},s') \quad \text{and} \quad u(x,\sigma) = \bar{\mathcal{U}}^T(k,\eta,\sigma(s))$$

Next using lemma 4.1.1 and lemma 4.1.2, we derive a coarse grained generator for the microscopic generator.

Proposition 4.1.3 *For $h \in L^{\infty}(\Sigma;\mathbb{R})$, $g \in L^{\infty}(\Xi,\mathbb{R})$, the coarse grained generators for the innovative and imitative case are given by*

$$
\begin{aligned}
L_C h(\eta) &= \sum_{k\epsilon\Lambda_C}\sum_{s'\in S}\sum_{s\in S}\eta_s(k)\,\mathbf{c}_C(k,s,s',\eta)\,(h(\eta^{k,s,s'})-h(\eta)) \\
L_C h(\eta) &= \sum_{k\epsilon\Lambda_C}\sum_{s'\in S}\sum_{s\in S}\Bigg[\bar{\mathcal{W}}(0)\eta_{s'}(k)\,\eta_s(k)\,\mathbf{c}_C(k,s,s',\eta) \\
&\quad + \sum_{l\epsilon\Lambda_C:l\neq k}\bar{\mathcal{W}}(k,l)\eta_{s'}(l)\,\eta_s(k)\,\mathbf{c}_C(k,s,s',\eta)\Bigg]\times(h(\eta^{k,s,s'})-h(\eta))
\end{aligned}
$$

where

$$
\mathbf{c}_C(k,s,s',\eta) = G(\bar{\mathcal{U}}^T(k,\eta^{k,s,s'},s') - \bar{\mathcal{U}}^T(k,\eta,s)).
$$

Proof. By noting $\sum_{s\in S} f(s)\delta_{\sigma(x)}(\{s\}) = f(\sigma(x))$, we compute the following.

$$
\begin{aligned}
\sum_{x\in\Lambda}\sum_{s'\in S}u(x,\sigma^{x,s'}) &= \sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\bar{\mathcal{U}}^T(k,\eta^{k,\sigma(x),s'},s') \\
&= \sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\sum_{s''\in S}\bar{\mathcal{U}}^T(k,\eta^{k,s'',s'},s')\delta_{\sigma(x)}(\{s''\}) \\
&= \sum_{k\in\Lambda_C}\sum_{s'\in S}\sum_{s''\in S}\bar{\mathcal{U}}^T(k,\eta^{k,s'',s'},s')\sum_{x\in C_k}\delta_{\sigma(x)}(\{s''\}) \\
&= \sum_{k\in\Lambda_C}\sum_{s'\in S}\sum_{s''\in S}\bar{\mathcal{U}}^T(k,\eta^{k,s'',s'},s')\eta_{s''}(k) \\
&= \sum_{k\in\Lambda_C}\sum_{s'\in S}\sum_{s\in S}\bar{\mathcal{U}}^T(k,\eta^{k,s,s'},s')\eta_s(k)
\end{aligned}
$$

Similarly we find

$$
\sum_{x\in\Lambda}\sum_{s'\in S}u(x,\sigma) = \sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\bar{\mathcal{U}}^T(k,\eta,\sigma(x)) = \sum_{k\in\Lambda_C}\sum_{s'\in S}\sum_{s\in S}\bar{\mathcal{U}}^T(k,\eta,s)\eta_s(k).
$$

Then using the generator, we can treat $h\left(\eta^{k,\sigma(x),s'}\right)$ similarly and we obtain the desired result. For the imitative case, we consider the following term:

$$\begin{aligned}
&\sum_{x\in\Lambda}\sum_{s'\in S} w\left(x,y,\sigma,s'\right) H\left(x,\sigma,s'\right) \\
= &\sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\sum_{y\in\Lambda} \mathcal{W}\left(x-y\right)\delta_{\sigma(y)}(\{s'\})H\left(x,\sigma,s'\right) \\
= &\sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\sum_{y\in C_k} \bar{\mathcal{W}}_m\left(0\right)\delta_{\sigma(y)}(\{s'\})H\left(x,\sigma,s'\right) \\
&+\sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{y\in C_l} \bar{\mathcal{W}}_m\left(k,l\right)\delta_{\sigma(y)}(\{s'\})H\left(x,\sigma,s'\right) \\
= &\sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S} \bar{\mathcal{W}}\left(0\right)\eta_k\left(s'\right)H\left(x,\sigma,s'\right) + \sum_{k\in\Lambda_C}\sum_{x\in C_k}\sum_{s'\in S}\sum_{\substack{l\in\Lambda_C\\ l\neq k}} \bar{\mathcal{W}}\left(k,l\right)\eta_l\left(s'\right)H\left(x,\sigma,s'\right)
\end{aligned}$$

Hence we see that manipulation can be separated and obtain the desired result. ■

Observe that when we set $Q_k = 1$ for all k, we have $\Lambda_C = \Lambda$ and there is only self-interaction within group interaction. In this setting

$$\begin{aligned}
\bar{\mathcal{U}}^B(k,\eta,s') \quad &: \quad = \sum_{\substack{l\in\Lambda_C\\ l\neq k}}\sum_{s\in S}\bar{\mathcal{W}}\left(k,l\right)a(s',s)\eta_s\left(l\right) = \sum_{\substack{y\in\Lambda\\ y\neq x}}\sum_{s\in S}\mathcal{W}\left(x-y\right)a(s',s)\eta_s\left(y\right) \\
&= \quad \sum_{\substack{y\in\Lambda\\ y\neq x}}\mathcal{W}\left(x-y\right)a(s',\sigma\left(y\right)) \\
\bar{\mathcal{U}}^W(k,\eta,\sigma(x)) \quad &: \quad = \sum_{s\in S}\bar{\mathcal{W}}\left(0\right)a\left(\sigma(x),s\right)\ \eta_s\left(k\right) = \sum_{s\in S}\mathcal{W}(0)a(\sigma(x),s)\delta_{\sigma(x)}(\{s\}) \\
&= \quad \bar{\mathcal{W}}(0)a(\sigma(x),\sigma(x)) \\
\bar{\mathcal{U}}^W(k,\eta^{k,\sigma(x),s'},s') \quad &: \quad = \sum_{s\in S}\bar{\mathcal{W}}\left(0\right)a\left(s',s\right)\ \eta_s^{k,\sigma(x),s'}\left(k\right) \\
&= \quad \sum_{s\in S}\bar{\mathcal{W}}\left(0\right)a\left(s',s\right)\ \left(\eta_s\left(k\right)-\delta_{\sigma(x)}(\{s\})+\delta_{s'}(\{s\}\right) \\
&= \quad \bar{\mathcal{W}}(0)a(s',s')
\end{aligned}$$

and from this we have

$$\bar{\mathcal{U}}^T(k,\eta,\sigma(x)) = u(x,\sigma), \quad \bar{\mathcal{U}}^T(k,\eta^{k,\sigma(x),s'},\sigma(x)) = u(x,\sigma^{x,s'})$$

so we recover our original microscopic payoffs for sites. In this case also the generator for the innovative case

