A Dynamic Theory of Fidelity Networks with an Application to the Spread of HIV/AIDS[1]

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Abstract: We study the dynamic stability of (in)fidelity networks, which are networks that form in a mating system of agents of two types (say men and women), where each agent desires direct links with opposite type agents, while engaging in multiple partnerships is considered an act of infidelity. Infidelity is punished more severely for women than for men. We consider two stochastic processes in which agents form and sever links over time based on the reward from doing so, but may also take non-beneficial actions with small probability. In the first process, an agent who invests more time in a relationship makes it stronger and harder to break by his/her partner; in the second, such an agent is perceived as weak. Under the first process, only egalitarian pairwise stable networks (in which all agents have the same number of partners) are visited in the long run, while under the second, only anti-egalitarian pairwise stable networks (in which all women are matched to a small number of men) are. Next, we apply these results to a new index of contagion to find that, in the long run, under the first process, HIV/AIDS is equally prevalent among men and women, while under the second, women bear a greater burden. We argue that the finding is qualitatively consistent with empirical observations.

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1 Introduction

We study a dynamic theory of *(in)fidelity networks*, which are networks that form in a mating system involving two types of agents (say *men* and *women*).[4] We consider bipartite graphs, which means that each link connects a member of one of the two sets of agents (a man) with a member of the other set (a woman). Each agent derives utility from having relationships with the other type, but having multiple partners is viewed as infidelity, which is punished if detected. There is *female discrimination* in that infidelity is punished more severely for women than for men.[5]

Our first aim is to precisely characterize both static and long-run predictions of fidelity networks. Next, using a simple process of information diffusion, we also analyze the effect of network configurations on the spread of a random infection shock, and derive implications for gender differences in the concentration of such an infection. Indeed, as a second goal of our study, we apply the theoretical analysis of dynamic fidelity networks to shed light on the social mechanism that underlies gender differences in HIV/AIDS prevalence in heterosexual cultures.[6]

According to UNAIDS (2008), globally the share of women among HIV infected adults has grown from 43% in 1990 to 50% in 2001 when it stabilized; in sub-Saharan Africa, it has grown from 53% in 1990 to 60% in 2007. Furthermore, surveys commissioned by the United States Agency for International Development show that, in most developing countries, women bear a disproportionate share of the HIV/AIDS burden (Mishra et al. (2009)).

Early on, it was hypothesized that the male-to-female transmission rate of the AIDS virus is greater than the female-to-male transmission rate, which was proposed as a leading plausible explanation for the higher prevalence of HIV/AIDS among women. That argument primarily rests on physiological factors (see WHO (2003)). But as pointed out in the same WHO's report, the evidence on this subject is incomplete. In fact, two ground-breaking studies (Quinn et al. (2000), Gray et al. (2001)) showed that the female-to-male transmission rate is greater than the male-to-female transmission rate, but the difference is not statistically significant. Thus, the question of the origins of gender differences in HIV infections remains open, and would benefit from alternative hypotheses.

The contribution of the current study is therefore twofold. First, with our assumptions, we are able to provide simple characterizations of static and dynamic predictions in fidelity networks; second, these results

[4] Pongou (2010) was the first to propose fidelity networks. The reader is referred to it for motivation of the model and other applications. The current paper is an outgrowth of its Chapter 3. Fidelity happens when an agent chooses to have only one partner, while multiple partners are instances of infidelity. The model allows for both. We stick to the "fidelity network" name having this in mind.

[5] Our model is an example of a repugnant market (Roth (2007)), where links cannot be sold or bought.

[6] Although our model is in principle applicable to other venereal diseases, we emphasize HIV transmission. A distinctive feature of HIV/AIDS is that it is not being cured yet, which provides a natural environment to test the predictions of our model. Other venereal diseases can be treated, and if there is any correlation between gender and health care utilization, the observed gender differences in those other diseases may not totally reflect the effect of sexual network configurations.

are used to provide a new hypothesis linking female discrimination to gender differences in an important public health problem.[7]

1.1 Fidelity Networks as an Example of Social Networks

HIV/AIDS has been studied in a wide range of literatures in the social sciences. However, the reasons why more women than men are affected by this epidemic in some societies and not in others have not been studied. From a methodological point of view, one aspect that has not been sufficiently explored is the application of the theory of social networks.[8] This is surprising, since one would expect that the structure and evolution of social networks, viewed as entities that describe sexual behavior, must have some bearing on the problem and may shed new light on this serious public health crisis.[9]

Our formal setting consists of a finite population of two equal-size sets of *men* and *women*. The trade-off between the benefit and the infidelity cost of having multiple partners results in each agent having a single-peaked utility function: each agent has an optimal number of partners. We assume peak homogeneity within each side. However, due to gender asymmetry in the punishment of infidelity, the optimal number of partners is strictly greater for each man than for each woman.

1.2 Static Analysis of Fidelity Networks: Pairwise Stability

We first characterize the *pairwise stable networks*. In a mating problem such as ours, individuals form new links or sever existing links based on the reward that the resulting network offers them relative to the current network. We say that a network is pairwise stable if: (*i*) no individual has an incentive to sever an existing link he or she is involved in, and (*ii*) no pair of a man and a woman have a strict incentive to form a new link between them while at the same time possibly severing some of the existing links they are involved in.[10]

We shall assume that our population is sufficiently large, which allows for a simple characterization of pairwise stable networks.[11] In particular, we find that a network is pairwise stable if and only if each woman has exactly her optimal number of partners, and each man has at most his optimal number of partners. Women supply a smaller number of links than the ones demanded by men, which in turn results in only men competing for female partners while each woman is sure of having the number of male partners she desires.

[7] We also note that while emphasis is placed on the HIV application, Pongou (2010) argues how a similar framework can apply to several bipartite economies outside of the fidelity context (e.g., the market between students and their advisors, buyers and sellers, faculty and their departments, or countries and their citizens). Our theory can therefore lend itself to different interpretations.

[8] Morris and Kretzschmar (1997) also use a network model to study the differential effects of serial monogamy and concurrent partnerships on the spread of HIV/AIDS, but they do not address the question of its gender gap prevalence.

[9] See Vega-Redondo (2007), Jackson (2008), and Easley and Kleinberg (2010) for authoritative monographs on social and economic networks.

[10] See Gale and Shapley (1962) for a first use of pairwise stability. Within networks, Jackson and Wolinsky (1996) provide the standard definition. Our definition is slightly different: they allow weak blocking in the pair, as opposed to strict blocking, as we do.

[11] Pongou (2010, Chapter 2) provides a full characterization of pairwise stable networks without the "large populations" assumption made here, but unlike the current paper, his analysis is only static.

