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A POLYTOPE COMBINATORICS FOR SEMISIMPLE GROUPS

JARED E. ANDERSON

ABSTRACT. Mirković and Vilonen discovered a canonical basis of algebraic cycles for the intersection homology of (the closures of the strata of) the loop Grassmannian. The moment map images of these varieties are a collection of polytopes, and they may be used to compute weight multiplicities and tensor product multiplicities for representations of a semisimple group. The polytopes are explicitly described for a few low rank groups.

(Mathematics subject classification numbers: 14L99, 20G05)

1. INTRODUCTION

Starting with a semisimple algebraic group, we construct a collection of polytopes. The central result is a method that uses these to decompose the tensor product of two irreducible representations. Each tensor product multiplicity (Littlewood-Richardson number) is the number of polytopes in a certain set.

The method is based on the geometry of the loop Grassmannian, and builds directly on the work of Mirković and Vilonen [MV]. But all algebraic geometry is deferred until section 5 since we may state the main result without it (Theorem 1). We do this in section 2, and follow with a lot of examples in sections 3 and 4. Much may be gained from just these first sections without ever understanding what the loop Grassmannian is.

Some background in geometry in representation theory is discussed in section 5. One begins by fixing an algebraic group, and constructing from it a space, the loop Grassmannian. The representation theory of the (Langlands dual) group is known to be closely related to the geometry of the loop Grassmannian. This relationship was made more explicit with Mirković and Vilonen's discovery of a collection of singular algebraic varieties in the loop Grassmannian, which we call MV-cycles. In terms of geometry, they provide a canonical basis for the intersection homology of the closure of each stratum of the loop Grassmannian. In terms of representation theory, they provide a canonical basis for each irreducible representation of the group.

Section 6 provides a definition of the polytopes as moment map images of MV-cycles. The rest of the paper consists mainly of the proof of Theorem 1. Section 8 contains the main geometric idea and is of interest in its own right (Theorem 7). The last section provides a glimpse at a closely related Hopf algebra.

2. STATEMENT OF RESULTS

Let \mathbf{G} be a connected semisimple complex algebraic group of rank n . Choose a maximal torus $\mathbf{T} \subset \mathbf{G}$. We will be considering polytopes in the real n -dimensional vector space in which pictures of roots and weights are usually drawn: this is the dual $\mathfrak{t}_{\mathbb{R}}^*$ of the Lie algebra of the split real form of \mathbf{T} .

We will define a collection of polytopes, $\mathcal{MV} = (P_{\phi})_{\phi \in \mathbb{B}}$ in $\mathfrak{t}_{\mathbb{R}}^*$ (Definition 5 on page 11). Let R^- denote the set of negative roots and Λ^- the semigroup they generate. The parameter set \mathbb{B} is naturally graded by Λ^- : $\mathbb{B} = \bigcup_{\nu \in \Lambda^-} \mathbb{B}_{\nu}$. It turns out that this parameter set is not important in the theorem below, since the parametrization $\phi \mapsto P_{\phi}$ is injective [AM]; so we will really be counting polytopes. We also write $\mathcal{MV}_{\nu} = \{P_{\phi} \mid \phi \in \mathbb{B}_{\nu}\}$ so that $\mathcal{MV} = \bigcup_{\nu \in \Lambda^-} \mathcal{MV}_{\nu}$.

Among our polytopes will be (shifts of) those familiar from representation theory: the convex hull of the weights in an irreducible representation of \mathbf{G} ; $\text{conv}(W \cdot \lambda)$ denotes the convex hull of the Weyl group orbit through a weight λ .

Theorem 1. *Weight multiplicities and tensor product multiplicities may be calculated according to the following rules.*

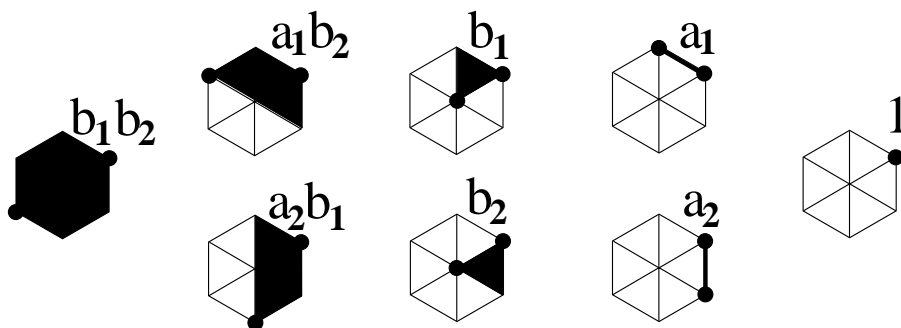
1. *If V_{λ} is an irreducible representation of \mathbf{G} with dominant weight λ , then the multiplicity of weight ν in V_{λ} equals the number of $\phi \in \mathbb{B}_{\nu-\lambda}$ for which $P_{\phi} + \lambda \subseteq \text{conv}(W \cdot \lambda)$.*
2. *If V_{λ} and V_{μ} are irreducible representations of \mathbf{G} with dominant weights λ and μ , and ν is any dominant weight, then the multiplicity of V_{ν} in $V_{\lambda} \otimes V_{\mu}$ equals the number of $\phi \in \mathbb{B}_{\nu-\mu-\lambda}$ for which $P_{\phi} + \lambda \subseteq \text{conv}(W \cdot \lambda) \cap (\text{conv}(W \cdot -\mu) + \nu)$.*

The polytopes in \mathcal{MV} are called MV-polytopes. The ‘‘MV’’ stands for Mirković and Vilonen, who discovered a collection of algebraic varieties called MV-cycles. MV-polytopes will be defined as moment map images of MV-cycles. Part 1 of the theorem is little more than a translation of some of the algebraic geometry of Mirković and Vilonen into the language of polytopes. Part 2, however, depends on some more geometry described in section 8.

3. EXAMPLES OF POLYTOPES

We explicitly describe the collection of polytopes \mathcal{MV} for a few low rank groups: Sl_2 , Sl_3 , Sp_4 , Sl_4 . This is not known for other groups, although Mirković and I have a conjecture that inductively constructs the polytopes for any group [AM].

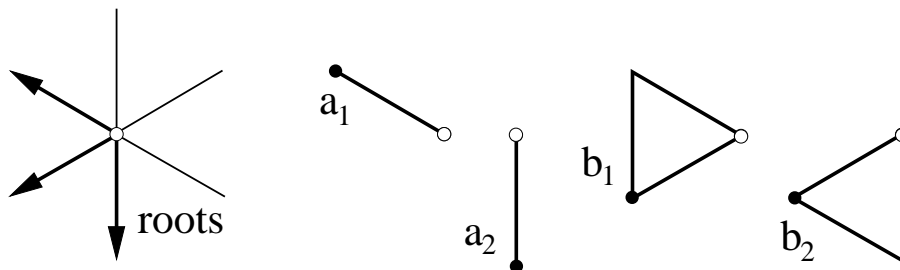
Before describing *all* the polytopes for these groups, we introduce them with a picture of eight of them, which count the weight multiplicities in the adjoint representation of Sl_3 . The weight multiplicity is 1 at each outer vertex, and 2 at the central vertex.