$$\begin{aligned} & \sum_{k \epsilon \Lambda_C} \sum_{s' \in S} \sum_{s \in S} \eta_s (k) \, G(\bar{\mathcal{U}}^T (k, \eta^{k,s,s'}, s') - \bar{\mathcal{U}}^T (k, \eta, s)) \\ = & \sum_{x \epsilon \Lambda} \sum_{s' \in S} \sum_{s \in S} \delta_{\sigma(x)} (\{s\}) \, G(\bar{\mathcal{U}}^T (k, \eta^{k,s,s'}, s') - \bar{\mathcal{U}}^T (k, \eta, s)) \\ = & \sum_{x \epsilon \Lambda} \sum_{s' \in S} G(u(x, \sigma^{x,s'}) - u(x, \sigma)) \end{aligned}$$

so again reproduce the microscopic generator. On the other hand we consider the case $Q_k = N$ for all k. In this case there is no between group effect and group level payoff from within group interaction $\bar{\mathcal{U}}^W (k, \eta, s') = \sum_{s \in S} \bar{\mathcal{W}} (0) \, a (s', s) \;\; \eta_s (k)$ becomes the average payoff using strategy s'. In this case it is easy to see that the coarse-grained generator become the same as the generator for the uniform interaction case. When $1 < Q_k < N$, this model contains both within group effect (uniform interaction) and between group effect (spatial interaction) and, hence capture both locally homogenous interaction but globally heterogenous interaction.

Remark. When we take $a(k, l, s, s')$ for $x \in C_k$, $y \in C_l$, $s \in S_R$, $s' \in S_C$ as an underlying payoff instead of $a(s, s')$, the same computations still hold. In this case,

$$u(x, \sigma) := \sum_{\substack{y \in \Lambda \\ y \neq x}} \mathcal{W}(x - y) a(x, y, s, \sigma(y)) + \mathcal{W}(0) a(x, x, s, s).$$

Then group payoffs can be written:

$$\begin{aligned} \bar{\mathcal{U}}^B (k, \eta, s') \; & : = \sum_{\substack{l \in \Lambda_C \\ l \neq k}} \sum_{s \in S} \bar{\mathcal{W}}(k, l) a(k, l, s', s) \eta_s(l) \\ \bar{\mathcal{U}}^W (k, \eta, s') \; & : = \sum_{s \in S} \bar{\mathcal{W}}(k, k) a(k, k, s', s) \eta_s(l) \end{aligned}$$

Under this setting bimatrix asymmetric game can be regarded as the case where $\Lambda = \{1, 2\}$ and $a(1, 1, s', s) = a(2, 2, s', s)$ for all s' and s.In addition the matrix model of evolutionary game with two groups is a special case of the Hierarchical coarse-grained stochastic processes (See Cressman, 1995).

4.1.4 Invariant Measures for the coarse-grained processes

Now we will find the expressions invariant measures for the coarse-grained process when the detailed balance condition is satisfied. First we find an invariant measure for the microscopic level when the detailed balance condition is satisfied. To do we introduce an energy function H at the microscopic level:

$$H(\sigma) := \frac{1}{2}\sum_{z\in\Lambda}\sum_{y\in\Lambda}\mathcal{W}(y-z)a(\sigma(y),\sigma(z)) + \frac{1}{2}\mathcal{W}(0)\sum_{y\in\Lambda}a(\sigma(y),\sigma(y)) \tag{4.6}$$

Then, we have the following lemma.

Lemma 4.1.4 *Suppose that a is a symmetric matrix. Then for all x, k*

$$H(\sigma^{x,k}) - H(\sigma) = u(x,\sigma^{x,k}) - u(x,\sigma)$$

First note that for $z \neq x$ and $y \neq x$, $a(\sigma^{x,k}(y),\sigma^{x,k}(z)) = a(\sigma(y),\sigma(z))$ and $a(\sigma^{x,k}(y),\sigma^{x,k}(y)) = a(\sigma(y),\sigma(y))$. Thus we have

$$\begin{aligned}
& H(\sigma^{x,k}) - H(\sigma) \\
= \;& \frac{1}{2}\sum_{z\in\Lambda}\sum_{y\in\Lambda}\mathcal{W}(y-z)a(\sigma^{x,k}(y),\sigma^{x,k}(z)) + \frac{1}{2}\mathcal{W}(0)\sum_{y\in\Lambda}a(\sigma^{x,k}(y),\sigma^{x,k}(y)) \\
& -\frac{1}{2}\sum_{z\in\Lambda}\sum_{y\in\Lambda}\mathcal{W}(y-z)a(\sigma(y),\sigma(z)) - \frac{1}{2}\mathcal{W}(0)\sum_{y\in\Lambda}a(\sigma(y),\sigma(y)) \\
= \;& \frac{1}{2}\sum_{z\in\Lambda,z\neq x}\mathcal{W}(x-z)a(k,\sigma^{x,k}(z)) - \frac{1}{2}\sum_{z\in\Lambda,z\neq x}\mathcal{W}(x-z)a(\sigma(x),\sigma(z)) \\
& +\frac{1}{2}\sum_{y\in\Lambda,y\neq x}\mathcal{W}(y-x)a(\sigma^{x,k}(y),k) - \frac{1}{2}\sum_{y\in\Lambda,y\neq x}\mathcal{W}(y-x)a(\sigma(y),\sigma(x)) \\
& +\frac{1}{2}\mathcal{W}(0)a(k,k) - \frac{1}{2}\mathcal{W}(0)a(\sigma(x),\sigma(x)) + \frac{1}{2}\mathcal{W}(0)a(k,k) - \frac{1}{2}\mathcal{W}(0)a(\sigma(x),\sigma(x)) \\
= \;& \sum_{y\in\Lambda,y\neq x}\mathcal{W}(x-y)a(k,\sigma^{x,k}(y)) - \sum_{y\in\Lambda,y\neq x}\mathcal{W}(x-y)a(\sigma(x),\sigma(y)) \quad \text{(by the symmetry of } A) \\
= \;& u(x,\sigma^{x,k}) - u(x,\sigma) \quad \text{(From the definition of } u)
\end{aligned}$$

Next we find the expression for the equilibrium measure. Let $d\rho$ be a uniform measure over a strategy set: i.e.,

$$\rho\left(\sigma(x)=k\right)=\frac{1}{|S|}\quad \text{for all } k\in S,\ \text{for all } x\in\Lambda$$

Then we define a prior measure over Ξ, $P_0(d\sigma)$: $P_0(d\sigma)=\bigotimes_{x\in\Lambda_n}\rho(d\sigma)$ and define a Gibbs measure P_β :

$$P_\beta(d\sigma):=\frac{1}{Z}\exp(\beta H(\sigma))P_0(d\sigma) \tag{4.7}$$

where $Z:=\int_\Lambda \exp(\beta H(\sigma))P_0(d\sigma)$. We recall that the innovative and comparing rate satisfies

$$\frac{c(x,\sigma^{x,k},\sigma(x))}{c(x,\sigma,k)}=\frac{G(u(x,(\sigma^{x,k})^{x,\sigma(x)})-u(x,\sigma^{x,k})}{G(u(x,\sigma^{x,k})-u(x,\sigma))}$$

and suppose that

$$\frac{G(t)}{G(-t)}=\exp(\beta t).$$

Note we have $(\sigma^{x,k})^{x,\sigma(x)}=\sigma$.Thus from lemma 4.1.4 we have

$$\begin{aligned}\frac{P_\beta(\{\sigma^{x,k}\})}{P_\beta(\{\sigma\})}\frac{c(x,\sigma^{x,k},\sigma(x))}{c(x,\sigma,k)} &= \exp(\beta(H(\sigma^{x,k})-\beta H(\sigma)))\exp(\beta(u(x,\sigma)-u(x,\sigma^{x,k})))\\ &= 1\end{aligned}$$

The the microscopic process is reversible with respect to $P(d\sigma)$.

