Mathematics and Statistics Department Faculty
Publication Series

Mathematics and Statistics

2002

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C I. Siettos

I G. Kevrekidis

PG Kevrekidis
University of Massachusetts - Amherst, kevrekid@math.umass.edu

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C.I. Siettos and I.G. Kevrekidis

Department of Chemical Engineering, Princeton University, 6 Olden Str. Princeton, NJ 08544

P.G. Kevrekidis

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-4515, USA

(February 8, 2008)

The nonlinear Schrödinger (NLS) equation is a ubiquitous example of an envelope wave equation for conservative, dispersive systems. We revisit here the problem of self-similar focusing of waves in the case of the focusing NLS equation through the prism of a dynamic renormalization technique (MN dynamics) that factors out self-similarity and yields a bifurcation view of the onset of focusing. As a result, identifying the focusing self-similar solution becomes a steady state problem. The discretized steady states are subsequently obtained and their linear stability is numerically examined. The calculations are performed in the setting of variable index of refraction, in which the onset of focusing appears as a supercritical bifurcation of a novel type of mixed Hamiltonian-dissipative dynamical system (reminiscent, to some extent, of a pitchfork bifurcation).

Self-focusing and wave collapse have received a considerable amount of attention in many diverse branches of physics, ranging from optics [1,2] to hydrodynamics [3] and from gravity [4] to plasma physics [5]. The interested reader can also consult [6,7].

The most frequently studied model equation in the context of self-focusing and the ensuing blow-up of the relevant solutions is the two-dimensional (2D) and three-dimensional (3D) nonlinear Schrödinger (NLS) equation. This benchmark example is well-known (see e.g., [1] and references therein) to exhibit unstable solitary wave solutions which collapse in finite time. Numerous publications have addressed the focusing properties of the solutions (Refs. [8,9] can only claim to give a number of representative examples of this large volume of work). Recently, a revived interest in the subject has led to numerous new insights [10].

Here we take a slightly different approach to the same problem, viewing it from the perspective of dynamical systems theory and making the connections to both earlier work on the NLS problem [8,9] as well as more recent work on self-similar blowup problems [11]. Many of the results that are presented here on NLS were previously known from the works of [8,9]. Such results include the effective factoring out of the self-similarity to view focusing as a steady state problem, as well as the continuation of the branch of focusing solutions. What we believe is new in our exposition is the intuition it provides by identifying the onset of focusing as a steady state bifurcation of a (transformed) partial differential-algebraic equation (PDAE). This appears to us to be a new type of bifurcation reminiscent of a supercritical pitchfork. The identification of focusing solutions as steady states of this system also allows us to linearize around them and examine (as well as justify) their stability and the ensuing structure of the bifurcation diagram. Here these computer-assisted tasks are performed for a finite discretization of the problem.

Finally, it is worth noting that this approach has been implemented in a somewhat different but equivalent form of the regular blow-up NLS problem. In particular, in previous studies see e.g., [1,8,9], the radial form of the NLS equation was analyzed,

$$R_{rr} + \frac{d-1}{r}R_r - R + |R|^2 R = 0, \tag{1}$$

where $R = R(r)$ denotes the radial part of the solution. The analysis (when performed parametrically) would then involve a slightly unphysical "bifurcation parameter", namely the spatial dimensionality d. In contrast to that, but equivalently for the purposes of the analysis, we use a different (but more physically realistic) bifurcation parameter. The latter is the power of the nonlinearity σ in the model:

$$iu_t = -\Delta u - |u|^{2\sigma} u; \tag{2}$$

u is the complex envelope of the electric field (in optical applications) and the present setting arises when the index of refraction has a power law dependence (σ being the relevant power) on the intensity of the field. It is well known that the model of Eq. (2) becomes critical (i.e., unstable to wave collapse) for $\sigma \geq 2/d$, where d is the spatial dimensionality of the Laplacian in (2) (see e.g., [1,12]). In view of that, we choose to study the model in *one spatial dimension*, using σ as a bifurcation parameter, with $\sigma_{cr} = 2$ in that case. Notice that for $\sigma = 1$, the model becomes the celebrated integrable example of the 1D cubic NLS [13].