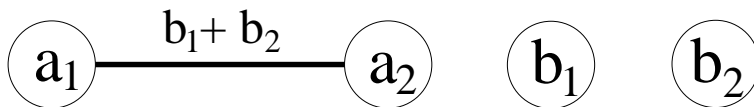
To describe the polytopes, it is easiest to first introduce a commutative algebra \mathcal{A} with basis \mathcal{MV} . This algebra is discussed briefly in section 10; conjecturally it is the algebra of functions on the unipotent radical of a Borel subgroup [A]. In these examples we give ad hoc definitions of \mathcal{A} by a natural set of generators and relations. Then a collection of monomials in the generators is defined. A polytope is associated to each generator, and then to each monomial by taking the Minkowski sum of the factors. (The Minkowski sum of two sets A and B is the set of sums $\{a + b | a \in A, b \in B\}$.)

3.1. Sl_2 . $\mathcal{A} = \mathbb{Z}[a]$ and the MV-polytopes $[k\alpha, 0]$ correspond to monomials a^k . (α is the negative root in $\mathfrak{t}_{\mathbb{R}}^* = \mathbb{R}$.)

3.2. Sl_3 . The algebra \mathcal{A} has four generators:



(The negative roots are indicated. For each polytope, the vertex with highest weight is at the origin.) There is a single relation, $a_1 a_2 = b_1 + b_2$, which we put in a diagram:

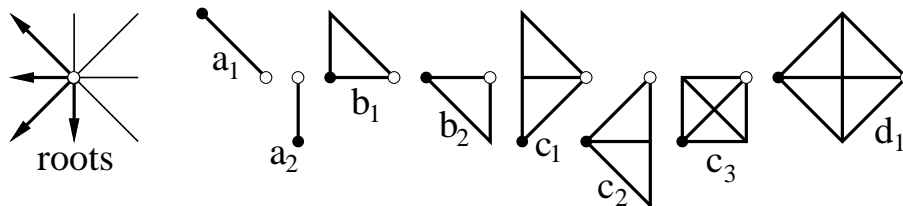


The MV-polytopes correspond to monomials of the form $a_1^i b_1^j b_2^k$ or $a_2^i b_1^j b_2^k$, i.e. those monomials in the generators such that not both a_1 and a_2 occur. These are the monomials that cannot be further simplified using the relations. Again, the MV-polytopes are found by taking Minkowski sums. For example, the regular hexagon $b_1 b_2$ is the Minkowski sum of the two triangles. In general, the monomials $b_1^j b_2^k$ give shifts of the symmetric hexagons $\text{conv}(W \cdot \lambda)$. One can think of an arbitrary MV-polytope as one of these hexagons stretched some length in the direction of either a_1 or a_2 .

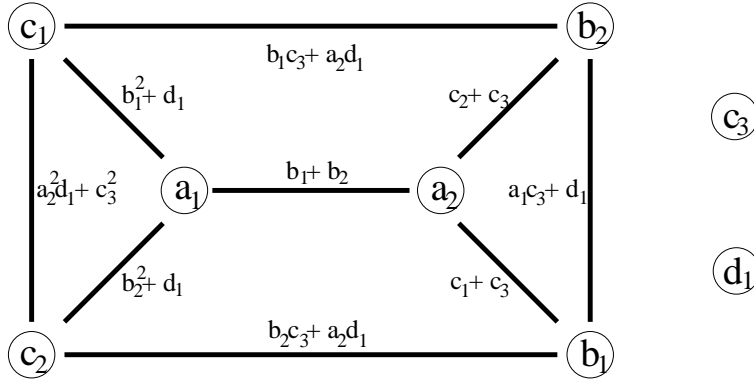
As an aside, note that the relation $a_1 a_2 = b_1 + b_2$ has an interpretation in terms of Minkowski sum: the Minkowski sum of the two line segments a_1 and a_2 is a parallelogram which equals the union of the two triangles b_1 and b_2 . In general, if $\alpha\beta = \sum_{\gamma \in \mathcal{MV}} n_\gamma \gamma$ then the Minkowski sum of α and β equals the union of the γ for which $n_\gamma \neq 0$. (See [A].)

The above definition places the highest weight vertex of $P \in \mathcal{MV}$ at the origin. Then the lowest weight vertex ν identifies the graded piece \mathcal{MV}_ν in which P lies.

3.3. Sp_4 . The algebra \mathcal{A} has eight generators:

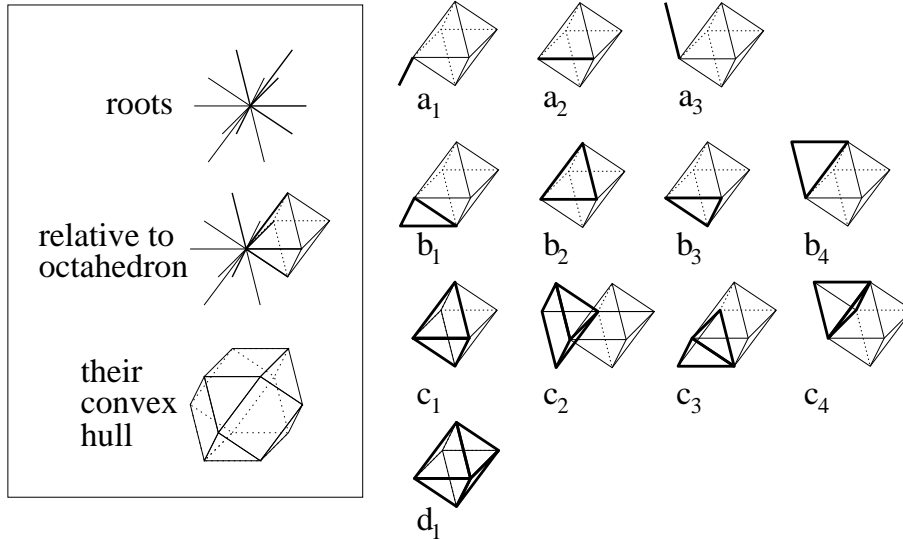


(Again the negative roots indicated, and the highest weight vertex is always at the origin.) There are nine relations:

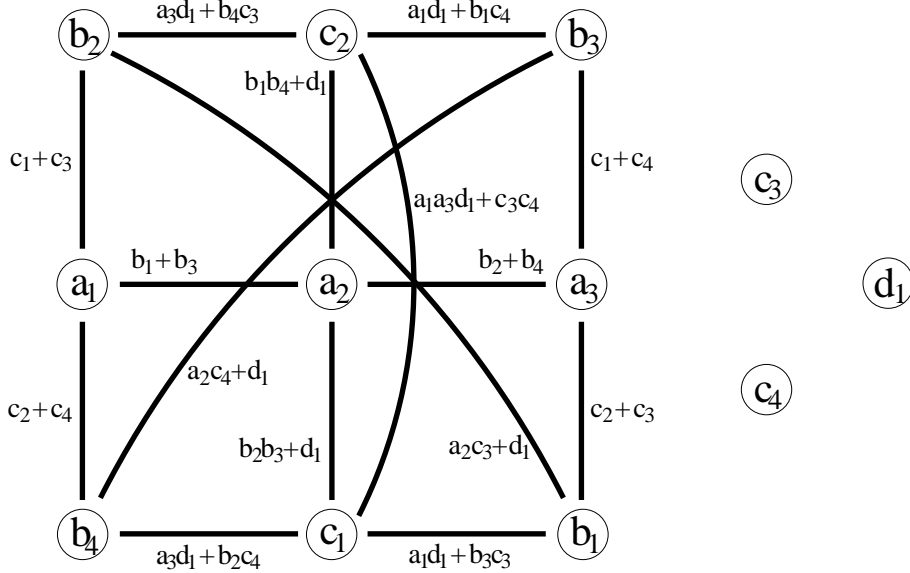


The MV-polytopes correspond to monomials of these forms: $a_1^i b_1^j c_3^k d_1^l$, $a_1^i b_2^j c_3^k d_1^l$, $a_2^i c_1^j c_3^k d_1^l$, $a_2^i c_2^j c_3^k d_1^l$, $b_1^i c_1^j c_3^k d_1^l$, $b_2^i c_2^j c_3^k d_1^l$. Equivalently, these are all monomials in the generators, no two factors of which are joined by a line in the diagram of relations. The monomials $c_3^k d_1^l$ give shifts of the symmetric octagons $\text{conv}(W \cdot \lambda)$. The grading is specified exactly as it was for Sl_3 .

3.4. Sl_4 . The example of Sl_4 is very rich, and harder to think about since the polytopes are three-dimensional. There are twelve generators:



These are drawn relative to an octahedron whose front left vertex is the origin of $t_{\mathbb{R}}^* = \mathbb{R}^3$. The three line segments a_1, a_2, a_3 identify the simple negative roots. The b_i are triangles; c_1, c_2 are square-based pyramids; c_3, c_4 are tetrahedra; d_1 is an octahedron. Here are the fifteen relations:

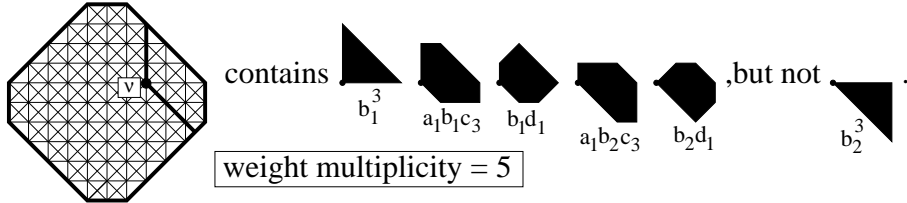


As before, the collection of monomials consists of those for which no two factors are joined by a line in this diagram.

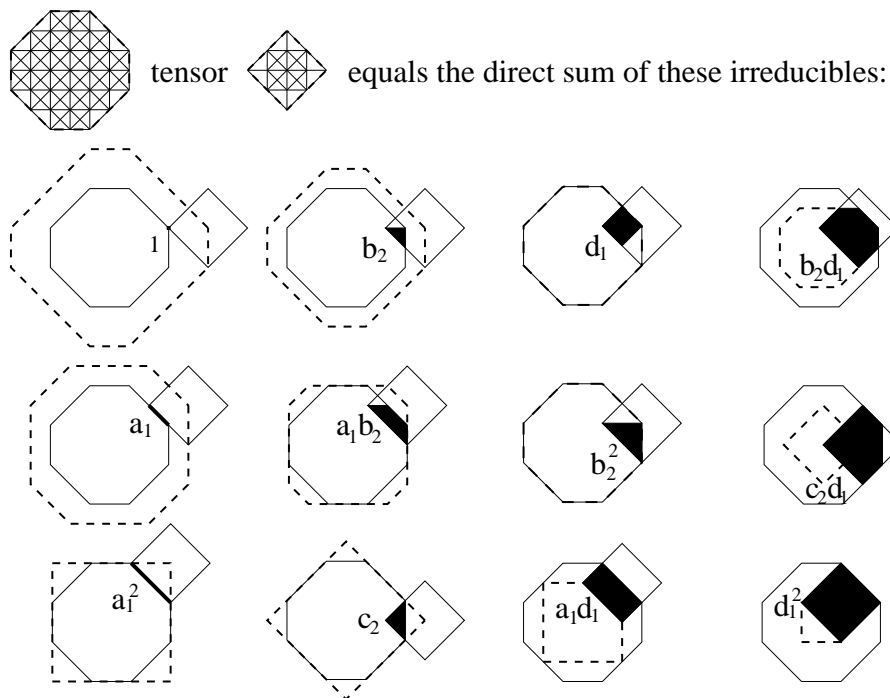
4. EXAMPLES OF MULTIPLICITY CALCULATIONS

We use the MV-polytopes for Sp_4 in two examples, illustrating the two parts of Theorem 1.

4.1. **Example: Sp_4 weight multiplicity calculation.** This illustrates part 1 of Theorem 1. The octagon below corresponds to an irreducible representation of Sp_4 . Suppose we want to know the weight multiplicity at the indicated weight ν . MV_ν consists of the six polygons pictured below. Each has one vertex, distinguished by a black dot, which must be placed at the weight ν . Five of the polygons are contained in the octagon, and one is not (its bottom right corner sticks out). So the weight multiplicity is 5.



4.2. **Example: Sp_4 tensor product multiplicity calculation.** This illustrates part 2 of Theorem 1. Suppose we want to decompose the tensor product of the two irreducible Sp_4 representations indicated in the picture.



The dashed lines are the outlines of the irreducible summands that occur with nonzero multiplicity. The MV-polygons that were counted to find the multiplicities are drawn in black; these were the ones contained in the intersection of the big octagon $\text{conv}(W \cdot \lambda)$ and the translated square $\text{conv}(W \cdot -\mu) + \nu$. Each summand has multiplicity 1, except for one which has multiplicity 2: d_1, b_2^2 .

Remark. This calculation has a similar flavour to the convolution of the characters of the two representations, where one computes a sum of products of weight multiplicities instead of counting polytopes. After convolving the characters, one uses the Weyl character formula to find the irreducible summands. Our method required no knowledge of weight multiplicities, no arithmetic, and no separate calculation to extract the irreducible summands. Of course, saying that this method is more efficient than using the Weyl character formula is not saying very much.

One might hope that this would provide an explicit formula for the Littlewood-Richardson numbers, but it seems only to provide an algorithm.

5. GEOMETRY IN REPRESENTATION THEORY

5.1. Loop Grassmannian. Our basic object of study is the loop Grassmannian (affine Grassmannian), which is an infinite dimensional

space associated to a complex algebraic group \mathbf{G} , taken to be connected and semisimple. It is the quotient of groups $\mathcal{G} = \mathbf{G}(\mathcal{K})/\mathbf{G}(\mathcal{O})$ where $\mathcal{O} = \mathbb{C}[[t]]$, the ring of formal power series, and $\mathcal{K} = \mathbb{C}((t))$, its field of fractions, the ring of formal Laurent series.