1.3 Unperturbed Dynamic Analysis: Steady-State Networks

The center of our analysis is a dynamic matching process for the mating problem, more precisely a Markov process. Random encounters between men and women are based on the incentives that agents have to form new links or sever existing ones. Specifically, the *unperturbed Markov process* assumes discrete time, and is defined as follows. In each period, a man and a woman chosen at random with arbitrary positive probability are given the opportunity to sever or add a link based on the improvement that the resulting network offers to each of them relative to the current network. If they are already linked in the current network, the decision is whether to sever the link; severance is a unilateral decision. Otherwise, the decision is whether to form a new link; link formation is a bilateral decision. While forming a new link, each agent is allowed to sever as many of the links he/she is involved in as possible in the current network (although, because of our simplifying single-peaked preferences assumption, without loss of generality, one can restrict attention to the case of severing only one link). The *long run predictions* –steady or recurrent states– of this process coincide with the set of pairwise stable networks, a very large set. Such a large set does not deliver any clear result in terms of gender differences in contagion.

1.4 Perturbed Dynamic Analysis: Stochastically Stable Networks

To gain predictive power in our analysis, the matching process is perturbed in two different ways, corresponding to two diametrically opposed sociological realities. Each perturbation consists of allowing a small probability of forming new links or severing existing ones when this action is not beneficial to the agents involved. We study the long-run predictions of these *perturbed processes* –their *stochastically stable networks*–, these predictions being the only networks that are visited a positive proportion of time in the very long run.[12]

In both perturbed dynamic processes, if a link formation is mutually beneficial or if a link severance is beneficial to its initiator, it occurs with probability 1. That is, this feature of the unperturbed dynamics is retained. However, the perturbed processes allow for more transitions. In both processes, an action that worsens its initiator, which we shall call a *mistake*, occurs with a small probability $\varepsilon > 0$. Key to our analysis are in-between actions that leave their initiators exactly indifferent. We shall refer to these as *utility neutral actions* or *neutral actions* for short. In the spirit of assuming that more serious mistakes are less likely, an agent's probability of taking a neutral action will always exceed ε. We now explain how.

In our models, neutral actions uniquely correspond to situations in which an agent severs an existing link with a current partner and forms a new link with another agent. We shall assume that the probability of

[12] In a perturbed process, one can no longer speak of "steady states," as by definition, there is always a positive probability of transiting from any state to any other. The notion of stochastic stability (Freidlin and Wentzell (1984)) provides a useful methodology to identify those states in which the perturbed process spends most of its time in the long run. It has been applied to study a number of problems in the economics literature (see, e.g., Foster and Young (1990), Kandori, Mailath and Rob (1993), Young (1993) for early contributions. Young (1998) presents many of its applications. The main shortcoming of stochastic stability is its associated slow speed of convergence, but it is very helpful in identifying long run trends, our main interest here. Also, the reader should keep in mind that the frequency of a random encounter between a man and a woman may be extremely high, perhaps every minute, thus allaying the concern.

taking such a neutral action is $\varepsilon^{f(\cdot)}$ (a number strictly greater than ε because the exponent will be smaller than 1). The exponent is the *strength of the existing link* so that stronger links $-f(\cdot)$ closer to 1– are harder to break.

In the first perturbed process, the strength $f(\cdot)$ of a severed link is *inversely* proportional to the number of partners that the old partner had in the existing network. The interpretation is that this link is as strong as the amount of time invested in it by the other partner. In this process, we find that networks are stochastically stable if and only if they are *egalitarian pairwise stable networks*. Men and women have the same number of partners, which is the optimal number of partners for women. Monogamous networks are a salient particular case, if such a number is 1.

In contrast, the second perturbed process assumes that the strength of a severed link $f(\cdot)$ is *directly* proportional to the number of partners that the old partner had in the existing network. Now, the interpretation is that in a relationship, the partner who invests more time on it is perceived as "weak" or dominated by the other partner; and thus, it is easier for the dominant partner to break the relationship.[13] For this case, we find that *anti-egalitarian pairwise stable networks*, which are networks in which each woman has her optimal number of partners, and the smallest possible set of men is matched, will be the only ones visited a positive proportion of time in the very long run. Each non-isolated man is matched to his optimal number of partners (except for at most one matched man, who will be matched to the remaining women). The rest of men will remain isolated. In the special case when each woman optimally has one partner, polygynous networks are selected.

There are several advantages to considering both perturbed processes.[14] Theoretically, studying polar opposites in the assumptions behind neutral actions offers a more complete understanding of the problem, and the rationale behind these actions offers an interesting alternative to justify the perturbations (in addition to mutations, experimentation or mistakes, invoked in previous literature). Empirically, the two approaches are consistent with different sociological realities, prevalent in different societies.

1.5 Back to Gender Differences in HIV/AIDS Prevalence

We now return to the problem posed at the outset, concerning gender differences in the spread of HIV/AIDS. To this end, we shall use the following framework, proposed in Pongou (2010). Assume that an agent is drawn at random from a network to receive a piece of information (interpreted here as an instance of HIV virus

[13] For a possible justification of the assumption in this second process, see Tertilt (2005) and further evidence from anthropologists (Pat Caldwell (1976), John C. Caldwell (1976, 1978), John C. Caldwell, Pat Caldwell and Orubuloye (1992), Quale (1992)). Some of these studies highlight the dominant role of men in male-female relationships as measured, for instance, by the small amount of time that men spend with their wives in societies like sub-Saharan Africa (a survey in Nigeria showed that "fewer than one-third of wives normally eat with their husbands or seat together on occasion" (John C. Caldwell (1976)), which markedly contrasts with what is observed in the West.) Our assumption differs in that it is gender-neutral: in a relationship between a man and a woman, the dominated partner, regardless of his/her gender, is the partner who invests more time in it. However, we will see that even this more general assumption leads to networks in which connected men dominate their female partners and invest less time in their relationships.

[14] Bergin and Lipman (1996) show that one can always construct processes with state-dependent perturbations that will select any subset of the steady-states as stochastically stable. An important implication of this result is that one should motivate the particular perturbed processes that one chooses to work with.

infection due to a random event). He/she then communicates it to his/her partners, who in turn communicate it to their other partners, and so on. If that agent has no partner, the information does not spread. Under the assumption that each agent is drawn with equal probability, one can define the *communication or contagion potential* of that network, which is the expected proportion of agents who will receive the information, and provide a formula for this notion. Similarly, one can also derive a formula for *gender difference in contagion potential* in a network. The key in these formulae is that the contagion potential in a gender is proportional to the sum of squares of the agents of that gender in each component of the network; see Section 7 for details.