When considering self-focusing solutions of Eq. (2) it is customary to make a straightforward appropriate scaling. In particular, using $u = L^{-1/\sigma}v(\xi, \tau)$, with $\xi = x/L$, $\tau = \int_0^t L^{-2}(t)dt$, one obtains

$$iv_\tau = -v_{\xi\xi} - |v|^{2\sigma}v - ia(\tau)(\frac{1}{\sigma}v + \xi v_\xi), \tag{3}$$

where $a(\tau) = -L_\tau/L$ (see e.g., [1]). A direct pointer to the criticality of $\sigma = 2$ is the fact that the norm is preserved under the rescaling only for that value of σ. Hence, dynamical (i.e., permitted by the time evolution) rescaling for the original NLS model of Eq. (2) is *only* permissible for $\sigma = 2$. On the other hand, Eq. (3) has appeared in various forms in a number of works on the focusing problem such as [8,9] (see [1] for a review of the relevant methods and results), in the context of what was called "dynamic rescaling". The idea behind such a rescaling is that the scaling is chosen in such a way that the rescaled time goes to infinity as the (finite) time of the singularity $t^\star$ is approached. The closure of the system (with an equation for $a(\tau)$) was given by imposing an arbitrary, but physically or numerically motivated, constraint, such as the preservation of a certain norm of the solution.

Here, we will follow a slightly different path, motivated from a general approach to self-similar problems recently presented in [11]. In particular, if we consider the equation

$$iu_t = F(u), \tag{4}$$

typically the potential for self-similar solutions arises when a scaling is inherent in the nonlinear operator of the right hand side (RHS) of (4). In general:

$$F\left(Bu(\frac{x}{L})\right) = L^c B^d F\left(u(\xi)\right), \tag{5}$$

where $\xi = x/L$ (keeping in line with the notation used in earlier works such as [1,8]). The dispersive operators with on-site nonlinearity considered here are however different from the ones of [11] in that the presence of the onsite potential imposes a certain restriction on the scaling through Eq. (5), namely for the F of the RHS of 4)

$$B = L^{-\frac{1}{\sigma}} \tag{6}$$

$$F\left(Bu(\frac{x}{L})\right) = \frac{1}{L^2}F\left(u(\xi)\right). \tag{7}$$

Using in the equation the ansatz $u = B(\tau)v(x/L(\tau), \tau)$, where τ is an, as of yet, undetermined function of the original time t, we obtain:

$$i\left(\frac{B_\tau}{B}v - \frac{L_\tau}{L}\xi v_\xi + v_\tau\right)\tau_t = -\frac{1}{L^2}\left(v_{\xi\xi} + |v|^{2\sigma}v\right), \tag{8}$$

where the subscripts denote respective partial derivatives. Notice also that using Eq. (6), $B_\tau/B = -L_\tau/(\sigma L)$ and hence the self-similarity has a single scaling factor, L. Time parametrization can be chosen on the basis of convenience and hence we use here τ such that $\tau_t = 1/L^2$. If we look, as is customary (see e.g., [1] and references therein) for a *standing wave* solution of Eq. (8), we use the so-called phase invariance to look for solutions of the form $v \to \exp(i\Lambda\tau)v$, which gives the same equation as before but with an extra $\Lambda v/L^2$ in the RHS. Without loss of generality, we set $\Lambda = 1$. Finally, one has the one-parameter family (L) freedom of the group orbit of self-similarity (see [14] for the case of traveling solutions; the case of self-similarity is treated in [11]). To determine the scaling factor, we lift the symmetry-induced degeneracy through a pinning condition [15] of the form

$$\int_{-\infty}^{\infty} Re(v(\xi, \tau))T(\xi)d\xi = C, \tag{9}$$

where C is a constant. $T(\xi)$ is an arbitrary, so-called "template" function. Alternatively, one can consider maximazing the inner product with the template (such an approach has been considered in [14] for settings with travelling wave solutions, as well as ones with self-similar solutions). Differentiating Eq. (9) and using eq. (8), we arrive to the following equations for the real and imaginary parts of the solution $v = U + iW$

$$U_\tau = -W_{\xi\xi} - (U^2 + W^2)^\sigma W + W - G(\frac{1}{\sigma}U + \xi U_\xi) \tag{10}$$

$$W_\tau = U_{\xi\xi} + (U^2 + W^2)^\sigma U - U - G(\frac{1}{\sigma}W + \xi W_\xi), \tag{11}$$

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where

$$G = -\frac{L_\tau}{L} = -\frac{\int_{-\infty}^{\infty} \left[W_{\xi\xi} + (U^2 + W^2)^\sigma W - W \right] T(\xi) d\xi}{\frac{C}{\sigma} + \int_{-\infty}^{\infty} \xi U_\xi T(\xi) d\xi}. \tag{12}$$

Eqs. (10)-(11) supplemented with Eq. (12) constitute the MN-dynamics formulation of the focusing problem for the NLS equation.