Next we will find the expressions for the invariant measures of the coarse-grained processes; we will derive this first by the aggregation of (4.7) and then check the obtained measures directly at the coarse level.

First by aggregating (4.6) under (4.1), we obtain

$$\begin{aligned}H(\sigma) &= \frac{1}{2}\sum_{l\in\Lambda_C}\sum_{k\in\Lambda_C}\sum_{s\in S}\sum_{s'\in S}\bar{\mathcal{W}}(k,l)\eta_s(k)a(s,s')\eta_{s'}(l)+\frac{1}{2}\bar{\mathcal{W}}(0)\sum_{k\in\Lambda_C}\sum_{s\in S}a(s,s)\eta_s(k)\\ &=: H^C(\eta)\end{aligned}$$

Thus Lemma 4.1.2 and 4.1.4 give

$$H^C(\eta^{k,s,s'})-H^C(\eta)=\bar{\mathcal{U}}^T\left(k,\eta^{k,s,s'},s'\right)-\bar{\mathcal{U}}^T\left(k,\eta,s\right).$$

We also aggregate the prior measure $P_0(d\sigma)$ and obtain

$$P_0^C(\{\eta\}) = \prod_{l\in\Lambda_C} \frac{Q_l!}{(Q_l\eta_1(l))!\cdots(Q_l\eta_{|S|}(l))!}\frac{1}{|S|^{Q_l}}.$$

So we can define

$$P_\beta^C(d\eta) = \frac{1}{Z^C}\exp(\beta H^C(\eta))P_0^C(d\eta). \tag{4.8}$$

Then (4.8) is the invariant measure for the coarse-grained process. In fact we can verify this directly.

Proposition 4.1.5 *Suppose that the innovative and comparing strategy revision rates is given by* $c_C(k,s,s',\eta) := \eta_s(k)G(\bar{\mathcal{U}}^T(k,\eta^{k,s,s'},s') - \bar{\mathcal{U}}^T(k,\eta,s))$ *and* G *satisfies* $G(t)/G(-t) = \exp(\beta t)$ *Then the Markov process defined by*

$$L_C g(\eta) = \sum_{k\epsilon\Lambda_C}\sum_{s'\in S}\sum_{s\in S} c_C(k,s,s',\eta)(g(\eta^{k,s,s'}) - g(\eta)), \qquad \textit{for } g\in L^\infty(\Delta^N;\mathbb{R})$$

is reversible with respect to P_β^C.

Proof. First we write $\alpha(\eta,\eta^{k,s,s'}) := c_C(k,s,s',\eta)$. Then we need to show that

$$P_\beta^C(\{\eta\})\alpha(\eta,\eta\prime) = P_\beta^C(\{\eta'\})\alpha(\eta',\eta) \text{ for all } \eta,\eta'\in\Sigma.$$

Let $\eta,\eta'\in\Sigma$. If $\alpha(\eta,\eta')=0$, then from the definition of α, we have $\alpha(\eta',\eta)=0$ and so we are done. So suppose that $\alpha(\eta,\eta')>0$. Then again from the definition of α, there exists $s,s'\in S$ and $k\in\Lambda_C$ such that $\eta'=\eta^{k,s,s'}$ and $\eta_s(k)>0$ and $\eta_{s'}^{k,s,s'}(k)>0$. So

$$\begin{aligned}
\frac{\alpha(\eta,\eta^{k,s,s'})}{\alpha(\eta^{k,s,s'},\eta)} &= \frac{\alpha(\eta,\eta^{k,s,s'})}{\alpha(\eta^{k,s,s'},(\eta^{k,s,s'})^{k,s',s})} = \frac{\eta_s(k)F(\bar{\mathcal{U}}^T\left(k,\eta^{k,s,s'},s'\right) - \bar{\mathcal{U}}^T(k,\eta,s))}{(\eta_{s'}^{k,s,s'})(k)F(\bar{\mathcal{U}}^T(k,(\eta^{k,s,s'})^{k,s',s},s') - \bar{\mathcal{U}}^T(k,\eta^{k,s,s'},s))} \\
&= \frac{\eta_s(k)}{\eta_{s'}(k)+\frac{1}{Q_k}}\frac{F(\bar{\mathcal{U}}^T\left(k,\eta^{k,s,s'},s'\right) - \bar{\mathcal{U}}^T(k,\eta,s))}{F(\bar{\mathcal{U}}^T(k,\eta,s') - \bar{\mathcal{U}}^T(k,\eta^{k,s,s'},s))} \\
&= \frac{Q_k\eta_s(k)}{Q_k\eta_{s'}(k)+1}\exp(\beta(\bar{\mathcal{U}}^T\left(k,\eta^{k,s,s'},s'\right) - \bar{\mathcal{U}}^T(k,\eta,s))) \\
&= \frac{Q_k\eta_s(k)}{Q_k\eta_{s'}(k)+1}\exp(\beta(H^C(\eta^{k,s,s'}) - H^C(\eta)))
\end{aligned}$$

Also

$$\frac{P^C_\beta(\{\eta\})}{P^C_\beta(\{\eta^{k,s,s'}\})} = \exp(H^C(\eta) - H^C(\eta^{k,s,s'}))\frac{P^C_0(\{\eta\})}{P^C_0(\{\eta^{k,s,s'}\})}$$

and

$$\begin{aligned}\frac{P^C_0(\{\eta\})}{P^C_0(\{\eta^{k,s,s'}\})} &= \frac{(Q_k\eta^{k,s,s'}_s(k))!(Q_k\eta^{k,s,s'}_{s'}(k))!}{(Q_k\eta_s(k))!(Q_k\eta_{s'}(k))!} = \frac{(Q_k\eta_s(k)-1)!(Q_k\eta_{s'}(k)+1)!}{(Q_k\eta_s(k))!(Q_k\eta_{s'}(k))!} \\ &= \frac{Q_k\eta_{s'}(k)+1}{Q_k\eta_s(k)}\end{aligned}$$

Therefore we have

$$\frac{P^C_\beta(\{\eta\})}{P^C_\beta(\{\eta^{k,s,s'}\})}\frac{\alpha(\eta,\eta^{k,s,s'})}{\alpha(\eta^{k,s,s'},\eta)} = 1$$

and obtain the desired result. ■

4.2 Deterministic Approximations of Hierarchical Stochastic Processes

4.2.1 Approximations of Stochastic Processes

In this section we suppose that $Q_k = Q_l := Q$ for all $k, l \in \Lambda_C$ and consider deterministic approximations of hierarchical stochastic processes when and $Q \to \infty$, $N \to \infty$. In this section we rescale the state space

$$\Sigma := \left\{\eta : \sum_{s\in S}\eta_s\left(k^{(l)}\right) = 1, k^{(l)} \in \Lambda_C\right\}$$

and rescale the time

$$L_C g(\eta) = \sum_{k\epsilon\Lambda_C}\sum_{s'\in S}\sum_{s\in S} Qc_C(k,s,s',\eta)(g(\eta^{k,s,s'}) - g(\eta)) \tag{4.9}$$