We show that under the first perturbed dynamic process, gender difference in contagion potential in any of the stochastically stable networks is zero. Under the second process, women's contagion potential is greater than men's, which shows the higher vulnerability of women to HIV/AIDS.[15]

Our results are consistent with trends in gender difference in HIV/AIDS in most regions of the world, especially sub-Saharan Africa and the Caribbean where UNAIDS (2008) has established that heterosexual relationships are the main mode of HIV transmission – see Figure 1. The global trend and the trend in the Caribbean are consistent with the long-run prediction of the first perturbed process. The prediction of the second process is consistent with sub-Saharan Africa.[16] Our results are also consistent with the possibility of HIV/AIDS being greater among men than women at certain periods of the transition process. When this is the case, according to our model, one should observe that the share of infected women is growing over time relative to the share of infected men. This seems to be qualitatively consistent with the levels of and trends in gender difference in HIV/AIDS in Asia, Latin America, Eastern Europe and Central Asia.

Our findings seem to shed new light on the origins of gender differences in HIV/AIDS, but our model is simply a first step in the application of network theory to these issues. For tractability reasons in the dynamic analysis, many complexities have been left out, and future research ought to address them. These include homosexuality, prostitution, drug injecting, the case in which an agent's well-being is affected by indirect links, or the presence of agents mindful of the disease.[17] All of these are important, but beyond our current scope.

1.6 Plan of the Paper

The remaining of this paper unfolds as follows. Section 2 introduces the model that forms the basis for our analysis. We characterize pairwise stable networks in Section 3. In Section 4, we define the unperturbed Markov process and characterize its recurrent or steady states. This process is perturbed in Section 5 and

[15]The unequivocally bad result for women in the stochastically stable networks of the second process may seem surprising, given that the definition of the perturbed process itself is "gender neutral". However, in combination with our assumption of asymmetric infidelity costs across genders, all the key transitions involve a woman severing a link to form a new one, and in doing so, the cost of breaking that link is a direct function of the dominant role of her former male partner, measured by the number of his links.

[16]It follows from these data, if one removes sub-Saharan Africa and the Caribbean from the sample, that the proportion of affected men exceeds that of women. Although other explanations have been given (e.g., according to UNAIDS (2008), male homosexuality and drug injecting with contaminated needles significantly contribute to transmission), our assumption of higher infidelity punishment against women is likely less compelling precisely in those societies, especially in developed countries.

[17]This last point is consistent with the fact that the AIDS virus has existed since at least 1959, but was only discovered in 1981 (Worobey et al. (2008)).

Section 6, and a characterization result of stochastically stable networks is provided for each of the two perturbed systems, respectively. In Section 7, we study the implications of our results for gender differences in HIV/AIDS. Section 8 discusses the related theoretical literature, and we conclude our study in Section 9. Section 10 collects all the proofs.

2 The Model

The social environment consists of a finite set of individuals $N = \{1, 2, \ldots, n\}$, partitioned into a set of men M and a set of women W, each of equal size. Each individual derives utility from direct links with opposite sex agents, but engaging in multiple links is an act of infidelity, and is punished if detected by the cheated partner. Detection occurs with positive probability. It is assumed that a woman whose infidelity is detected is more severely punished than a man in a similar situation. Networks that arise from this environment are called fidelity networks.

2.1 Utility Functions

Let $\overline{M} = M \cup \{\emptyset\}$ be the expanded set of men, and $\overline{W} = W \cup \{\emptyset\}$ the expanded set of women. A network g is a subset of $\overline{M} \times \overline{W}$ such that $\forall m \in M, (m, \emptyset) \in g \Longrightarrow \forall w \in W$, $(m, w) \notin g$ and $\forall w \in W, (\emptyset, w) \in g \Longrightarrow \forall m \in M$, $(m, w) \notin g$. Here, $(m, \emptyset) \in g$ means that m is connected to no woman in g, and similarly, $(\emptyset, w)$ means that w is connected to no man in g. If an agent is not connected in a network g, we say that such an agent is isolated in that network.

Let g be a network. Since we are dealing with undirected graphs, if $(i, j) \in g$, we will abuse notation and consider that $(j, i) \in g$ (in fact, (i, j) and (j, i) represent the same relationship). Let $i \in N$ be an individual, and $s_i(g)$ the number of opposite sex partners that i has in the network g. The utility that i derives from g is expressed by the following function (we distinguish between men's an women's utility, i.e., $\lambda = m, w$):

$$u_i^{\lambda}(s_i(g)) = v(s_i(g)) - c^{\lambda}(s_i(g))$$

where $v(s_i(g))$ is the utility derived from direct links with opposite sex partners in g, and is concave and strictly increasing in $s_i(g)$; and $c^{\lambda}(s_i(g))$ the cost of infidelity.

Let us define a possible cost function more precisely. Let $j, k \in N$ be such that $(i, j) \in g$ and $(i, k) \in g$. Let π be the probability that j detects the liaison (i, k), and c^{λ} the cost incurred by i if j detects that liaison. Because i has $s_i(g)$ partners, he/she will be detected $s_i(g)(s_i(g) - 1)$ times with probability π, incurring an expected cost of $s_i(g)(s_i(g) - 1)\pi c^{\lambda}$. So we define the cost function as:

$$c^{\lambda}(s_i(g)) = s_i(g)(s_i(g) - 1)\pi c^{\lambda}$$

An agent will thus maximize the following utility function:

$$u_i^\lambda(s_i(g)) = v(s_i(g)) - s_i(g)(s_i(g)-1)\pi c^\lambda$$

The utility function admits the following interpretation. We may assume that there is a "social stigma" associated with being caught in an act of infidelity. Upon each infidelity detection, the cheater then incurs or internalizes a certain cost c^λ, perhaps psychological, perhaps a social sanction inflicted to him/her by others in society.

Denote the extension of u_i^λ to the non-negative reals as $\overline{u_i^\lambda}(s_i)$. Without loss of generality, let $\overline{u}_i^\lambda$ be twice continuously differentiable. The following remark is straightforward:

Remark 1 *(1)* $\exists s^* > 0$ *such that* $\overline{u}^{\lambda\prime}(s^*) = 0$*;* $\forall s \in [0, s^*)$, $\overline{u}^{\lambda\prime}(s) > 0$*; and* $\forall s \in (s^*, +\infty)$, $\overline{u}^{\lambda\prime}(s) < 0$.

(2) $\frac{\partial s^*}{\partial c^\lambda} \leq 0$

Remark 1 implies that u_i^λ is single-peaked. Given that the cost incurred per detection is equal for all individuals of the same sex, they have the same optimal number of partners. Further, assuming that $c^w > c^m$, the optimal number of partners for women is smaller than the optimal number of partners for men because the former are more severely punished than the latter if their infidelity is detected. Note that if s^* is not an integer, then the optimal number of partners will be either the largest integer smaller than s^* $\lfloor s^* \rfloor$ or the smallest integer greater than s^* $\lceil s^* \rceil$. The fact that $c^\lambda(0) = c^\lambda(1) = 0$ is a good justification of the "infidelity cost" terminology.[18]

More generally, all that matters for the analysis is the location of peaks. Indeed, one can drop all previous assumptions on the utility and cost functions. Preferences may differ within each side away from the peak, one can simply assume that agents on the same side have the same peak, and that the men's peak is larger than the women's. A host of different hypotheses may justify this. We also postulate that for no $s \geq 0$, $u_i^\lambda(s) = u_i^\lambda(s+1)$.