The regular solitary wave solutions of the equation (2) exist as stationary solutions of the equation ensuing from (2) upon the substitution $u = \exp(it)v$, or equivalently as "trivial" focusing solutions of the MN equations (10)-(11) with a zero blowup rate, $G = 0$. These solutions are stable for $\sigma < 2$. Their mechanism of instability involves a pair of linearization eigenvalues that bifurcate from the band edge of the continuous spectrum, for $\sigma > 1$. The dispersion relation describing the continuous spectrum is $\lambda = \pm i(1 + k^2)$, where λ is the eigenvalue corresponding to wavenumber k, for our case of $\Lambda = 1$. This pair of eigenvalues moves towards the origin of the spectral plane, where 2 more pairs of eigenvalues lie due to the 2 invariances (translational and phase) of Eq. (2). At $\sigma = 2$, the additional pair "arrives" at the origin, thereafter exiting along the real axis, rendering the soliton branch unstable as is shown in Fig. 1.

Beyond the instability threshold, the soliton is well-known to be unstable towards focusing, resulting in a self-similar blowup for $\sigma > 2$. Such solutions are most commonly obtained through integration of the dynamic rescaling equations (or even possibly of the original equations in an adaptively refined mesh), see e.g., [1,8,9] and references therein. Here we use an alternative perspective[1]. We identify the soliton branch as a special solution (with $G = 0$) of Eqs. (10)-(11). We then use a "soliton-motivated" pinning condition. In particular, we use as our arbitrary template a Dirac mass, centered at a given point x_0, forcing the real part of the solution to have at x_0 the same value as the soliton does for $\sigma = 2$. Other pinnings (for the imaginary part, or for combination of the real and the imaginary part) have also been used with the same results. The pinning condition in conjunction with Eqs. (10)-(11) allows us to construct for $\sigma > 2$ the branch of self-focusing solutions as steady states through the use of Newton iteration and pseudo-arclength continuation [16].

The algebraic problem is solved using the boundary value solver of the package gPROMS[2], a commercial simulation tool for solving systems of ordinary, and/or partial differential equations and/or of partial differential-algebraic equations of index 1. The pseudo-arclength continuation method was incorporated within gPROMS. The computational domain was chosen to be the right half-axis $\xi = 0$ (due to symmetry of the solutions sought). The length and the tessellation of the domain were chosen appropriately to ensure reliability of the solutions, i.e., robustness against further discretization refinement and domain enlargement. The solutions were approximated using a centered finite difference method of order two in a domain of length equal to 14 and a total of 700 discretization intervals, while the absolute accuracy was set to $1E - 06$. The solutions we obtained are shown in Fig. 2.

An additional advantage of effectively "factoring out" the self-similarity through probing the solutions as steady states of the MN problem, stems from our ability to perform linear stability analysis computations for these solutions. It should be noted that linear stability within a renormalization group approach to self-similar solutions was also considered in gravitational problems, see e.g., [4]. Such numerical computations for the truncated problem are shown in Fig. 2. These computations are for $G > 0$, and clearly indicate that the dynamical system of Eqs. (10)-(11) is a dissipative (a non-Hamiltonian) one. This is also structurally clear in the equations, as terms of the form $-iv$ are well-known dissipative perturbations of NLS. Analyzing the structure of the "umbrella-shaped" spectrum in comparison with the linearization spectrum of the Hamiltonian problem for $G = 0$, we recognize that the genuinely complex eigenvalues constitute the continuous spectrum. In addition, we (typically) obtain 3 eigenvalues along the real line. In the Hamiltonian untransformed problem, there are 6 eigenvalues that are near the origin for σ close to 2, 4 of which are at the origin (two with a corresponding symmetric, zero node eigenvector, related to phase invariance and two with an antisymmetric, one node eigenvector, corresponding to translational invariance). The additional pair that eventually leads to the instability has a corresponding symmetric eigenvector with two nodes. In the dissipative MN system, we are left with only 3 eigenvalues, since 2 out of the 6 are absent due to considering only half the domain (where only symmetric eigenfunctions will be present). An additional eigenmode is absent due to imposing the pinning condition (decreasing by 1 the degrees of freedom of the system). One of the three eigenvalues is at the origin, as the MN dynamics also possess a conservation law, namely Eq. (9). The phase invariance is, however,