The result follows from the well-known result of ODE approximations of the stochastic processes. We will use the approximation theorems by Kurtz (1982). We will provide heuristic explanations (see Kurtz (1982); Ethier and Kurtz (1986) for the detailed discussion and proofs; also see Gardiner (2004); Van Kampen (1981).) We consider here the innovative and comparing case, however the similar computation holds for the imitative comparing case. First we represent the system more succinctly by using the following notations. First we define a set $\mathcal{I}$

$$\mathcal{I} := \{l \in \mathbb{Z}^d = \mathbb{Z}^{|S|+|\Lambda_C|} : l(s,k) = -1, \quad l(s',k) = 1 \text{ for some } s \text{ and } s' \in S \text{ and some } k \in \Lambda_C\}.$$

For example, when $S = \{1,2\}$ and $\Lambda_C = \{k_1, k_2\}$,

$$\mathcal{I} = \left\{ \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right], \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right] \right\}.$$

Then $|\mathcal{I}| = |S| \times (|S|-1) \times |\Lambda_C|$. Also for $l \in \mathcal{I}$, there exists unique k, s, s' such that $l(s,k) = -1$ and $l(s',k) = 1$. Thus $\eta^{k,s,s'}$ can be written as $\eta^{k,s,s'} = \eta + \frac{1}{Q} l$ for some l. So we can define

$$\beta_l(\eta) := \eta_s(k)\mathbf{c}_C(k,s,s',\eta) = \eta_s(k) F(\bar{\mathcal{U}}^T(k, \eta + \frac{1}{Q} l, s') - \bar{\mathcal{U}}^T(k,\eta,s))$$

and the generator for the coarse-grained process (4.9) can be written

$$Lf(\eta) = \sum_{l \in \mathcal{I}} \beta_l(\eta) Q(f(\eta + \frac{1}{Q} l) - f(\eta)).$$

Then from Kurtz (1981), we have

$$\mathcal{N}(t) = \mathcal{N}(0) + \sum_{l \in \mathcal{I}} \frac{1}{Q} l Y_l (Q \int_0^t \beta_l(\mathcal{N}(s)) ds$$

where $\{Y_l(t)\}_l$ are independent Poisson processes. We define the compensated Poisson process $\hat{Y}_l(t) := Y_l(t) - t$ and obtain

$$\mathcal{N}(t) = \mathcal{N}(0) + \sum_{l \in \mathcal{I}} \frac{1}{Q} l \hat{Y}_l (Q \int_0^t \beta_l(\mathcal{N}(s)) ds + \sum_{l \in \mathcal{I}} l \int_0^t \beta_l(\mathcal{N}(s)) ds. \qquad (4.10)$$

Then since $\lim_{Q\to\infty} \sup_{t \le T} \frac{1}{Q} \left| \hat{Y}_l(Qt) \right| = 0$ a.s. for all $T > 0$. We obtain the following ODE:

$$\frac{d\eta}{dt} = \sum_{l \in \mathcal{I}} l \beta_l(\eta(t)) := F(\eta(t)). \qquad (4.11)$$

Also since $\frac{1}{\sqrt{Q}}\hat{Y}_l(Q\cdot) \Rightarrow W_l(\cdot)$ in distribution where W_l is an independent standard Brownian motion. This suggests approximating $\mathcal{N}$ by a solution of

$$\mathcal{N}_D(t) = \mathcal{N}(0) + \sum_{l\in\mathcal{I}} \frac{1}{\sqrt{Q}} l W_l(\int_0^t \beta_l(\mathcal{N}_D(s)))ds + \int_0^t F(\mathcal{N}_D(s))ds.$$

Note from the time change, we have $W_l(\int_0^t \beta_l(\mathcal{N}_D(s))) = \int_0^t \sqrt{\beta_l(\eta(s))}dB_l(s)$. Thus we find the following diffusion approximation:

$$d\mathcal{N}_D = F(\mathcal{N}_D(t))dt + \frac{1}{\sqrt{Q}} \sum_{l\in\mathcal{I}} l\sqrt{\beta_l(\mathcal{N}_D(t))}dB_l, \tag{4.12}$$

and this approximation is justified by the following theorem (Theorem 8.4 in Kurtz (1982)).

Proposition 4.2.1 (Diffusion Approximation) *Suppose that* $\mathcal{N}_D(t)$ *satisfies (4.12). Then*

$$\mathcal{N}(t) = \mathcal{N}_D(t) + \mathcal{O}\left(\frac{\log Q}{Q}\right).$$

Next, by subtracting ODE (4.11) from (4.12), we obtain

$$\begin{aligned}\mathcal{N}_D(t) - \eta(t) &= \mathcal{N}(0) - \eta(0) + \sum_{l\in\mathcal{I}} \frac{1}{\sqrt{Q}} l W_l(\int_0^t \beta_l(\mathcal{N}_D(s)))ds + \int_0^t F(\mathcal{N}_D(s)) - F(\eta(s))ds \\ &\approx \sum_{l\in\mathcal{I}} \frac{1}{\sqrt{Q}} l W_l(\int_0^t \beta_l(\eta(s)))ds + \int_0^t \partial F(\eta(s))(\mathcal{N}_D(s) - \eta(s))ds\end{aligned}$$

where ∂F is the Jacobian matrix of F. Thus we have

$$\sqrt{Q}(\mathcal{N}_D(t) - \eta(t)) \approx \int_0^t \partial F(\eta(s))\sqrt{Q}(\mathcal{N}_D(s) - \eta(s))ds + \sum_{l\in\mathcal{I}} l W_l(\int_0^t \beta_l(\eta(s)))ds. \tag{4.13}$$

Since $\mathcal{N}_D$ approximates $\mathcal{N}(t)$, we define

$$\mathcal{V} := \sqrt{Q}(\mathcal{N} - \eta)$$

and expect that the limiting distribution of V_Q would satisfy (4.13) (Theorem 8.2 in Kurtz (1982)).

Proposition 4.2.2 (Centeral Limit Theorem) *Suppose that $\mathcal{V}$ is the solution of*

$$\begin{cases} \frac{d\eta}{dt} = F(\eta) \\ d\mathcal{V} = \partial F(\eta(t))\mathcal{V}dt + \sum_{l\in\mathcal{I}} l\sqrt{\beta_l(\eta(t))}dBl \end{cases}.$$

Then

$$\sqrt{Q}(\mathcal{N} - \eta) \Rightarrow \mathcal{V}.$$

These approximation methods provide a nice and succinct representation, however it does not show the specific forms of equations. To find the more concrete expressions for the approximation equations, we proceed as follows. First, from a martingale representation, we can approximate the Markov process $\mathcal{N}(t)$,

$$\begin{aligned} & h\left(\eta\left(t\right)\right) - h\left(\eta\left(0\right)\right) \approx \\ & + \int_0^t d\tau \sum_{k\epsilon\Lambda_C} \sum_{s'\in S} \sum_{s\in S} \bar{\mathcal{W}}_m(0)\eta_{s'}\left(k\right)\eta_s\left(k\right)\mathbf{c}_C\left(k,s,s',\eta\right)Q(h(\eta^{k,s,s'}) - h(\eta)) \\ & + \int_0^t d\tau \sum_{k\epsilon\Lambda_C} \sum_{s'\in S} \sum_{s\in S} \sum_{l\epsilon\Lambda_C: l\neq k} \bar{\mathcal{W}}_m(k,l)\eta_{s'}\left(l\right)\eta_s\left(k\right)\mathbf{c}_C\left(k,s,s',\eta\right)Q(h(\eta^{k,s,s'}) - h\left(\eta\right)) \\ : & \quad = I + II \end{aligned}$$