In order to derive our results, we will make a "large populations" assumption. As it turns out, this will imply that the unique integer that is optimal for men is greater than the unique integer optimal for women, our key assumption. Specifically:

Assumption A1. We assume $|M|$ to be large enough.[19]

For the kinds of applications we have in mind, A1 is an appropriate assumption.

2.2 Definitions of Concepts in Networks

Let g be a fidelity network. The elements of N are called vertices. A path in g connecting two vertices i_1 and i_n is a set of distinct nodes in $\{i_1, i_2, \ldots, i_n\} \subset N$ such that for any k, $1 \leq k \leq n-1$, $(i_k, i_{k+1}) \in g$.

[18] Note that Remark 1 would still hold if we had assumed a more general cost function that is convex and increasing in the number of partners. In this sence, the function $c^\lambda(s) = s(s-1)\pi c^\lambda$ is just an example of a cost function that has an intuitive interpretation.

[19] Denoting by s_m^* and s_w^* the unique optimal integer number of partners for men and women, Respectively, A1 actually implies the inequality used in the different proofs, which is $\frac{|M|-(s_w^*-1)}{|M|} > \frac{s_w^*}{s_m^*}$, which itself implies that $s_m^* > s_w^*$.

Let i be an individual. We denote by $g(i) = \{j \in N : (i,j) \in g\}$ the set of individuals who have i as a partner in the network g. The cardinality of $g(i)$ is called the degree of i. If a set A is included either in M or W, then the image of A in the network g is $g(A) = \bigcup_{i \in A} g(i)$.

We denote respectively by $M(g) = \{i \in M : \exists j \in W, (i,j) \in g\}$ and by $W(g) = \{i \in W : \exists j \in M, (i,j) \in g\}$ the set of men and women who are matched in the network g. We pose $N(g) = M(g) \cup W(g)$.

A subgraph $g' \subset g$ is a component of g if for any $i \in N(g')$ and $j \in N(g')$ such that $i \neq j$, there is a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$ such that $(i,j) \in g$, $(i,j) \in g'$. A network g can always be partitioned into its components. This means that if $C(g)$ is the set of all components of g, then $g = \bigcup_{g' \in C(g)} g'$, and for any $g' \in C(g)$ and $g'' \in C(g)$, $g' \cap g'' = \emptyset$.

3 Pairwise Stable Networks

In a society such as the one we are describing, individuals form new links or sever existing links based on the improvement that the resulting network offers them relative to the current network. We say that a network g is pairwise stable if: (i) no individual has an incentive to sever an existing link he/she is involved in, and (ii) no pair of a man and a woman have an incentive to form a new link between them while possibly at the same time severing some of the existing links they are involved in.

More formally, given a profile of utility functions $u = (u_i)_{i \in N}$, a network g is pairwise stable with respect to u if:

(i) $\forall i \in N$, $\forall (i,j) \in g$, $u_i(s_i(g)) \geq u_i(s_i(g \setminus \{(i,j)\}))$

(ii) $\forall (i,j) \in (M \times W) \setminus g$, if network g' is obtained from g by adding the link (i,j) and perhaps severing other links involving i or j, $u_i(s_i(g')) > u_i(s_i(g)) \Longrightarrow u_j(s_j(g')) \leq u_j(s_j(g))$ and $u_j(s_j(g')) > u_j(s_j(g')) \Longrightarrow u_i(s_i(g')) \leq u_i(s_i(g))$.

According to (ii), (i,j) is a blocking pair whenever the two parties involved strictly benefit from the union. In this sense, link formation is driven only by self-interest, and so, an agent does not enter a relationship if he/she has no incentives to do so.[20] In this, our definition is different from the one introduced by Jackson and Wolinsky (1996), where two agents form a link if one is willing to do so and the other is indifferent.

To illustrate this definition, consider the following examples. A network in which a woman is matched to $s > s_w^*$ men is not pairwise stable as she can unilaterally sever $s - s_w^*$ links. A network in which a man is matched to $s_m^* + 2$ women and a woman not matched to him matched to fewer than s_w^* men is not stable, as they could form a link while the man could sever three of his former links (alternatively, the man alone could sever only one of his links). Finally, a network in which a man and a woman who are unmatched have fewer than their optimal partners is not pairwise stable either, as they could form a link without severing any other.

[20] In the absence of side payments, the strict improvement of each individual in the pair is a natural assumption (see, e.g., Aumann (1959)).

3.1 Characterization of the Pairwise Stable Networks

In this subsection, under our "large populations" assumption, we characterize the pairwise stable networks. This characterization will be useful in our dynamic analysis later on. It says that a network is pairwise stable if and only if each woman has exactly her optimal number of partners and each man has at most his optimal number of partners.

Theorem 1 *Assume A1, and let g be a network. Then, g is pairwise stable if and only if $\forall(m, w) \in M \times W$, $0 \leq s_m \leq s_m^*$ and $s_w = s_w^*$.*

The intuition for the theorem is simple enough. One could view men making offers to women in sequence, who accept offers until they reach their peak.

Let us illustrate Theorem 1 with the following example.

Example 1 *Consider a mating problem in which there are 10 men and 10 women. Assume that their utility functions are such that $s_w^* = 2$ and $s_m^* = 4$. The three networks represented respectively by Figure 2-1, Figure 2-2 and Figure 2-3 are pairwise stable. In fact, in each graph, each woman has 2 partners (the optimal number of partners for each woman), and each man has at most 4 partners. In the first network component configuration $[(2,2);(5,5);(3,3)]$[21], all agents have 2 partners, thus this network is egalitarian; in the second network component configuration $[(7,6);(2,4);(1,0)]$, 2 men have 1 partner each, 5 men have 2 partners each, 2 men have 4 partners each, and 1 man has no partner; in the third network component configuration $[(2,4);(2,2);(2,4),(1,0),(1,0),(1,0),(1,0)]$, 2 men have 2 partners each, 4 men have 4 partners each, and 4 men have no partner. An interesting feature of the last two graphs is the uneven share of female partners among men, which reveals a sharp competition in the latter group.*

4 A Dynamic Network Formation Process

In this section we turn to dynamics. First, we shall define a Markov process for any given mating problem as previously defined, to describe the formation and severance of links over time. Later on, given the lack of predictive power of this process, we shall resort to perturbing it in two different ways, leading to two perturbed Markov processes, studied in Sections 5 and 6, respectively.