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[1] Our work has a somewhat similar flavor to Phys. Rev. A **38**, 3837 (1988), but is posed in a different, we believe slightly more general framework that allows us to perform a systematic bifurcation analysis of the results.

[2] see e.g., www.psenterprise.com

explicitly broken here due to the presence of the term proportional to G. In fact, one can show that unless one is "exactly on" a steady state of Eqs. (10)-(11), neither the invariance $v \rightarrow v \exp(i\phi)$ (for ϕ an arbitrary constant), nor the corresponding conservation law $d||u||_2/dt = 0$ (where $||u||_2$ is the L^2 norm of the solution) are respected, for the pinning of Eq. (9). In fact, it can be straighforwardly predicted within the realm of (leading order) perturbation theory that the bifurcation of the phase eigenvalue for $\sigma \rightarrow 2^+$ can be approximated by

$$\lambda \sim G \frac{\sigma - 2}{2\sigma}. \tag{13}$$

This shows that, as is indeed verified in numerical computations, the bifurcation of the phase eigenvalue is in the real positive semi-axis of the spectral plane, for $\sigma > 2$. However, such an instability (which is a genuine one for the non-phase invariant MN dynamics) will *not* be relevant for the original norm preserving (Hamiltonian) dynamics and hence the self-focusing solution will be a stable blowup solution of Eq. (2). I.e., the presence of a conservation law in the original NLS model does not permit the dynamical development of this instability, or, equivalently, does not permit the system to "explore" this unstable eigendirection.

As σ is increased from the critical value, the generation of the blowup branches can be seen to occur in a form reminiscent of a supercritical pitchfork-like bifurcation in Fig. 1. Notice the steepness of the variation of G in the neighborhood of the critical point. This very abrupt increase of the value of G, as well as earlier asymptotic arguments seem to support a dependence beyond all algebraic orders of G on $\sigma - 2$ (i.e., an exponential dependence). However, the rigorous proof of such an estimate is beyond the scope of the present work and will be addressed in a future publication. However, it should be highlighted that this is a rather unusual, and most probably novel form of a dynamical system, possessing solutions with as well as without focusing, the corresponding dynamics however being significantly modified (from dissipative to Hamiltonian) between the two cases. Identifying the normal form behavior of such dynamical systems close to criticality (close to the onset of focusing) is an interesting open problem. Furthermore, it should be noted that a natural setting for the study of the bifurcations of such dynamical systems appears to be the framework of the relevant differential-algebraic equations (due to the presence of the constraint).

Notice that for $\sigma \rightarrow 2^+$, $G \rightarrow 0$ very abruptly. For $\sigma = 2$, there is no solution with $G \neq 0$ [8]. Dynamically, this is manifested as $\lim_{\tau \to \infty} G(\tau) = 0$, the approach to the limit being logarithmic [8]. It would be interesting to explore whether travelling problems can exhibit similar critical behavior.

In summary, in this work we have presented a different, bifurcation-based perspective to the focusing problem in the ubiquitous NLS equation. We have "factored out" the self-similarity group action through the MN dynamics approach and set the problem up as a steady state problem in the latter framework. This allowed us to identify stable self-focusing solutions for the original problem (which actually are unstable for MN dynamics). We were able continue the relevant branches, constructing the supercritical pitchfork-like bifurcation diagram that constitutes the focusing instability for this novel (mixed Hamiltonian-dissipative) dynamical system. We were able to identify the eigenvalues of the focusing steady states of the MN dynamics and to explain their spectrum on the basis of the knowledge of the "soliton branch" spectrum. The theoretical study of the normal form of the MN-dynamics that could provide analytical justification of the scaling behavior near the transition is currently under study and will be reported in future publications.