Note that this approximation is essentially the same as (4.10). Then we use an evaluation map, $h : \Sigma \to \mathbb{R},\ \eta \mapsto \eta_{s''}\left(m\right)$ for a given $s'' \in S$ and $m \in \Lambda_C$. Then we find

$$\begin{aligned} \frac{d(II)}{dt} &= \sum_{k\epsilon\Lambda_C} \sum_{s'\in S} \sum_{s\in S} \sum_{l\epsilon\Lambda_C: l\neq k} \bar{\mathcal{W}}_m(k,l)\eta_{s'}\left(l\right)\eta_s\left(k\right)\mathbf{c}_C\left(k,s,s',\eta\right)Q(\eta_{s''}^{k,s,s'}\left(m\right) - \eta_{s''}\left(m\right)) \\ &= \sum_{l\epsilon\Lambda_C: l\neq m} \sum_{s'\in S} \bar{\mathcal{W}}_m(m,l)\eta_{s'}\left(l\right)\eta_{s''}\left(m\right)\mathbf{c}_C\left(m,s'',s',\eta\right)(-1) \\ &\quad + \sum_{l\epsilon\Lambda_C: l\neq m} \sum_{s\in S} \bar{\mathcal{W}}_m(m,l)\eta_{s''}\left(l\right)\eta_s\left(k\right)\mathbf{c}_C\left(k,s,s'',\eta\right)(+1) \end{aligned}$$

So by changing notations $m \to k$, $s'' \to s$, and $s \to s'$ the second part of ODE for $\eta_s\left(k\right)$ is given by

$$\sum_{l\epsilon\Lambda_C: l\neq k} \sum_{s'\in S} \bar{\mathcal{W}}_m(k,l)\eta_s\left(l\right)\eta_{s'}\left(k\right)\mathbf{c}_C\left(k,s',s,\eta\right) - \sum_{l\epsilon\Lambda_C: l\neq k} \sum_{s'\in S} \bar{\mathcal{W}}_m(k,l)\eta_{s'}\left(l\right)\eta_s\left(k\right)\mathbf{c}_C\left(k,s,s',\eta\right)$$

Similarly we find

$$\frac{d(I)}{dt} = \sum_{s' \in S} \bar{\mathcal{W}}_m(0)\eta_s(k)\,\eta_{s'}(k)\,\mathbf{c}_C(k, s', s, \eta) - \sum_{s' \in S} \bar{\mathcal{W}}_m(0)\eta_{s'}(k)\,\eta_s(k)\,\mathbf{c}_C(k, s, s', \eta).$$

Thus we obtain the following ODE equations:

- Innovative and comparing case:

$$\frac{d\eta_s(k)}{dt} = \sum_{s' \in S} \mathbf{c}_C(k, s', s, \eta)\,\eta_{s'}(k) - \eta_s(k) \sum_{s' \in S} \mathbf{c}_C(k, s, s', \eta) \text{ for each } k \in \Lambda_C,\ s \in S \tag{4.14}$$

- Imitative and comparing case:

$$\begin{aligned} &\frac{d\eta_s(k)}{dt} \\ = \;& \bar{\mathcal{W}}_m(0)\,\eta_s(k) \sum_{s' \in S} \mathbf{c}_C(k, s', s, \eta)\eta_{s'}(k) - \bar{\mathcal{W}}_m(0)\,\eta_s(k) \sum_{s' \in S} \mathbf{c}_C(k, s, s', \eta)\eta_{s'}(k) \\ + \;& \sum_{l \neq k} \bar{\mathcal{W}}_m(k,l)\eta_s(l) \sum_{s' \in S} \mathbf{c}_C(k, s', s, \eta)\eta_{s'}(k) - \sum_{l \neq k} \bar{\mathcal{W}}_m(k,l)\eta_s(k) \sum_{s' \in S} \mathbf{c}_C(k, s, s', \eta)\eta_{s'}(l) \end{aligned} \tag{4.15}$$

where $\mathbf{c}_C(k, s, s', \eta) = G(\bar{\mathcal{U}}^T(k, \eta, s') - \bar{\mathcal{U}}^T(k, \eta, s))$.

Also it is easy to see that (4.14) and (4.15) are the same ODEs as (4.11).

Next we will consider the diffusion approximation. Recall that the innovative coarse-grained generator is given by

$$L_C h(\eta) = \sum_{k \epsilon \Lambda_C} \sum_{s' \in S} \sum_{s \in S} \eta_s(k)\,\mathbf{c}_C(k, s, s', \eta)\,Q_k(h(\eta^{k,s,s'}) - h(\eta)).$$

We expand h around η and obtain:

$$h(\eta^{k,s,s'}) - h(\eta) \approx \frac{1}{N_2}(\mathbf{e}_{s'} - \mathbf{e}_s) \cdot \bigtriangledown_k h + \frac{1}{2}\frac{1}{N_2^2}(\mathbf{e}_{s'} - \mathbf{e}_s) \cdot \bigtriangledown_k^2 h\ (\mathbf{e}_{s'} - \mathbf{e}_s)$$

where $(\bigtriangledown_k h)_i = \partial h / \partial \eta_i(k)$, $(\bigtriangledown_k^2 h)_{ij} = \partial^2 h / \partial \eta_i(k) \partial \eta_j(k)$, $\mathbf{e}_s$ is a $|S|-$dimensional standard basis. So we define for $f \in L^\infty(\Sigma; \mathbb{R})$

$$Lf(\rho) = \sum_{k \epsilon \Lambda_C} \sum_{s' \in S} \sum_{s \in S} \eta(s)\mathbf{c}_C(k, s, s', \eta)((\mathbf{e}_{s'} - \mathbf{e}_s) \cdot \bigtriangledown_k f + \frac{1}{2}\frac{1}{N_2}(\mathbf{e}_{s'} - \mathbf{e}_s) \cdot \bigtriangledown_k^2 f\ (\mathbf{e}_{s'} - \mathbf{e}_s)) \tag{4.16}$$

Then the generator (4.16) defines an Ito diffusion process. To find the explicit expression for this process, we define a coordinate map $\phi_s : \mathbb{R}^{|S|} \to \mathbb{R}$, $\phi(\mathbf{a}) = \mathbf{a}(s)$. Then for given $k \in \Lambda_C$

$$\phi_s(\sum_{s'\in S}\sum_{s''\in S}\eta_{s''}(k)\mathbf{c}_C(k,s'',s',\eta)(\mathbf{e}_{s'}-\mathbf{e}_{s''})) = \sum_{s''\in S}\eta_{s''}(k)\mathbf{c}_C(k,s'',s,\eta)-\sum_{s'\in S}\eta_s(k)\mathbf{c}_C(k,s,s',\eta)$$

which gives us a drift term in the diffusion process (ODE part, (4.14) and (4.15)). Our next goal is to find a $|S| \times |S|$ matrix M such that

$$\sum_{s'\in S}\sum_{s\in S}\eta(s)\mathbf{c}_C(k,s,s',\eta)(\mathbf{e}_{s'}-\mathbf{e}_s)\cdot \bigtriangledown_k^2 f\ (\mathbf{e}_{s'}-\mathbf{e}_s) = \sum_{s'\in S}\sum_{s\in S}(M)_{s,s'}\frac{\partial^2 f}{\partial\eta_s(k)\partial\eta_{s'}(k)}. \tag{4.17}$$

First we show the following lemma.