The unperturbed Markov process P^0 is as follows. Time is discrete. In each period, a man and a woman chosen at random with arbitrary positive probability are given the opportunity to sever or add a link based on the improvement that the resulting network offers to them relative to the current network (think of a random encounter between the two individuals at a party). If they are already linked in the current network, the decision is whether to sever the link. Otherwise, the decision is whether to form a new link. While forming a

[21] $[(2,2);(5,5);(3,3)]$ refers to a network component configuration with 3 components, the first containing 2 men and 2 women, the second 5 men and 5 women, and the third containing 3 men and 3 women. This notation is a simplification that abstracts from the complete network structure as represented by the graph.

new link, each agent is allowed to sever as many of the links he/she is involved in as possible in the current network. Link severance is unilateral, while link formation is bilateral.

Let g and g' be two networks. They are said to be adjacent if g' is obtained from g by agent severing an existing link he/she is involved in in g, and possibly forming a new link with an agent of the opposite type. More formally, g and g' are adjacent if there exist $i \in M$ and $j \in W$ such that $g' \in \{g+ij, g+ij-ik, g+ij-ik-jm, g+ij-jm, g-ij\}$.[22] Let x and y be two networks. An $(x, y)-path$ is a finite sequence of networks $(g^0, g^1, \ldots, g^k)$ such that $g^0 = x$, $g^k = y$ and for any $t \in \{0, 1, \ldots, k-1\}$, g^t and g^{t+1} are adjacent.

An improving path from x to y is a finite sequence $x = g^0, g^1, \ldots, g^k = y$ such that for any $t \in \{0, 1, \ldots, k-1\}$, the transition from g^t to g^{t+1} strictly benefits its initiator(s). More formally:

- (i) $g^{t+1} = g^t - ij$ for some ij such that $u_i(s_i(g^{t+1})) > u_i(s_i(g^t))$ or $u_j(s_j(g^{t+1})) > u_j(s_j(g^t))$; or
- (ii) $g^{t+1} \in \{g^t+ij, g^t+ij-ik, g^t+ij-ik-jm, g^t+ij-jm\}$ for some ij such that $u_i(s_i(g^{t+1})) > u_i(s_i(g^t))$ and $u_j(s_j(g^{t+1})) > u_j(s_j(g^t))$. Here, without loss of generality, we do not allow for an agent severing more than one link when forming a new link.

Recurrent classes of a Markov process are those sets of states such that, if reached, the process cannot get out of them, and which do not contain a smaller set with the same property. We next characterize the recurrent classes of the unperturbed markov process P^0.

Theorem 2 *The recurrent classes of the unperturbed markov process P^0 are singletons, whose union coincides with the set of pairwise stable networks.*

Thus, the set of long run predictions of the unperturbed dynamics is quite large (recall the characterization in Theorem 1). We proceed by perturbing this process in the sequel. We shall define below two such perturbed processes.[23]

5 The First Perturbed Markov Process P_1^ε

In this section, we define and analyze the first perturbed process. In each period, the revision opportunity offered at random to a male-female pair is the same as described in the process P^0. However, now agents may make decisions that do not necessarily lead to an immediate individual improvement. We describe these events in detail.

- If the two agents are linked in the current network:

[22] We simplify notation here and write ij instead of (i, j), $g + ij$ instead of $g \cup \{(i, j)\}$, and $g - ij$ instead of $g \setminus \{(i, j)\}$, etc.

[23] One could consider a related model that avoids perturbations of the basic Markov process. In it, agents' preferences are lexicographic with respect to number of links and neutral actions (in that order). However, we note that the models are not equivalent. For instance, Example 4 in Pongou and Serrano (2009) (an earlier version of the current paper) shows that the network represented by Figure 5-1 is pairwise stable under the lexicographic specification, yet it is not stochastically stable in our processes.

– Link severance takes place with probability 1 if it benefits either of the two agents, just as before.

– Otherwise, while in the unperturbed process no severance of this link was taking place, now if it makes the two agents worse off, severance takes place with probability ε (note that in our model, link severance cannot make an agent indifferent). Recall that link severance is a unilateral decision, and thus it takes one "mistake" to sever such a good link: an agent making a mistake with probability $\varepsilon > 0$.

- If the two agents are not linked in the current network, the decision is whether to form a new link:

  – This link formation takes place with probability 1 if it is mutually beneficial, just as before. All other transitions did not happen in the unperturbed process, while now they will.

  – If forming the link makes one agent worse off and the other better off –one "mistake"–, it occurs with probability ε.

  – If the link formation makes the two agents worse off –two "mistakes"–, it occurs with probability ε^2.

  – If the transition makes one agent better off and the other agent, say j, indifferent, agent j may take this "neutral action" and looks at considerations other than his/her well-being. Indifference in the transition happens because, while forming a new link with i, j severs an existing link, say with agent k in the current network. Then, the resistance of this transition amounts essentially to the strength of the severed link. Specifically, we assume that the transition occurs with probability $\varepsilon^{f(\frac{1}{s_k})}$ where the link strength f is a strictly increasing function of $\frac{1}{s_k}$ mapping into $(0, 1)$. Here, s_k is the number of partners that k has in the current network. We offer an interpretation below, at the end of the description of the process.

  – If the transition makes one agent worse off and the other agent indifferent (one "mistake" and one "neutral action"), the transition occurs with probability $\varepsilon * \varepsilon^{f(\frac{1}{s_k})} = \varepsilon^{1+f(\frac{1}{s_k})}$.

  – Finally, if it makes the two agents indifferent (two "neutral actions"), meaning that while forming a new link, i and j severed links with, say h and k, respectively in the current network, it occurs with probability $\varepsilon^{f(\frac{1}{s_h})} * \varepsilon^{f(\frac{1}{s_k})} = \varepsilon^{f(\frac{1}{s_h})+f(\frac{1}{s_k})}$.

We emphasize our assumption on the resistance of transitions involving indifferences or "neutral actions", the key transitions for our results. The function $f(\frac{1}{s_k})$ can be viewed as the strength of the link that is being severed by j. If we assume for instance that each agent is endowed with 1 unit of time that he/she splits equally among all his/her partners, then it makes sense to assume that the strength of a link is inversely proportional to the number of partners.[24]

[24] Although for simplicity we assume that j observes s_k, slightly weaker assumptions would do, as j could evaluate the strength $f(\frac{1}{s_k})$, for instance through a noisy signal, such as the amount of time spent by the partner out of the house. We do not model

It follows from the description above that the probabilities of the perturbed processes can be seen as being "utility driven." When there is a strict improvement following a link formation, it happens with probability 1, while the infinitesimal probability comes into play only when there is no increase in utility.

5.1 Resistance of a Path and Stochastic Stability

For any adjacent networks g and g', the resistance of the transition from g to g' , denoted $r(g, g')$, is the weighted number of agents directly involved in the transition who do not find this change profitable; it is the exponent of ε in the corresponding transition probability. We explicitly define $r(g, g')$ in the table below, as a function of the possible frictions –"mistakes" or "neutral actions"– found in a randomly chosen pair (i, j). To read the table, note that there are only three actions that either i or j can take, some combinations of which might not be possible:

A- Forming a new link without severing an existing link.