One can show that if we use the rings $\mathbb{C}[t, t^{-1}]$ and $\mathbb{C}[t]$ instead of \mathcal{K} and \mathcal{O} in this definition, then we get the same set. Since the numerator $\mathbf{G}(\mathbb{C}[t, t^{-1}])$ is a group of maps from the unit circle to \mathbf{G} (let $t = e^{i\theta}$), we understand the use of the word *loop*. There is a model of \mathcal{G} due to Lusztig which describes it as a set of subspaces of an infinite dimensional vector space, which explains the use of *Grassmannian* [L].

\mathcal{G} may be realized as an increasing union of finite dimensional complex projective varieties by filtering $\mathbf{G}(\mathcal{K})$ by order of pole [L]. The orbits of the action of $\mathbf{G}(\mathcal{O})$ on \mathcal{G} by left multiplication provide a stratification of \mathcal{G} , and are indexed by the dominant coweights of \mathbf{G} . Here is our notation for this: Fix a maximal torus $\mathbf{T} \subseteq \mathbf{G}$. Any coweight $\lambda \in \text{Hom}(\mathbb{C}^*, \mathbf{T})$ defines an element of $\mathbf{T}(\mathcal{K}) \subseteq \mathbf{G}(\mathcal{K})$, and hence one of \mathcal{G} , which we denote $\underline{\lambda}$. Let $\mathcal{G}_\lambda = \mathbf{G}(\mathcal{O})\underline{\lambda}$. Any point in \mathcal{G} is in some \mathcal{G}_λ and this λ is determined up to the action of the Weyl group.

Each stratum \mathcal{G}_λ is a complex vector bundle over a flag manifold for \mathbf{G} . For a coweight λ in the interior of the positive Weyl chamber this is the full flag manifold, but if λ lies on a Weyl chamber wall then it is some partial flag manifold. The closure $\overline{\mathcal{G}_\lambda}$ of a stratum consists of the union of all \mathcal{G}_μ with $\mu \preceq \lambda$, where λ and μ are dominant. ($\mu \preceq \lambda$ means that $\lambda - \mu$ is in the semigroup generated by the positive coroots.) $\overline{\mathcal{G}_\lambda}$ is a finite dimensional complex projective algebraic variety, and is almost always singular. (See [PS].)

5.2. Relation to representation theory. It turns out that what gives information about representation theory is the intersection homology of Goresky and MacPherson [GM1, GM2]. This is a homology theory defined on a stratified space, using cycles that satisfy bounds on the allowed dimensions of intersections with lower dimensional strata. Work of Drinfeld, Ginzburg, Lusztig, Mirković, and Vilonen [BD, G, MV] shows that the intersection homology $\text{IH}(\overline{\mathcal{G}_\lambda})$ of the closure of a stratum may be regarded as the vector space underlying the representation V_λ of the Langlands dual group $\mathbf{G}^\vee = \mathbf{G}$. (Here λ is both a coweight of \mathbf{G} and a weight of \mathbf{G} .) They actually show much more: the category of $\mathbf{G}(\mathcal{O})$ -equivariant perverse sheaves on \mathcal{G} is given a tensor product, and is equivalent to the tensor category of representations of \mathbf{G} .

5.3. MV-cycles. Mirković and Vilonen discovered a canonical basis of algebraic cycles for the intersection homology of stratum closures

$\overline{\mathcal{G}}_\lambda$ [MV]. These MV-cycles are projective varieties in the loop Grassmannian. The intersection homology of $\overline{\mathcal{G}}_\lambda$ is represented by those MV-cycles contained in it, but not contained in any $\overline{\mathcal{G}}_\mu$ with $\mu \prec \lambda$.

To define MV-cycles, we must first make some choices. Fix opposite Borel subgroups of \mathbf{G} that intersect in the maximal torus \mathbf{T} . Denote by N and N^- their unipotent radicals. We choose the positive roots to be roots in N^- . Then for any coweights λ, μ , we let $S_\lambda = N(\mathcal{K})\underline{\lambda}$ and $T_\mu = N^-(\mathcal{K})\underline{\mu}$. Any $N(\mathcal{K})$ -orbit contains a unique $\underline{\lambda}$, as does any $N^-(\mathcal{K})$ -orbit. These orbits have both infinite dimension and infinite codimension in \mathcal{G} . There are simple closure relations: $\overline{S}_\lambda = \bigcup_{\xi \succeq \lambda} S_\xi$ and $\overline{T}_\mu = \bigcup_{\eta \preceq \mu} T_\eta$ [MV].

Definition 2. Let $\overline{\mathcal{G}}_\lambda$ be the closure of a stratum of the loop Grassmannian, where λ is chosen dominant. Let ν be a coweight of \mathbf{G} with $\underline{\nu} \in \overline{\mathcal{G}}_\lambda$. The MV-cycles for $\overline{\mathcal{G}}_\lambda$ at ν , relative to N , are the irreducible components of $\overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda$. Equivalently, they are those irreducible components of $\overline{S}_\nu \cap \overline{T}_\lambda$ contained in $\overline{\mathcal{G}}_\lambda$.

The equivalence of the two definitions is a consequence of the dimension calculations of [MV]: both $\overline{S}_\nu \cap \overline{T}_\lambda$ and $\overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda$ have pure (complex) dimension equal to the height of $\lambda - \nu$.

Proposition 3. The two definitions are equivalent.

Proof. Suppose $a \in \text{Irr}(\overline{S}_\nu \cap \overline{T}_\lambda)$ (i.e. a is an irreducible component of this variety) and $a \subseteq \overline{\mathcal{G}}_\lambda$. Evidently $a \subseteq \overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda$. Since $\overline{S}_\nu = \bigcup_{\xi \succeq \nu} S_\xi$ and $\overline{\mathcal{G}}_\lambda = \bigcup_{\text{dom. } \eta \preceq \lambda} \mathcal{G}_\eta$, we see that $\overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda = \overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda \cup X$ where every component of X has dimension strictly less than $\text{height}(\lambda - \nu)$, the dimension of a . So $a \in \text{Irr}(\overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda)$.

Conversely, suppose $a \in \text{Irr}(\overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda)$. According to [MV], $\dim(\mathcal{G}_\lambda \cap T_\lambda) = \dim(\mathcal{G}_\lambda)$; therefore $\overline{\mathcal{G}}_\lambda \subseteq \overline{T}_\lambda$. So $a \subseteq \overline{S}_\nu \cap \overline{T}_\lambda$. Since $\overline{S}_\nu = \bigcup_{\xi \succeq \nu} S_\xi$ and $\overline{T}_\lambda = \bigcup_{\eta \preceq \lambda} T_\eta$, we see that $\overline{S}_\nu \cap \overline{T}_\lambda = \overline{S}_\nu \cap \overline{T}_\lambda \cup Y$ where every component of Y has dimension strictly less than $\text{height}(\lambda - \nu)$, the dimension of a . (In fact, I believe Y is empty.) So $a \in \text{Irr}(\overline{S}_\nu \cap \overline{T}_\lambda)$ and clearly $a \subseteq \overline{\mathcal{G}}_\lambda$. \square

When $\nu = \lambda$ there is one MV-cycle, the point $\underline{\lambda}$. When $\nu = w_0\lambda$ (where w_0 is the longest element of the Weyl group), there is also one MV-cycle, the whole variety $\overline{\mathcal{G}}_\lambda$. In general, the number of irreducible components of $\overline{S}_\nu \cap \overline{\mathcal{G}}_\lambda$ is the dimension of the weight space ν in the irreducible representation V_λ [MV].