Lemma 4.2.3 *We have*

$$(\mathbf{e}_{s'}-\mathbf{e}_s)\cdot \bigtriangledown_k^2 f\ (\mathbf{e}_{s'}-\mathbf{e}_s) = \sum_{i\in S}\sum_{j\in S}((\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}})_{i,j}\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)}.$$

Proof. We denote the dot product between $\mathbf{a}$ and $\mathbf{b}$ by $\langle \mathbf{a}, \mathbf{b}\rangle$ more explicitly. We first note that for $\mathbf{x} = (\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}}$, $(\mathbf{x}\mathbf{x}^{\mathbf{T}})_{i,j} = \mathbf{x}_i\mathbf{x}_j$.Then we have

$$\begin{aligned}
\left\langle(\mathbf{e}_{s'}-\mathbf{e}_s), \bigtriangledown_k^2 f\ (\mathbf{e}_{s'}-\mathbf{e}_s)\right\rangle &= \sum_i\sum_j\left\langle(\mathbf{e}_{s'}-\mathbf{e}_s), \mathbf{e}_i\mathbf{e}_j^T(\mathbf{e}_{s'}-\mathbf{e}_s)\right\rangle\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)} \\
&= \sum_i\sum_j\left\langle(\mathbf{e}_{s'}-\mathbf{e}_s), \mathbf{e}_i\left\langle\mathbf{e}_{s'}-\mathbf{e}_s, \mathbf{e}_j\right\rangle\right\rangle\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)} \\
&= \sum_i\sum_j\left\langle(\mathbf{e}_{s'}-\mathbf{e}_s), \mathbf{e}_i\right\rangle\left\langle\mathbf{e}_{s'}-\mathbf{e}_s, \mathbf{e}_j\right\rangle\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)} \\
&= \sum_i\sum_j((\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}})_{i,j}\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)}
\end{aligned}$$

■

Thus by applying Lemma 4.2.3, for $\alpha(s,s') := \eta(s)\mathbf{c}_C(k,s,s',\eta)$, we have

$$\begin{aligned}
&\sum_{s'\in S}\sum_{s\in S}\alpha(s,s')(\mathbf{e}_{s'}-\mathbf{e}_s)\cdot\bigtriangledown_k^2 f(\mathbf{e}_{s'}-\mathbf{e}_s)\\
= &\sum_{s'\in S}\sum_{s\in S}\alpha(s,s')\sum_{i\in S}\sum_{j\in S}((\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}})_{i,j}\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)}\\
= &\sum_{i\in S}\sum_{j\in S}\sum_{s'\in S}\sum_{s\in S}\alpha(s,s')((\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}})_{i,j}\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)}\\
= &\sum_{i\in S}\sum_{j\in S}\left[\sum_{s'\in S}\sum_{s\in S}\alpha(s,s')(\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}}\right]_{i,j}\frac{\partial^2 f}{\partial\eta_i(k)\partial\eta_j(k)}
\end{aligned}$$

Thus the matrix M in (4.17) is given by

$$(M)_{i,j} = \left[\sum_{s'\in S}\sum_{s\in S}\alpha(s,s')(\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}}\right]_{i,j}.$$

To find an explicit expression for the matrix M, we define a similar coordinate map $\phi_{i,j} : \mathbb{R}^{|S|}\times\mathbb{R}^{|S|}\to\mathbb{R}$, $\phi_{i,j}(M) = M_{i,j}$. First note the diagonal elements of $(\mathbf{e}_{s'}-\mathbf{e}_s)\cdot(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}}$ are either 0 or 1 and the off-diagonal elements are either 0 or -1. Also

$$\phi_{i,i}((\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}}) = 1 \text{ if and only if } i = s' \text{or } i = s$$

and $\phi_{i,j}((\mathbf{e}_{s'}-\mathbf{e}_s)(\mathbf{e}_{s'}-\mathbf{e}_s)^{\mathbf{T}}) = -1$ if and only if $(i = s' \text{and } j = s)$ or $(i = s$ and $i = s')$. For example,

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{pmatrix}.$$

Then for $k\in\Lambda_C$ and $s=\bar{s}$,

$$\begin{aligned}
(M)_{s,s} &= \phi_{s,s}(\sum_{s'\in S}\sum_{s''\in S}\eta_{s''}(k)\mathbf{c}_C(k,s'',s',\eta)(\mathbf{e}_{s'}-\mathbf{e}_{s''})\cdot(\mathbf{e}_{s'}-\mathbf{e}_{s''})^{\mathbf{T}})\\
&= \sum_{\substack{s''\in S\\ s''\neq s}}\eta_{s''}(k)\mathbf{c}_C(k,s'',s,\eta) + \sum_{\substack{s'\in S\\ s'\neq s}}\eta_s(k)\mathbf{c}_C(k,s,s',\eta)
\end{aligned}$$

For $k \in \Lambda_C$ and $s \neq \bar{s}$,

$$\begin{aligned}(M)_{s,\bar{s}} &= \phi_{s,\bar{s}}(\sum_{s' \in S}\sum_{s'' \in S} \eta_{s''}(k)\mathbf{c}_C(k, s'', s', \eta)(\mathbf{e}_{s'} - \mathbf{e}_{s''}) \cdot (\mathbf{e}_{s'} - \mathbf{e}_{s''})^{\mathbf{T}}) \\ &= -\eta_{\bar{s}}(k)\mathbf{c}_C(k, \bar{s}, s, \eta) - \eta_s(k)\mathbf{c}_C(k, s, \bar{s}, \eta)\end{aligned}$$

Note that the $M(k)$ is symmetric, thus has an orthonormal basis of eigenvectors. Also note that since $M_{s,s}(k) \geq 0$ and $|M_{s,s}(k)| \geq \sum_{s' \neq s} |M_{s,s'}(k)|$ for all s, from the diagonally dominant condition (see for example Horn and Johnson (1985)), $M(k)$ is positive semi-definite. So we can write

$$M = VDV^{-1}$$

and define $|S| \times |S|$ matrix, $M^{\frac{1}{2}}(k) := VD^{\frac{1}{2}}V^{-1}$. Then we find the following explicit formula for the Ito process.