B- Forming a new link while severing an existing link.[25]

C- Severing an existing link.

Let (a_i, a_j) be the pair of actions taken by i and j, respectively. Then $(a_i, a_j) \in \{(A, A), (A, B), (B, B), (C, C)\}$. A pair of actions (a_i, a_j) might make either agent better off (b), lose (l), or indifferent (i). Transition probabilities and resistances are summarized in Table 1 below.

Table 1

(a_i, a_j)	*Outcomes*	*Probability*	$r(g, g') = \log_\varepsilon(\textit{probability})$
(A, A)	(b, b)	1	0
(A, A)	(b, l)	ε	1
(A, A)	(l, l)	ε^2	2
(A, B)	(b, i)	$\varepsilon^{f(\frac{1}{s_k})}$	$f(\frac{1}{s_k})$
(A, B)	(l, i)	$\varepsilon^{1+f(\frac{1}{s_k})}$	$1 + f(\frac{1}{s_k})$
(B, B)	(i, i)	$\varepsilon^{f(\frac{1}{s_h})+f(\frac{1}{s_k})}$	$f(\frac{1}{s_h}) + f(\frac{1}{s_k})$
(C, C)	(b, b)	1	0
(C, C)	(b, l)	1	0
(C, C)	(l, l)	ε	1

The resistance of an (x, y)-path $q = (g^0, g^1, \ldots, g^k)$ is the sum of the resistances of its transitions: $r(q) = \sum_{t=0}^{k-1} r(g^t, g^{t+1})$.

Let $Z^0 = \{g^0, g^1, \ldots, g^l\}$ be the set of absorbing states of the unperturbed process (the pairwise stable networks, in our case).[26] Consider the complete directed graph with vertex set Z^0, denoted ∇. The resistance of the edge (g^i, g^j) in ∇ is the minimum resistance over all the resistances of the $(g^i, g^j) - \textit{paths}$: $r(g^i, g^j) = \textit{minimum}\{r(q) \mid q \text{ is an } (g^i, g^j)\text{-path}\}$.

incomplete information in this paper: a next step in the analysis of the fidelity model would be not to assume observability of the number of your partner's partners. For the use of stochastic stability, the agent may not be aware of the exact probability of each event happening, which is just a parameter of the overall dynamics.

[25] Forming a new link while severing more than one link, if not utility improving, is a transition with strictly higher resistance than the one severing only one link, and hence, it can be safely ignored in the subsequent analysis.

[26] Absorbing states are those in singleton recurrent classes.

Let g be an absorbing state. A g-tree is a tree whose vertex set is Z^0 and such that from any vertex g' different from g, there is a unique directed path in the tree to g. The resistance of a g-tree is the sum of the resistances of the edges that compose it. The stochastic potential of g, denoted $r(g)$, is the minimum resistance over all the $g - trees$.

The set of stochastically stable networks is the set $\{g \mid r(g) \leq r(g') \text{ for all } g'\}$ (Young (1993), Kandori, Mailath and Rob (1993)). Intuitively, this set is the set of states (or networks in our case) that are visited a positive proportion of time in the long run.

5.2 The Result

We shall now characterize the set of stochastically stable states of the perturbed process P_1^{ε}. The following definitions and lemmas are needed.

Let g be a network. We shall say that g is egalitarian if all vertices have the same degree; that is, if all individuals have the same number of partners.

Pose $I(g) = \{i \in M : s_i(g) \geq s_j(g) \ \forall j \in M\}$, i.e., the set of men who are linked to the highest number of women in the network g.

Let $J(g) = \{i \in M : s_i(g) \leq s_j(g) \ \forall j \in M\}$, i.e., the set of men who are linked to the lowest number of women in the network g.

And call $I^*(g) = \{i \in M : s_i(g) \geq s_w^*\}$, i.e., the set of men who have at least a number of partners no less than the women's optimal number.

It is obvious that, if g is pairwise stable, $I(g)$, $J(g)$ and $I^*(g)$ are non-empty. Let $L(g) = \sum_{i \in I^*(g)} (s_i(g) - s_w^*)$.

The following lemma states that, under our large populations assumption, any non-egalitarian pairwise stable network is such that any man in $I(g)$ is matched with more than s_w^* partners, and any man in $J(g)$ is matched with less than s_w^* partners.

Lemma 1 *Assume A1, and let g be a non-egalitarian pairwise stable network. Then, $\forall (i,j) \in I(g) \times J(g)$, $s_i(g) > s_w^* > s_j(g)$ (and therefore, $s_i(g) \geq s_j(g) + 2$).*

The following lemma describes a simple way to reach egalitarian networks travelling through pairwise stable networks from any initial pairwise stable network.

Lemma 2 *Let g be a pairwise stable network. Then, there exists a finite sequence of pairwise stable networks $(g^0, g^1, \ldots, g^k)$ such that $g^0 = g$, $g^k = g^{L(g)}$, and g^k is egalitarian.*

In addition, any two egalitarian pairwise stable networks are "connected". This is shown in the following connectivity lemma:

Lemma 3 *Let g and g' be two distinct egalitarian pairwise stable networks. Then, there exists a finite sequence of pairwise stable networks $(g^0, g^1, \ldots, g^{2k})$ such that $g^0 = g$, $g^{2k} = g'$, and for any t such that $0 \leq t \leq k$, g^{2t} is egalitarian.*

We are now ready to state the main result of the section:

Theorem 3 *Assume A1. A network is stochastically stable in the perturbed process P_1^{ε} if and only if it is egalitarian and pairwise stable.*

The interested reader may find illustrations of the workings of Theorem 3, and Theorem 4 below, in Pongou and Serrano (2009), which provides examples to show how networks that are not stochastically stable transition into stochastically stable ones.

6 The Second Perturbed Process P_2^{ε}

In this section, we define and analyze the second perturbed process. This process is defined as the first perturbed process in Section 5, the only difference being the definition of the probability of a "neutral action", an action that leaves an agent indifferent. Recall that that probability was based on the strength of the link to be broken to form the new link. Now, the strength of such a link will now be inversely proportional to the amount of time invested in it. This corresponds to a situation in which an agent who invests too much time in a relationship might be perceived as weak or dominated in that relationship. [27] We describe next more formally the only change in assumptions with respect to the previous perturbed process:

- A person who is indifferent in a particular transition, and in it, breaks an existing link with another person who has s_k partners in order to form a new link looks at the strength of the link he/she severs. That strength $f(s_k)$ is strictly increasing in s_k and strictly bounded between 0 and 1.

6.1 Resistance of a Path

All the definitions of resistance provided earlier apply to this section as well. For completeness, for each adjacent transition in the perturbed process P_2^{ε}, its probability and resistance are summarized in Table 2 below. It uses the same notation employed in Table 1:

[27] As noted in the introduction (footnote 13), a possible justification of this assumption comes from the anthropological literature.