6. THE MOMENT MAP AND MV-POLYTOPES

The moment map μ , for the action of the torus \mathbf{T} on \mathcal{G} , is a map from \mathcal{G} to $\mathrm{Lie}(\mathbf{T})^*$. We use the Killing form to identify $\mathrm{Lie}(\mathbf{T})^*$ with $\mathrm{Lie}(\mathbf{T})$, which in turn is canonically identified with $\mathrm{Lie}(\mathbf{T})^*$. We will view $\mathrm{Lie}(\mathbf{T})^*$ as the codomain of μ , since moment map images will have to do with the representation theory of \mathbf{G} , which is usually pictured in the real subspace $\mathfrak{t}_{\mathbb{R}}^*$ of $\mathrm{Lie}(\mathbf{T})^*$.

We define μ as the restriction of the moment map on a projective space. Let \mathcal{L} be an ample line bundle on \mathcal{G} and $\Gamma(\mathcal{G}, \mathcal{L})$ the vector space of global sections. Then \mathcal{G} naturally embeds in the projective space $\mathbb{P}(V)$ where $V = \Gamma(\mathcal{G}, \mathcal{L})^*$ by mapping $x \in \mathcal{G}$ to the point determined by the line in V dual to the hyperplane $\{s \in \Gamma(\mathcal{G}, \mathcal{L}) \mid s(x) = 0\}$. The action of the torus \mathbf{T} on V decomposes it into eigenspaces: $V = \bigoplus_{\nu \in X^*(\mathbf{T})} V_{\nu}$, where $X^*(\mathbf{T})$ denotes the weights. Choose an inner product on V that is invariant under the action of the maximal compact subgroup of T , so that this decomposition is orthogonal. Then, given $v = \sum v_{\nu} \in V$, we define $\mu([v]) = \sum \frac{|v_{\nu}|^2}{|v|^2} \nu$, the usual moment map on a projective space.

Proposition 4.

1. *The fixed points of the torus action are the $\underline{\nu} \in \mathcal{G}$ (ν a coweight of \mathbf{G}); and $\mu(\underline{\nu}) = \nu$.*
2. *If X is a one-dimensional torus orbit then $\mu(X)$ is a line segment in a root direction joining two weights.*
3. *If a is an MV-cycle then $\mu(a)$ is a convex polytope. Its vertices are a collection of weights parametrized by the Weyl group, possibly in a degenerate way.*
4. *$\mu(\overline{\mathcal{G}_{\lambda}}) = \mathrm{conv}(W \cdot \lambda)$*
5. *If $X \subseteq \mathcal{G}$ is any compact torus-invariant variety and η is any coweight then $\mu(\eta X) = \eta + \mu(X)$.*

Proof. (1) It is easy to check that any $\underline{\nu}$ is a fixed point. That there aren't others follows, for instance, from decomposing \mathcal{G} into $N(\mathcal{K})$ orbits. For a coweight ν , we have $v = v_{\nu}$ so that $\mu(\underline{\nu}) = \nu$. (2) $\mu(X)$ is certainly a line segment joining two weights [GM3]. That it lies in a root direction follows from viewing \mathcal{G} as a partial flag variety and knowing the \mathbf{T} -invariant curves in a flag variety [C]. (3) The moment map image of any compact irreducible torus-invariant variety is the convex hull of the images of its \mathbf{T} -fixed points [B, GM3]. How the vertices are parametrized by the Weyl group is explained in [A]; this fact is not used here. (4) $\mu(\overline{\mathcal{G}_{\lambda}})$ is the convex hull of the images of its fixed points. Since the fixed points of \mathcal{G}_{λ} are the $W \cdot \lambda$, we have $\mu(\overline{\mathcal{G}_{\lambda}}) \supseteq \mathrm{conv}(W \cdot \lambda)$. But by the closure relations for strata, all the

other fixed points of $\overline{\mathcal{G}_\lambda}$ are also in this convex set. (5) For each fixed point $\underline{\xi}_i$ of X , we have $\boldsymbol{\mu}(\eta\underline{\xi}_i) = \boldsymbol{\mu}(\eta + \underline{\xi}_i) = \eta + \underline{\xi}_i = \eta + \boldsymbol{\mu}(\underline{\xi}_i)$. The $\eta\underline{\xi}_i$ are the fixed points of ηX . The statement follows since $\boldsymbol{\mu}(X)$ and $\boldsymbol{\mu}(\eta X)$ are the convex hulls of the $\boldsymbol{\mu}(\underline{\xi}_i)$ and $\boldsymbol{\mu}(\eta\underline{\xi}_i)$ respectively. \square

Definition 5. For each $\nu \in \Lambda^-$, let $\mathbb{B}_\nu = \text{Irr}(\overline{S_\nu \cap T_0})$, and for each irreducible component ϕ in \mathbb{B}_ν , let P_ϕ be its moment map image $\boldsymbol{\mu}(\phi)$. So $\mathcal{MV} = \bigcup_\nu \mathcal{MV}_\nu$ where $\mathcal{MV}_\nu = \{\boldsymbol{\mu}(a) | a \in \mathbb{B}_\nu\}$.

7. PROOF OF WEIGHT MULTIPLICITY CALCULATION (PART 1 OF THEOREM 1)

Proposition 6. Let $\underline{\xi}$ be any fixed point in the closure of the stratum $\overline{\mathcal{G}_\eta}$, where η is chosen dominant. An irreducible component a of $\overline{S_\xi \cap T_\eta}$ is an MV-cycle if and only if its moment map image $\boldsymbol{\mu}(a)$ is contained in $\boldsymbol{\mu}(\overline{\mathcal{G}_\eta})$.

Proof. If a is an MV-cycle then it is a component of $\overline{S_\xi \cap \mathcal{G}_\eta}$, which is contained in $\overline{\mathcal{G}_\eta}$. So $\boldsymbol{\mu}(a)$ is contained in $\boldsymbol{\mu}(\overline{\mathcal{G}_\eta})$.

To see the other direction, we need to use a larger torus action of $\mathbb{C}^* \times \mathbf{T}$ on $\mathcal{G} = \mathbf{G}(\mathcal{K})/\mathbf{G}(\mathcal{O})$: the first factor acts on the indeterminate that occurs in \mathcal{K} , and the \mathbf{T} acts just as before. All of the varieties in question are preserved by this action. Let us assume that $\boldsymbol{\mu}(a) \subseteq \boldsymbol{\mu}(\overline{\mathcal{G}_\eta})$. Suppose $x \in a$. We know that x is contained in some stratum, say \mathcal{G}_ϵ . Like all strata, this is a vector bundle over a flag manifold, $\mathbf{G}(\mathbb{C})_{\underline{\epsilon}}$, sometimes called the core. The action of small $t \in \mathbb{C}^*$ (the first factor in $\mathbb{C}^* \times \mathbf{T}$) on x sends it close to the core. Choosing a sequence of such t_n converging to 0, we can construct a point $y = \lim t_n x$ contained in both a and the core. Then acting by the second factor \mathbf{T} will allow us to move y arbitrarily close to some fixed point in the core; such points are the Weyl translates of $\underline{\epsilon}$. Since a is closed and preserved by the torus, this fixed point is contained in a . By assumption, it follows that ϵ lies in $\boldsymbol{\mu}(\overline{\mathcal{G}_\eta})$. By the closure relations for strata, this implies $\mathcal{G}_\epsilon \subseteq \overline{\mathcal{G}_\eta}$. Hence $x \in \overline{\mathcal{G}_\eta}$ and we have $a \subseteq \overline{\mathcal{G}_\eta}$. So a is an MV-cycle for \mathcal{G}_η . \square