- Variance Processes

$$d\mathcal{N}_D(k) = F(\mathcal{N}_D(k))dt + \frac{1}{\sqrt{N_2}} M(k)^{\frac{1}{2}} dB \quad \text{for each } k \in \Lambda_C \tag{4.18}$$

where B is $|S|-$dimensional Brownian motion, and

$$\begin{aligned}(M)_{s,s} &= \sum_{\substack{s'' \in S \\ s'' \neq s}} \eta_{s''}(k)\mathbf{c}_C(k, s'', s, \eta) + \sum_{\substack{s' \in S \\ s' \neq s}} \eta_s(k)\mathbf{c}_C(k, s, s', \eta) \\ (M)_{s,\bar{s}} &= -\eta_{\bar{s}}(k)\mathbf{c}_C(k, \bar{s}, s, \eta) - \eta_s(k)\mathbf{c}_C(k, s, \bar{s}, \eta).\end{aligned}$$

Next we need to check whether the solution to (4.18) has the same law as the solution to (4.12). To do this, we suppose that $|\Lambda_C| = 1$. For example, when $|S| = 2$, we have

$$\begin{aligned}\sum_{l \in \mathcal{I}} l\sqrt{\beta_l(\eta(t))}dB_l &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sqrt{c_{12}}dB_{l_{12}} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sqrt{c_{21}}dB_{l_{21}} \\ &= \begin{pmatrix} -\sqrt{c_{12}} & \sqrt{c_{21}} \\ \sqrt{c_{12}} & -\sqrt{c_{21}} \end{pmatrix} \begin{pmatrix} dB_{l_{12}} \\ dB_{l_{21}} \end{pmatrix},\end{aligned}$$

where we set $c_l := \beta_l(\eta)$. Then note that

$$C := \begin{pmatrix} -\sqrt{c_{12}} & \sqrt{c_{21}} \\ \sqrt{c_{12}} & -\sqrt{c_{21}} \end{pmatrix} = \sqrt{c_{12}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \sqrt{c_{21}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix},$$

and from this we have

$$CC^T = c_{12} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} + c_{21} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = M. \tag{4.19}$$

Proposition 4.2.4 *The solution to (4.12) has the same law as (4.18).*

Proof. We first associate an ordered index to $\mathcal{I}$ and denote this index $i(l)$. Recall that $|\mathcal{I}| = |S|(|S|-1)$.Then there exists $|S| \times |S|(|S|-1)$ matrix C such that

$$\sum_{l\in\mathcal{I}} l\sqrt{\beta_l(\eta)}dB_l = C \begin{pmatrix} dB_{l_1} \\ \vdots \\ dB_{l_{|S|(|S|-1)}} \end{pmatrix}.$$

So we need to show that $CC^T = M$. Note that

$$C = \sum_{l\in\mathcal{I}} l\sqrt{c_l}\mathbf{e}^{\mathbf{T}}_{i(l)} \quad \text{where } c_l = \sqrt{\beta_l(\eta)}$$

Then since $\mathbf{e}^{\mathbf{T}}_{i(l)}\mathbf{e}_{i(l')} = 0$ if $l \neq l'$ and $\mathbf{e}^{\mathbf{T}}_{i(l)}\mathbf{e}_{i(l')} = 1$ if $l = l'$, we have

$$(\sum_{l\in\mathcal{I}} \sqrt{c_l} l\mathbf{e}^{\mathbf{T}}_{i(l)})(\sum_{l'\in\mathcal{I}} \sqrt{c_{l'}}\mathbf{e}_{i(l')}l^T_{i(l')}) = \sum_{l\in\mathcal{I}} c_l l l^T = \sum_{s'\in S}\sum_{s\in S} c_l(\mathbf{e}_{s'} - \mathbf{e}_s)(\mathbf{e}_{s'} - \mathbf{e}_s)^{\mathbf{T}} = Q$$

■

From this we also obtain the explicit expression for variance processes.

$$\begin{cases} \frac{d\eta(k)}{dt} = F(\eta(k)) & \text{for each } k \in \Lambda_C \\ d\mathcal{V}(k) = \partial F(\eta(k))\mathcal{V}dt + \frac{1}{\sqrt{N_2}}M(k)^{\frac{1}{2}}dB & \text{for each } k \in \Lambda_C \end{cases}$$

where B is $|S|-$dimensional Brownian motion, and

$$\begin{aligned} (M)_{s,s} &= \sum_{\substack{s''\in S \\ s''\neq s}} \eta_{s''}(k)\mathbf{c}_C(k,s'',s,\eta) + \sum_{\substack{s'\in S \\ s'\neq s}} \eta_s(k)\mathbf{c}_C(k,s,s',\eta) \\ (M)_{s,\bar{s}} &= -\eta_{\bar{s}}(k)\mathbf{c}_C(k,\bar{s},s,\eta) - \eta_s(k)\mathbf{c}_C(k,s,\bar{s},\eta). \end{aligned}$$

4.3 Hybrid Models: examples

4.3.1 Examples of Approximations

Here we will focus on two group dynamics with two strategy set where we call two groups E and R. We suppose that the sizes of each group are $N^{(E)}$ and $N^{(R)}$ and first suppose that $N = N^{(E)} = N^{(R)}$. We use η for the population fraction using strategy 1 in the group α and ρ for the population fraction using strategy 2 in the group β. We consider the following games for within group interactions and between group interactions:

	Group Eta	Group Rho
Group Eta	$\begin{pmatrix} \beta_W^{(E)}(1-\zeta_W^{(E)}) & 0 \\ 0 & \beta_W^{(E)}\zeta_W^{(E)} \end{pmatrix}$	$\begin{pmatrix} \beta_B^{(E)}(1-\zeta_B^{(E)}) & 0 \\ 0 & \beta_B^{(E)}\zeta_B^{(E)} \end{pmatrix}$
Group Rho	$\begin{pmatrix} \beta_B^{(R)}(1-\zeta_B^{(R)}) & 0 \\ 0 & \beta_B^{(R)}\zeta_B^{(R)} \end{pmatrix}$	$\begin{pmatrix} \beta_W^{(R)}(1-\zeta_W^{(R)}) & 0 \\ 0 & \beta_W^{(R)}\zeta_W^{(R)} \end{pmatrix}$

- ODE approximations

In this setting the logit dynamics are given by

$$\begin{aligned} \frac{d\eta}{dt} &= l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\rho-\zeta_B^{(E)})) - \eta \\ \frac{d\rho}{dt} &= l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(R)}(\rho-\zeta_W^{(R)}) + \bar{\mathcal{W}}_{(2,1)}\beta_B^{(R)}(\eta-\zeta_B^{(R)})) - \rho \end{aligned}$$

And because $r_\kappa(t) - r_\kappa(-t) = t$ and when no imitation between groups is considered (i.e., $\bar{\mathcal{W}}_m(1,2) = \bar{\mathcal{W}}_m(2,1) = 0$), the replicator dynamics are

$$\begin{aligned} \frac{d\eta}{dt} &= \bar{\mathcal{W}}_{(0)}\beta_W^{(E)}\eta(1-\eta)(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}\eta(1-\eta)(\rho-\zeta_B^{(E)}) \\ \frac{d\rho}{dt} &= \bar{\mathcal{W}}_{(0)}\beta_W^{(R)}\rho(1-\rho)(\rho-\zeta_W^{(R)}) + \bar{\mathcal{W}}_{(2,1)}\beta_B\rho(1-\rho)(\eta-\zeta_B) \end{aligned}$$

- Diffusion approximations

Next we find the diffusion approximation; we will derive three versions of it. First focus on the case of innovative and group E. From 4.19 (or by directly verifying) we see that the matrix M is given by

$$M = (\eta_1 \mathbf{c}(1,2,\eta) + \eta_2 \mathbf{c}(2,1,\eta)) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then by noting that

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]^{\mathbf{T}},$$

we find

$$\begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix} = \begin{pmatrix} F_1(\eta,\rho) \\ F_2(\eta,\rho) \end{pmatrix} dt + \frac{1}{\sqrt{N}} \sqrt{\frac{1}{2}(\eta_1 \mathbf{c}(1,2,\eta,\rho) + \eta_2 \mathbf{c}(2,1,\eta,\rho))} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix} \tag{4.20}$$