Table 2

(a_i, a_j)	*Outcomes*	*Probability*	$r(g, g') = \log_\varepsilon(probability)$
(A, A)	(b, b)	1	0
(A, A)	(b, l)	ε	1
(A, A)	(l, l)	ε^2	2
(A, B)	(b, i)	$\varepsilon^{f(s_k)}$	$f(s_k)$
(A, B)	(l, i)	$\varepsilon^{1+f(s_k)}$	$1 + f(s_k)$
(B, B)	(i, i)	$\varepsilon^{f(s_h)+f(s_k)}$	$f(s_h) + f(s_k)$
(C, C)	(b, b)	1	0
(C, C)	(b, l)	1	0
(C, C)	(l, l)	ε	1

6.2 The Result

We shall now characterize the set of stochastically stable states of the perturbed process P_2^ε. The following definition is needed.

Let g be a network. We shall say that g is anti-egalitarian if $\left\lfloor \frac{s_w^*}{s_m^*}|M| \right\rfloor$ men are matched to s_m^* women each, at most one man is matched to the remaining women (if there is such a remainder), and all other men have no partner.

To understand this definition, the idea is that all women are matched to a set of men that is as small as possible; hence the name "anti-egalitarian." Thus, if $\frac{s_w^*}{s_m^*}|M|$ happens to be an integer, each of those men is matched to s_m^* women and the rest of men are unmatched. Note that if $\frac{s_w^*}{s_m^*}|M|$ is not an integer, one can assign the remaining women to only one man and have a pairwise stable network. This is because, if one calls K the integer part of that fraction, the total number of links from the set of men not matched to their optimal number must be less than s_m^*: otherwise, the number of links coming from the men side would be at least $Ks_m^* + s_m^*$, but this number is strictly greater than $s_w^*|M|$, the number of links coming from the women side, and both numbers must always be equal.

Equipped with this definition, we state our next result:

Theorem 4 *Assume A1. A network is stochastically stable in the perturbed process P_2^ε if and only if it is anti-egalitarian and pairwise stable.*

7 Gender Discrimination and Contagion Diffusion

In this section, we study the implications of our analysis for gender differences in communication or contagion potential in stochastically stable networks, with a particular focus on HIV/AIDS.The exercise here is one of comparative statics: what are the gender differences in contagion potential as a function of the shape of the (stochastically stable) network? In doing so, we draw on the theoretical framework proposed in Pongou (2010).

Let g be a network. Assume that an agent $i \in N$ is drawn at random to receive a piece of information γ that he/she communicates to his/her partners in $g(i)$, who in turn communicate it to their other partners, and so on. This "piece of information" might also represent becoming infected with the HIV/AIDS virus through blood transfusion or any other exogenous random event. If i is not matched with any agent, the information does not spread. Suppose that with equal probability, $\frac{1}{|N|}$, each agent receives the information (i.e., is infected due to a random event). We define the communication or contagion potential of g as the expected proportion of agents who will receive the information. We also define the gender difference in contagion potential as the difference in the expected proportion of men and women who will receive the information. To formally define these notions, we first need a few definitions.

Let $i \in N$ be an agent such that $g(i) = \emptyset$. We say that i is isolated in the network g. We abuse language and call $\{i\}$ an isolated component of g, thus consisting only of one agent. We denote by $\mathcal{I}(g)$ and $\mathcal{J}(g)$ respectively the set of isolated and non-isolated components of g. Clearly, the set of components of g $C(g) = \mathcal{I}(g) \cup \mathcal{J}(g)$.

Assume that g is a k-component network, and let $C(g) = \{g_1, \ldots, g_k\}$ be the set of its components. Pose $I_k = \{1, \ldots, k\}$. To simplify notation, we write $N(g_i) = N_i$, $M(g_i) = M_i$, $W(g_i) = W_i$, and $|N_i| = n_i$ for $i \in I_k$. We associate each component g_i with the number n_i and its bipartite component vector $(|M_i|, |W_i|)$, and g with the vector $[(n_i)]_{i\in I_k}$ and its bipartite vector $[(|M_i|, |W_i|)]_{i\in I_k}$. Also, if g_i is an isolated component, its associated vector is either $(1, 0)$ or $(0, 1)$.

Denote by $\rho(z, \gamma)$ the status of an agent z with respect to the information γ. We pose $\rho(z, \gamma) = 1$ if z has received the information and 0 if he/she has not. For any set $B = N, M, W$, let $\Pr(\gamma|B) = \frac{|\{z\in B:\rho(z,\gamma)=1\}|}{|B|}$ be the proportion of agents who have received the information in the population B. We provide below a formula for the expected value of $\Pr(\gamma|N)$ and $\Pr(\gamma|M) - \Pr(\gamma|W)$. We have the following result.

Claim 1 (Pongou 2010)**:**

- $E[\Pr(\gamma|N)] = \frac{1}{n^2}\sum_{i\in I_k} n_i^2$.
- $E[\Pr(\gamma|M) - \Pr(\gamma|W)] = \frac{2}{n^2}\sum_{i\in I_k}(|M_i|^2 - |W_i|^2)$.

This result provides the foundation for the following definition:

Definition 1 *Let g be a k-component network with the corresponding component vector $[(n_i)]_{i\in I_k}$.*

(1) The communication or contagion potential of g is defined as

$$\mathcal{P}(g) = \frac{1}{n^2}\sum_{i\in I_k} n_i^2.$$

(2) If g is a bipartite graph with the corresponding component vector $[(|M_i|, |W_i|)]_{i\in I_k}$, the gender difference in the contagion potential of g is defined as

$$\mathcal{F}(g) = \frac{2}{n^2}\sum_{i\in I_k}(|M_i|^2 - |W_i|^2).$$

Note that our contagion model assumes that the transmission probability of HIV is 1. This assumption is motivated by the fact that we are studying HIV transmission in "equilibrium" or "stable" networks, which implies that sexual interactions are repeated over time, causing the transmission probability to approach 1. In fact, let us assume that the transmission probability per coital act is $\lambda < 1$, and that transmission is independent across acts. Then the transmission probability after k coital acts is $1-(1-\lambda)^k$, which effectively goes to 1 as k goes to infinity.[28] This logic is justified in our model. Since our comparative statics is on stochastically stable networks, once a stochastically stable network is reached, the system stays there a very long time, only getting out of it after extremely unlikely events.

Consider the following illustrative example of the above definition.