Proof of Part 1 of Theorem 1. According to [MV], the weight multiplicity at weight ν in an irreducible representation V_λ equals the number of MV-cycles at weight ν , i.e. the number of components of $\overline{S_\nu \cap T_\lambda}$ that are contained in $\overline{\mathcal{G}_\lambda}$. By the preceding proposition, this is the number of $a \in \text{Irr}(\overline{S_\nu \cap T_\lambda})$ such that $\boldsymbol{\mu}(a)$ is contained in $\boldsymbol{\mu}(\overline{\mathcal{G}_\lambda}) = \text{conv}(W \cdot \lambda)$. But this is the same as the number of

$a \in \text{Irr}(\overline{S_{\nu-\lambda} \cap T_0})$ such that $\boldsymbol{\mu}(a) + \lambda$ is contained in $\text{conv}(W \cdot \lambda)$. Therefore the weight multiplicity is the number of $\phi \in \mathbb{B}_{\nu-\lambda}$ such that $P_\phi + \lambda \subseteq \text{conv}(W \cdot \lambda)$. \square

8. FIBERS OF THE CONVOLUTION MAP

This section describes the geometric idea that, when translated into the language of polytopes, yields the tensor product calculation of Theorem 1. Given two stratum closures $\overline{\mathcal{G}_\lambda}, \overline{\mathcal{G}_\mu}$ of the loop Grassmannian, we recall the construction of their twisted product $\overline{\mathcal{G}_\lambda} \tilde{\times} \overline{\mathcal{G}_\mu}$ and a convolution map π from this to $\overline{\mathcal{G}_{\lambda+\mu}}$. We show that the relevant irreducible components of the fibers of this map are MV-cycles.

8.1. Relative position convolution. We describe the relative position convolution of the closures of two strata in the loop Grassmannian. Given two points $a\mathbf{G}(\mathcal{O})$ and $b\mathbf{G}(\mathcal{O})$ in \mathcal{G} , one can ask: what is the relative position of $b\mathbf{G}(\mathcal{O})$ with respect to $a\mathbf{G}(\mathcal{O})$? By definition, this means: in what stratum is $a^{-1}b\mathbf{G}(\mathcal{O})$? So, given closures of strata $\overline{\mathcal{G}_\lambda}$ and $\overline{\mathcal{G}_\mu}$, we can define an algebraic variety often called the twisted product:

$$\overline{\mathcal{G}_\lambda} \tilde{\times} \overline{\mathcal{G}_\mu} = \{(a\mathbf{G}(\mathcal{O}), b\mathbf{G}(\mathcal{O})) \in \overline{\mathcal{G}_\lambda} \times \overline{\mathcal{G}_{\lambda+\mu}} \mid a^{-1}b\mathbf{G}(\mathcal{O}) \in \overline{\mathcal{G}_\mu}\}$$

This space maps to $\overline{\mathcal{G}_{\lambda+\mu}}$ by projection π onto the second factor, a stratified map of algebraic varieties. The decomposition theorem [M, BBD] applied to this map decomposes the intersection homology of $\overline{\mathcal{G}_\lambda} \tilde{\times} \overline{\mathcal{G}_\mu}$. The map is known to be semi-small [MV], which means that the dimension of a fiber over a stratum \mathcal{G}_ν is not larger than half the codimension of the stratum in $\overline{\mathcal{G}_{\lambda+\mu}}$. Because of this, the decomposition theorem has a particularly simple form in this case:

$$\text{IH}(\overline{\mathcal{G}_\lambda}) \otimes \text{IH}(\overline{\mathcal{G}_\mu}) \cong \text{IH}(\overline{\mathcal{G}_\lambda} \tilde{\times} \overline{\mathcal{G}_\mu}) \cong \bigoplus_{\mathcal{G}_\nu \subseteq \overline{\mathcal{G}_{\lambda+\mu}}} F_\nu \otimes \text{IH}(\overline{\mathcal{G}_\nu})$$

Here we take F_ν to be the vector space spanned by the fundamental classes of each component of the fiber over $\underline{\nu} \in \mathcal{G}_\nu$ that has maximum possible complex dimension—in this case the height of $\lambda + \mu - \nu$. These are called the relevant components since they are the ones that appear in the above decomposition. IH means the global intersection homology. It always has coefficients in the trivial local system since each stratum is simply connected [BD].

The isomorphism on the left, due to [BD, G, MV], allows one to relate this geometry to a tensor product of representations: $V_\lambda \otimes V_\mu = \bigoplus V_\nu^{\oplus \dim F_\nu}$.

8.2. Fibers are MV-cycles. Since the twisted product $\overline{\mathcal{G}_\lambda} \tilde{\times} \overline{\mathcal{G}_\mu}$ is a subset of $\overline{\mathcal{G}_\lambda} \times \overline{\mathcal{G}_{\lambda+\mu}}$ that maps to $\overline{\mathcal{G}_{\lambda+\mu}}$ by projection onto the second factor, we may view any fiber as a subset of the first factor. Viewed as such, we show that the relevant components of the fiber at $\underline{\nu} \in \overline{\mathcal{G}_{\lambda+\mu}}$ are MV-cycles for $\overline{\mathcal{G}_\lambda}$.

Theorem 7. *For the map $\pi : \overline{\mathcal{G}_\lambda} \tilde{\times} \overline{\mathcal{G}_\mu} \rightarrow \overline{\mathcal{G}_{\lambda+\mu}}$, the relevant irreducible components of the fiber $\pi^{-1}(\underline{\nu})$, for $\nu \preceq \lambda + \mu$ dominant, are MV-cycles for $\overline{\mathcal{G}_\lambda}$. They are precisely those MV-cycles that are also contained in $\nu \overline{\mathcal{G}_{-\mu}}$. If a is such a cycle then $\nu^{-1}a$ is an MV-cycle for $\overline{\mathcal{G}_{-\mu}}$ relative to the opposite choice of unipotent subgroup.*

Proof. As mentioned above, we view the fiber as a subset of $\overline{\mathcal{G}_\lambda}$:

$$\begin{aligned} \pi^{-1}(\underline{\nu}) &= \pi^{-1}(\nu \mathbf{G}(\mathcal{O})) \cong \{a \mathbf{G}(\mathcal{O}) \in \overline{\mathcal{G}_\lambda} \mid a^{-1} \nu \mathbf{G}(\mathcal{O}) \in \overline{\mathcal{G}_\mu}\} \\ &= \{a \mathbf{G}(\mathcal{O}) \in \overline{\mathcal{G}_\lambda} \mid \nu^{-1} a \mathbf{G}(\mathcal{O}) \in \overline{\mathcal{G}_{-\mu}}\} \\ &= \overline{\mathcal{G}_\lambda} \cap \nu \overline{\mathcal{G}_{-\mu}} \end{aligned}$$