Note equation 4.20 ensures the invariance of the simplex for the solution; $\eta_1 + \eta_2 = 1$. Thus we further reduce (4.20) and obtain the following expression for the logit dynamics:

$$\begin{aligned} d\eta &= F(\eta,\rho)dt + \frac{1}{\sqrt{N}} \sqrt{\frac{1}{2} G(\eta,\rho)} (dB_1 - dB_2) \\ \text{where } F(\eta,\rho) &= l_\kappa(\bar{\mathcal{W}}_{(0)} \beta_W^{(E)} (\eta - \zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)} \beta_B^{(E)} (\rho - \zeta_B^{(E)})) - \eta \\ \text{and } G(\eta,\rho) &= \eta l_\kappa(\bar{\mathcal{W}}_{(0)} \beta_W^{(E)} (\zeta_W^{(E)} - \eta) + \bar{\mathcal{W}}_{(1,2)} \beta_B^{(E)} (\zeta_B^{(E)} - \rho)) \\ &\quad + (1-\eta) l_\kappa(\bar{\mathcal{W}}_{(0)} \beta_W^{(E)} (\eta - \zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)} \beta_B^{(E)} (\rho - \zeta_B^{(E)})). \end{aligned}$$

Next we find the expressions for β_l :

$$\begin{aligned} \beta_{(1,-1)}(\eta_1,\eta_2) &= \eta_2 l_\kappa(\bar{\mathcal{W}}_{(0)} \beta_W^{(E)} (\eta_1 - \zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)} \beta_B^{(E)} (\rho_1 - \zeta_B^{(E)})) \\ \beta_{(-1,1)}(\eta_1,\eta_2) &= \eta_1 l_\kappa(\bar{\mathcal{W}}_{(0)} \beta_W^{(E)} (\zeta_W^{(E)} - \eta_1) + \bar{\mathcal{W}}_{(1,2)} \beta_B^{(E)} (\zeta_B^{(E)} - \rho_1)). \end{aligned}$$

In this case we have

$$\begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix} = \begin{pmatrix} F_1(\eta,\rho) \\ F_2(\eta,\rho) \end{pmatrix} dt$$
$$+\frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sqrt{\beta_{(1,-1)}(\eta_1,\eta_2)} dB_1 + \frac{1}{\sqrt{N}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sqrt{\beta_{(-1,1)}(\eta_1,\eta_2)} dB_2.$$

Therefore we find the second SDE:

$$\begin{aligned} d\eta &= F(\eta,\rho)dt + \frac{1}{\sqrt{N}}\sqrt{G_1(\eta,\rho)}dB_1 - \frac{1}{\sqrt{N}}\sqrt{G_2(\eta,\rho)}dB_2 \\ \text{where } F(\eta,\rho) &= l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\rho-\zeta_B^{(E)})) - \eta \\ G_1(\eta,\rho) &= (1-\eta)l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\rho-\zeta_B^{(E)})) \\ G_2(\eta,\rho) &= \eta l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\zeta_W^{(E)}-\eta) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\zeta_B^{(E)}-\rho)). \end{aligned}$$

Finally observe that

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

Therefore we have

$$\begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix} = \begin{pmatrix} F_1(\eta,\rho) \\ F_2(\eta,\rho) \end{pmatrix} dt + \frac{1}{\sqrt{N}}\sqrt{\eta_1\mathbf{c}(1,2,\eta,\rho)+\eta_2\mathbf{c}(2,1,\eta,\rho))} \begin{pmatrix} 1 \\ -1 \end{pmatrix} dB$$

and we obtain:

$$\begin{aligned} d\eta &= F(\eta,\rho)dt + \frac{1}{\sqrt{N}}\sqrt{G(\eta,\rho)}dB \\ \text{where } F(\eta,\rho) &= l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\rho-\zeta_B^{(E)})) - \eta \\ \text{and } G(\eta,\rho) &= \eta l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\zeta_W^{(E)}-\eta) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\zeta_B^{(E)}-\rho)) \\ &\quad +(1-\eta)l_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\rho-\zeta_B^{(E)})). \end{aligned}$$

We can derive the similar expression for the replicator dynamics:

$$\begin{aligned}
d\eta &= F(\eta,\rho)dt + \frac{1}{\sqrt{N}}\sqrt{G(\eta,\rho)}dB \\
\text{where } F(\eta,\rho) &= \bar{\mathcal{W}}_{(0)}\beta_W^{(E)}\eta\,(1-\eta)\,(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}\eta\,(1-\eta)\,(\rho-\zeta_B^{(E)}) \\
\text{and } G(\eta,\rho) &= \eta(1-\eta)\Big[r_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\eta-\zeta_W^{(E)}) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\rho-\zeta_B^{(E)})) \\
&\quad + r_\kappa(\bar{\mathcal{W}}_{(0)}\beta_W^{(E)}(\zeta_W^{(E)}-\eta) + \bar{\mathcal{W}}_{(1,2)}\beta_B^{(E)}(\zeta_B^{(E)}-\rho))\Big].
\end{aligned}$$

- Variance Processes

  To simplify the notation, we suppose that there is no between group interaction, so $\bar{\mathcal{W}}_{(1,2)} = 0$ and $\bar{\mathcal{W}}_{(0)} = 1$ and we will drop the index for groups. To find variance processes, we need to find the Jacobian matrix for the vector field of the ODE. In the case of the logit dynamic, this is given by

$$\partial F = \begin{pmatrix} \kappa\beta(1-\zeta)l_\kappa(1-l_\kappa) - l_\kappa & -\kappa\beta\zeta l_\kappa(1-l_\kappa) + l_\kappa \\ -\kappa\beta(1-\zeta)l_\kappa(1-l_\kappa) + l_\kappa & \kappa\beta\zeta l_\kappa(1-l_\kappa) - l_\kappa \end{pmatrix}.$$

  Thus the variance process is given by

$$\begin{aligned}
\begin{pmatrix} dV_1 \\ dV_2 \end{pmatrix} &= \begin{pmatrix} \kappa\beta(1-\zeta)l_\kappa(1-l_\kappa) - l_\kappa & -\kappa\beta\zeta l_\kappa(1-l_\kappa) + l_\kappa \\ -\kappa\beta(1-\zeta)l_\kappa(1-l_\kappa) + l_\kappa & \kappa\beta\zeta l_\kappa(1-l_\kappa) - l_\kappa \end{pmatrix} \begin{pmatrix} dV_1 \\ dV_2 \end{pmatrix} dt \\
&\quad + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sqrt{\beta_{(1,-1)}(\eta_1,\eta_2)}dB_1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sqrt{\beta_{(-1,1)}(\eta_1,\eta_2)}dB_2.
\end{aligned}$$

  Note that we have $V_1 = -V_2$. Hence we can obtain the following scalar equations.

$$dV = (\kappa\beta l_\kappa(1-\kappa_\kappa) - 2l_\kappa)dV + \sqrt{\eta l_\kappa(\beta(\zeta-\eta)) + (1-\eta)l_\kappa(\beta(\eta-\zeta))}dB.$$

  Similarly we find the following expression for the replicator equations:

$$dV = 2\beta\eta(1-\eta)l_\kappa V dt + \sqrt{\eta(1-\eta)((r_\kappa(\beta(\zeta-\eta)) + r_\kappa(\beta(\eta-\zeta)))}dB.$$

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