[1] C. Sulem and P.L. Sulem, *The Nonlinear Scrödinger Equation* (Springer-Verlag, New York 1999).

[2] J. Juul Rasmussen and K. Rypdal, Phys. Scr. **33**, 481 (1986).

[3] G.I. Barenblatt, *Scaling, Self-Similarity and Intermediate Asymptotics* (Cambridge University Press, Cambridge 1996).

[4] see e.g., M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993); T. Koike, T. Hara and S. Adachi, Phys. Rev. Lett. **74**, 5170 (1995).

[5] L. Bergé, Phys. Rep. **303**, 259 (1998).

[6] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Boston 1992).

[7] M.P. Brenner *et al.*, Nonlinearity **12**, 1071 (1999).

[8] see e.g., B.J. LeMesurier, G.C. Papanicolaou, C. Sulem and P.L. Sulem, Phys. **31D**, 78 (1986); *ibid.* **32D**, 210 (1988); D.W. McLaughlin, G.C. Papanicolaou, C. Sulem and P.L. Sulem, Phys. Rev. A **34**, 1200 (1988); M.J. Landman, G.C.

Papanicolaou, C. Sulem and P.L. Sulem, Phys. Rev. A **38**, 3837 (1988); N. Kopell and M.J. Landman, SIAM J. Appl. Math. **55**, 1297 (1995).

[9] V.E. Zakharov and V.F. Shvets, JETP Lett. **47**, 275 (1988); N.E. Kosmatov, V.F. Shvets and V.E. Zakharov, Phys. **52D**, 16 (1991); S.N. Vlasov, L.V. Piskunova and V.I. Talanov, Sov. Phys. JETP **68**, 1125 (1986).

[10] G. Fibich and G. Papanicolaou, SIAM J. Appl. Math. **60**, 183 (1999); P. Plechak and V. Sverak, Comm. Pure Appl. Math. **54**, 1215 (2001); C.J. Budd, SIAM J. Appl. Math. **62**, 801 (2001).

[11] D.G. Aronson, S.I. Betelu and I.G. Kevrekidis, nlin.AO/0111055 (2001).

[12] M.I. Weinstein, Nonlinearity **12**, 673 (1999). It is worth noting here that the model of variable σ has been considered in numerous other works such as B.A. Malomed and M.I. Weinstein, Phys. Lett. A **220**, 91 (1996) or S. Flach, K. Kladko and R.S. MacKay, Phys. Rev. Lett. **78**, 1207 (1997). The variable exponent in the nonlinearity may be physically reasonable in the case of a thermal nonlinearity [B. Malomed, personal communication].

[13] M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia (1981).

[14] C.W. Rowley and J.E. Marsden, Physica **142D**, 1 (2000); see also C.W. Rowley, K. Lust and I.G. Kevrekidis (in preparation).

[15] This is a standard technique in finding travelling waves, where the relevant symmetry that creates the degeneracy is translational invariance. It was only recently, however, appreciated that this can be applied to any one parameter family of solutions induced by the action of a symmetry group.

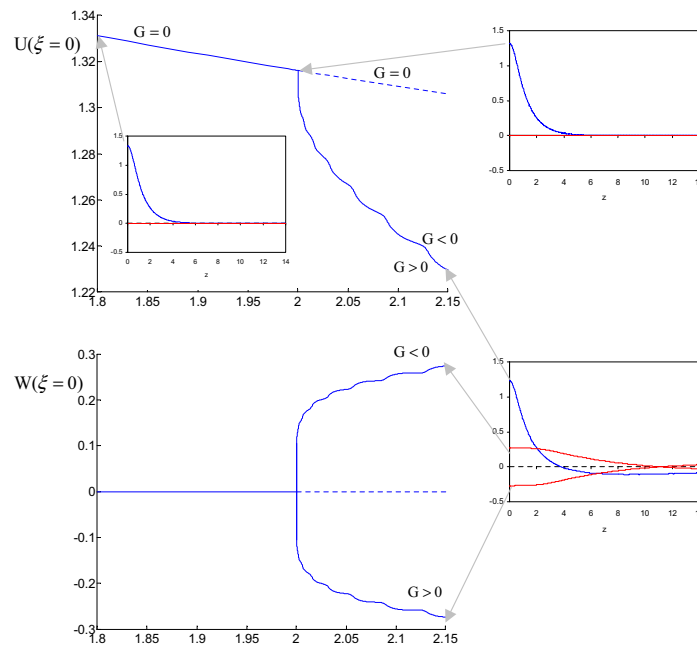[16] E.J. Doedel, Int. J. Bifurcat. Chaos **1**, 493 (1991).