We claim that

$$(1) \quad \overline{\mathcal{G}_\lambda} \cap \nu \overline{\mathcal{G}_{-\mu}} = (\overline{\mathcal{G}_\lambda} \cap \overline{S_{\nu-\mu}}) \cap \nu(\overline{\mathcal{G}_{-\mu}} \cap \overline{T_{-\nu+\lambda}}).$$

Obviously the left side contains the right side. The reverse containment is because $\nu \overline{\mathcal{G}_{-\mu}} \subseteq \overline{S_{\nu-\mu}}$ and $\overline{\mathcal{G}_\lambda} \subseteq \nu \overline{T_{-\nu+\lambda}}$. Let us check the first of these; the second is similar. We need the closure relations for strata and for unipotent orbits:

$$\begin{aligned} \mathcal{G}_\xi \subseteq \overline{\mathcal{G}_\eta} \quad (\xi, \eta \text{ dominant}) & \quad \text{iff } \xi \preceq \eta \quad \text{or, equivalently,} \\ \mathcal{G}_\xi \subseteq \overline{\mathcal{G}_\eta} \quad (\xi, \eta \text{ anti-dominant}) & \quad \text{iff } \xi \succeq \eta. \\ S_\xi \subseteq \overline{S_\eta} \quad (\xi, \eta \text{ any weights}) & \quad \text{iff } \xi \succeq \eta. \end{aligned}$$

So $\overline{S_{\nu-\mu}} = \bigcup S_{\nu-\delta} = \bigcup \nu S_{-\delta} = \nu \bigcup S_{-\delta} = \nu \overline{S_{-\mu}}$, each union taken over $\delta \preceq \mu$. Therefore it suffices to show $\overline{\mathcal{G}_{-\mu}} \subseteq \overline{S_{-\mu}}$. Suppose $x \in \overline{\mathcal{G}_{-\mu}}$. Then $x \in \mathcal{G}_{-\epsilon}$ for some $\epsilon \preceq \mu$; here $-\mu, -\epsilon$ are anti-dominant. Now, $x \in S_\delta$ for some $\delta \in \overline{\mathcal{G}_{-\epsilon}}$ since x lies in some MV-cycle for $\mathcal{G}_{-\epsilon}$ [MV]; so $\delta \succeq -\epsilon \succeq -\mu$. Hence $S_\delta \subseteq \overline{S_{-\mu}}$, and $x \in \overline{S_{-\mu}}$ as required.

Now, MV-cycles for $\overline{\mathcal{G}_\lambda}$ at weight $\nu - \mu$ are the irreducible components of $\overline{\mathcal{G}_\lambda} \cap \overline{S_{\nu-\mu}}$; similarly MV-cycles for $\overline{\mathcal{G}_{-\mu}}$ at weight $-\nu + \lambda$ relative to the opposite Borel are components of $\overline{\mathcal{G}_{-\mu}} \cap \overline{T_{-\nu+\lambda}}$. These sets may be smaller than the sets $\overline{\mathcal{G}_\lambda} \cap \overline{S_{\nu-\mu}}$ and $\overline{\mathcal{G}_{-\mu}} \cap \overline{T_{-\nu+\lambda}}$ in equation 1, but only by lower dimensional components. For instance, if we write $\overline{\mathcal{G}_\lambda} = \bigcup \mathcal{G}_{\lambda'}$ ($\lambda' \preceq \lambda$ dominant) and $\overline{S_{\nu-\mu}} = \bigcup S_{\nu-\mu'}$ ($\mu' \preceq \mu$), we see that

$$\overline{\mathcal{G}_\lambda} \cap \overline{S_{\nu-\mu}} = \bigcup_{\lambda', \mu'} \mathcal{G}_{\lambda'} \cap S_{\nu-\mu'} = \bigcup_{\lambda', \mu'} \overline{\mathcal{G}_{\lambda'} \cap S_{\nu-\mu'}}.$$

(To see the second equality, note that each closure in the third expression is a subset of the first expression.) The dimension of $\overline{\mathcal{G}_{\lambda'} \cap S_{\nu-\mu'}}$ is the height of $\lambda' + \mu' - \nu$, which is strictly smaller than the height of $\lambda + \mu - \nu$ unless $\lambda = \lambda'$, $\mu = \mu'$. Therefore $\overline{\mathcal{G}_{\lambda} \cap S_{\nu-\mu}}$ equals $\overline{\mathcal{G}_{\lambda} \cap S_{\nu-\mu}}$ possibly together with some lower dimensional MV-cycles.

We have shown that $\pi^{-1}(\underline{\nu})$ equals the intersection of $\overline{\mathcal{G}_{\lambda} \cap S_{\nu-\mu}}$ and $\nu(\overline{\mathcal{G}_{-\mu} \cap T_{-\nu+\lambda}})$ possibly together with some components of dimension less than $\lambda + \mu - \nu$. All statements in the theorem are proved. \square

9. PROOF OF TENSOR PRODUCT MULTIPLICITY CALCULATION (PART 2 OF THEOREM 1)

The preceding section gave a geometric interpretation to the decomposition of the tensor product of two irreducible representations V_{λ} and V_{μ} into irreducibles: the tensor product multiplicities are the dimensions of the F_{ν} . Theorem 7 suggests a method for their calculation: count the number of MV-cycles in each fiber. Part 2 of Theorem 1 describes this count in terms of moment map images.

For another point of view on using the loop Grassmannian to decompose tensor products, see [BG], where a construction of Kashiwara's crystal bases is given.

Proposition 8. *Let λ, μ, ν be dominant weights with $\nu \preceq \lambda + \mu$. An irreducible component a of $\overline{S_{\nu-\mu} \cap T_{\lambda}}$ is contained in $\overline{\mathcal{G}_{\lambda} \cap \nu \overline{\mathcal{G}_{-\mu}}}$ if and only if its moment map image $\mu(a)$ is contained in $\mu(\overline{\mathcal{G}_{\lambda}}) \cap (\mu(\overline{\mathcal{G}_{-\mu}}) + \nu)$.*

Proof. Obviously containment of the varieties implies containment of the moment map images. Conversely, if $\mu(a) \subseteq \mu(\overline{\mathcal{G}_{\lambda}}) \cap (\mu(\overline{\mathcal{G}_{-\mu}}) + \nu)$, then $\mu(a) \subseteq \mu(\overline{\mathcal{G}_{\lambda}})$ with $a \subseteq \overline{S_{\nu-\mu} \cap T_{\lambda}}$, and $\mu(\nu^{-1}a) \subseteq \mu(\overline{\mathcal{G}_{-\mu}})$ with $\nu^{-1}a \subseteq \overline{S_{-\mu} \cap T_{-\nu+\lambda}}$. Proposition 6 applied twice (once for the opposite unipotent) implies that a is contained in $\overline{\mathcal{G}_{\lambda}}$ and in $\nu \overline{\mathcal{G}_{-\mu}}$ as required. \square