FIG. 1. The figure shows the variation as a function of σ of the real (top panel) and imaginary (bottom panel) parts of the field evaluated at $\xi = 0$. The soliton branch is stable for $\sigma < 2$ (solid line), while it becomes unstable (dashed line) for $\sigma \geq 2$, giving rise to the blowup branches (with $G \neq 0$). The insets show the spatial profile of the solution for the positive semi-axis.
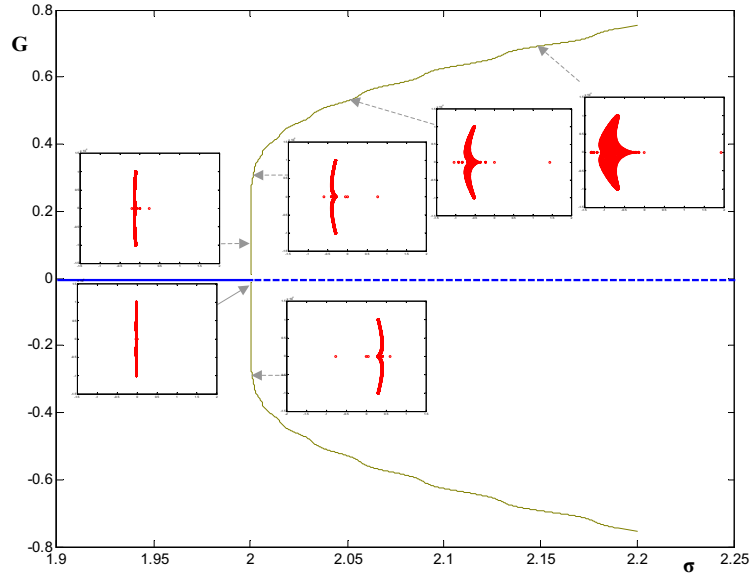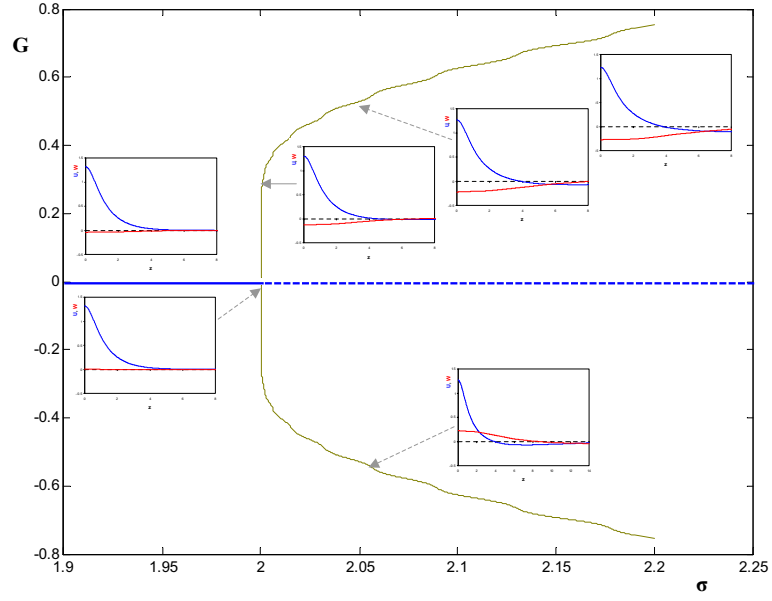
FIG. 2. The top panel shows the MN-dynamics bifurcation diagram of G vs. σ. At $\sigma_{cr} = 2$, the new branch of focusing solutions is born. The panel insets show the profile of the solution in different points along the branch. The bottom panel insets show the corresponding MN-dynamics stability analysis through the linearization eigenvalues. Notice the non-Hamiltonian nature of the dynamics, as well as the presence of one unstable (real and positive) eigenvalue in the upper branch (for details see text). The spectrum of the lower branch is the mirror symmetric of the upper one with respect to the imaginary axis.