Proof of Part 2 of Theorem 1. By Proposition 7, the multiplicity with which V_{ν} occurs in $V_{\lambda} \otimes V_{\mu}$ equals the number of MV-cycles for $\overline{\mathcal{G}_{\lambda}}$ contained in $\nu \overline{\mathcal{G}_{-\mu}}$. This is the number of irreducible components of $\overline{S_{\nu-\mu} \cap T_{\lambda}}$ contained in $\overline{\mathcal{G}_{\lambda} \cap \nu \overline{\mathcal{G}_{-\mu}}}$. By Proposition 8, this is the number of $a \in \text{Irr}(\overline{S_{\nu-\mu} \cap T_{\lambda}})$ such that $\mu(a)$ is contained in $\mu(\overline{\mathcal{G}_{\lambda}}) \cap (\mu(\overline{\mathcal{G}_{-\mu}}) + \nu) = \text{conv}(W \cdot \lambda) \cap (\text{conv}(W \cdot -\mu) + \nu)$. This is the same as the number of $a \in \text{Irr}(\overline{S_{\nu-\mu-\lambda} \cap T_0})$ such that $\mu(a) + \lambda$ is contained in $\text{conv}(W \cdot \lambda) \cap (\text{conv}(W \cdot -\mu) + \nu)$. Therefore the tensor product multiplicity is the number of $\phi \in \mathbb{B}_{\nu-\mu-\lambda}$ such that $P_{\phi} + \lambda \subseteq \text{conv}(W \cdot \lambda) \cap (\text{conv}(W \cdot -\mu) + \nu)$. \square

Remark. The two parts of Theorem 1 suggest that weight multiplicities and tensor product multiplicities are closely related. This is known, but since MV-cycles provide such a simple explanation of this, we highlight it as a theorem:

Theorem 9. *Suppose λ, μ are dominant weights and $\delta \succeq 0$ is such that $\lambda + \mu - \delta$ is dominant. Then the multiplicity of $V_{\lambda+\mu-\delta}$ in $V_\lambda \otimes V_\mu$ is less than or equal to the multiplicity of the weight $\lambda - \delta$ in V_λ . By Kostant's formula, this in turn is less than or equal to the number of ways of writing δ as a sum of positive roots; moreover, this bound is sharp in the sense that, given δ , if λ and μ are chosen sufficiently large, then for all $0 \preceq \epsilon \preceq \delta$ with $\lambda + \mu - \epsilon$ dominant, the multiplicity of $V_{\lambda+\mu-\epsilon}$ in $V_\lambda \otimes V_\mu$ exactly equals the number of ways of writing ϵ as a sum of positive roots.*

Proof. The multiplicity of the weight $\lambda - \delta$ in V_λ equals the number of $\phi \in \mathbb{B}_{-\delta}$ for which $P_\phi + \lambda \subseteq \text{conv}(W \cdot \lambda)$. (Part 1 of Theorem 1). The multiplicity of $V_{\lambda+\mu-\delta}$ in $V_\lambda \otimes V_\mu$ equals the number of these that are also contained in $\text{conv}(W \cdot -\mu) + \lambda + \mu - \delta$ (Part 2 of Theorem 1). Sharpness is because, given δ , we can choose λ and μ large enough that $\mu(\overline{\mathcal{G}_\lambda}) \cap (\mu(\overline{\mathcal{G}_{-\mu}}) + \lambda + \mu - \delta) = \mu(\overline{S_{\lambda-\delta} \cap T_\lambda})$ and that this contains only dominant weights. \square

10. HOPF ALGEBRA OF MV-CYCLES

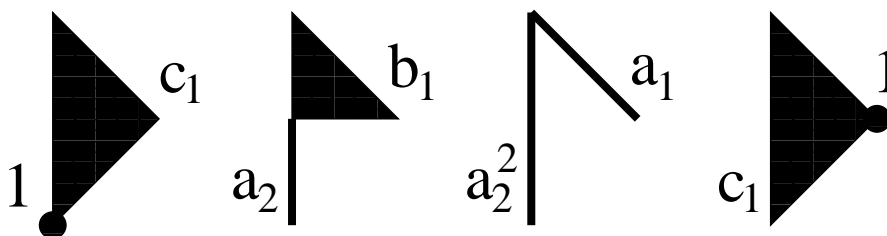
We close with a brief discussion of some related topics. This is mostly conjectural and is discussed in much more detail in [A]. There, we define a product on $\mathcal{A} = \text{span}(\mathcal{MV})$ and give a conjectural definition of a coproduct. Conjecturally, \mathcal{A} is isomorphic to the Hopf algebra of polynomial functions on the unipotent radical of a Borel subgroup of G . Another loop Grassmannian approach to the (dual) Hopf algebra may be found in [FFKM].

The product in \mathcal{A} is defined by a deformation of varieties over a curve, using an idea of Drinfeld's. There seem to be canonical generators and relations (described for some low rank groups in section 3), but little is understood about them. Positive integer coefficients appear in the relations because they are multiplicities of irreducible components.

The coproduct is not well understood. As an example, the coproduct and antipode for Sp_4 are given in the following table. (It is only necessary to specify them on the generators since these functions are multiplicative.)

Name x	Antipode $S(x)$	Coproduct $\Delta(x)$
1	1	$1 \otimes 1$
a_1	$-a_1$	$a_1 \otimes 1 + 1 \otimes a_1$
a_2	$-a_2$	$a_2 \otimes 1 + 1 \otimes a_2$
b_1	b_2	$b_1 \otimes 1 + a_1 \otimes a_2 + 1 \otimes b_1$
b_2	b_1	$b_2 \otimes 1 + a_2 \otimes a_1 + 1 \otimes b_2$
c_1	$-c_2$	$c_1 \otimes 1 + 2b_1 \otimes a_2 + a_1 \otimes a_2^2 + 1 \otimes c_1$
c_2	$-c_1$	$c_2 \otimes 1 + a_2^2 \otimes a_1 + 2a_2 \otimes b_2 + 1 \otimes c_2$
c_3	$-c_3$	$c_3 \otimes 1 + b_2 \otimes a_2 + a_2 \otimes b_1 + 1 \otimes c_3$
d_1	d_1	$d_1 \otimes 1 + c_1 \otimes a_1 + 2b_1 \otimes b_2 + a_1 \otimes c_2 + 1 \otimes d_1$

Just as the product in \mathcal{A} corresponds to Minkowski sum of polytopes, the coproduct has a (conjectural) interpretation in terms of polytopes as well: Write $\Delta(x) = \sum k_{ij} x_i \otimes x'_j$ where each x_i is an MV-cycle relative to N and each x'_j is an MV-cycle relative to N^- . If $k_{ij} \neq 0$ then (1) x_i and x'_j are contained in x ; (2) x_i and x'_j are associated to the same weight in x , and this is their only point of intersection; (3) k_{ij} is a positive integer. The picture



illustrates this for $\Delta(c_1) = c_1 \otimes 1 + 2b_1 \otimes a_2 + a_1 \otimes a_2^2 + 1 \otimes c_1$.

What are the meanings of the k_{ij} and why are they positive? If x is the largest MV-cycle in a stratum, they seem to give the intersection form in intersection homology [A]. If not (as in the above example) their meaning is mysterious.

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