Example 2 *Consider the networks given in Example 1 and represented respectively by Figure 2-1, Figure 2-2 and Figure 2-3. Call them respectively g, g' and g''. The contagion potential of each of these networks is: $\mathcal{P}(g) = \frac{1}{20^2}(4^2+10^2+6^2) = \frac{152}{400} = 0.38$; $\mathcal{P}(g') = 0.515$; and $\mathcal{P}(g'') = 0.2$. In the event of a random HIV/AIDS infection in, say g, $\mathcal{P}(g) = 0.38$ means that 38% of the population would end up being infected following the diffusion of the virus.*

The gender difference in the contagion potential of each of these networks is: $\mathcal{F}(g) = \frac{2}{20^2}[(2^2-2^2)+(5^2-5^2)+(3^2-3^2)] = 0$; $\mathcal{F}(g') = 0.01$; and $\mathcal{F}(g'') = -0.12$. Following the diffusion of a random HIV/AIDS infection in these networks, these numbers imply that HIV/AIDS prevalence will be: equal for men and women in g; 1 percentage point greater among men than women in g'; and 12 percentage points greater for women than men in g''.

Note how the contagion potential varies across networks despite the fact that the number of links supplied by women and received by men is the same in all networks. This clearly shows the effect of network structure in the propagation of information, including diseases like HIV/AIDS. We see that g is gender neutral in contagion potential; but in network g', men are more vulnerable to infection than women, while in network g'', it is the opposite.

This example also shows that *female discrimination*, captured in our model by the higher cost of infidelity for women relative to men, does not necessarily cause women to be more vulnerable to infection, when one considers only pairwise stable networks (in g', which is a pairwise stable network in the mating problem defined in Example 1, men are more vulnerable than women to HIV/AIDS despite women being discriminated against). But we next show that in the networks that are visited a positive proportion of time in the long run (under our perturbed processes P_1^{ε} and P_2^{ε}), the ones we are concerned with in the current paper, men are never more vulnerable than women.

We now state below the main result of this section.

Theorem 5 *Assume A1.*

(1) For any stochastically stable network g of the perturbed process P_1^{ε}, $\mathcal{F}(g) = 0$.

[28] Looking more carefully into the probability per coital act pushes us beyond the scope of the current model, in particular forcing one to consider the use of protection to avoid infection. The Demographic and Health Surveys show that only a very small number of individuals use protection consistently in most relevant regions, though.

(2) For any stochastically stable network g of the perturbed process P_2^ε, $\mathcal{F}(g) < 0$.

Theorem 5 is illustrated in the following example.

Example 3 *There are 3 men and 3 women; $s_m^* = 3$ and $s_w^* = 1$. Consider the networks g_1, g_2 and g_3 represented respectively by Figures 3-1, 3-2 and 3-3 and by the following component configurations:* $[(1,1),(1,1),(1,1)]$, $[(1,2),(1,0),(1,1)]$ *and* $[(1,3),(1,0),(1,0)]$.

In g_1, each man is matched to a woman; in g_2, man m_1 is matched with two women, while m_2 is unmatched and m_3 is still matched with one; in g_3, m_1 is matched with all three women while the other men are unmatched. We note that only the egalitarian pairwise stable network g_1 is stochastically stable under the process P_1^ε, while only the anti-egalitarian pairwise stable g_3 is stochastically stable under the process P_2^ε. In addition, we have $\mathcal{F}(g_1) = 0$ and $\mathcal{F}(g_3) = -\frac{1}{3} < 0$.

Applied to the spread of HIV/AIDS, Theorem 5 implies that any initial network g, if not stochastically stable under P_1^ε or P_2^ε, will transition to a network g' that is stochastically stable, in which HIV prevalence is at least as high among women as among men, even if in the initial network g, the prevalence was higher among men than women. Furthermore, in the case of the second process, which under our assumptions may be viewed as a description of male-dominant societies, more women than men are infected by HIV/AIDS.[29]

8 Related Theoretical Literature

Aumann and Myerson (1988) and Jackson and Wolinsky (1996) pioneered the study of endogenous formation of links among agents. Aumann and Myerson (1988) examine a two-stage game. In its first stage, players form bilateral links resulting in a communication and cooperative structure, to which the Myerson value (Myerson (1977)) is applied in the second stage.[30] Jackson and Wolinsky (1996) introduce a framework for the study of the stability of networks among self-interested individuals. They develop a notion of pairwise stability of networks, and analyze its relationship with efficiency.[31] They also define the coauthor model, related to ours, in which indirect links have a negative effect (indirect links do not affect our agents' utility).

As for dynamics within two-sided markets, the type of dynamics in which at each period a pair of individuals can form and sever links goes back to Roth and Vande Vate (1990). More recently, several papers have also studied the dynamics of network formation using the notion of stochastic stability. Some of these papers include Jackson and Watts (2002) and Feri (2007).

Our paper also studies the endogenous formation of links, but there are some significant differences with previous work. First, our notion of pairwise stability allows for simultaneous link formation and severance and

[29] In fact, one can show in general that the set of anti-egalitarian pairwise stable networks includes those networks that maximize the gender difference in contagion potential, although the inclusion is strict.

[30] For extensions and variants, see Dutta, van den Nouweland and Tijs (1996), and Slikker and van den Nouweland (2001a, b).

[31] Other studies on strategic network formation include Dutta and Mutuswami (1997), Bala and Goyal (2000), Watts (2001), Jackson and Watts (2002), Jackson and van den Nouweland (2005), Page, Wooders and Kamat (2005), Dutta, Ghosal and Ray (2005), Bloch and Jackson (2007).

therefore differs from pairwise stability à la Jackson and Wolinsky (1996). Second, our focus is confined to fidelity networks, yielding simple characterizations given our assumptions. And third, our dynamic analysis rests on the notion that different transitions in link formation or severance have different probabilities (the different likelihood of our neutral actions), as opposed to uniform mistakes as is customary in the literature.

Another distinctive feature of our model is that we avoid the standard coordination problem by looking at a continuous problem rather than a discrete one. Agents maximize in a continuous way their utility function to determine their optimal number of partners. A similar approach is adopted in Cabrales, Calvó-Armengol and Zenou (2011).[32] As in this study, agents in our model do not direct their links but decide the number of partners. What is key is the fact that the link formation process is not equivalent to elaborating a nominal list of intended relationships, as is the case in the literature on network formation. Network formation is therefore not the result of an earmarked socialization process, which enables us to totally characterize the pairwise stable matchings, something that has proved rather difficult in the standard framework.

Furthermore, compared to the dynamic network formation literature, our analysis innovates in that individuals do not form links at random as it has often been assumed (see, e.g., the preferential attachment model à la Jackson and Rogers (2007)), but choose links that maximize their myopic utility. Similarly, individuals do not delete links at random but in a strategic way. Indirect links, however, do not matter in our model, as *a priori* agents do not know their partners' other partners, which enables a clean though not trivial characterization of long-run equilibria.

Our results on pairwise stability also relate to the literature on stability in many-to-many matching markets (e.g., Echenique and Oviedo (2006)). Indeed, if one defines the core with respect to strict coalitional improvements, the core will coincide with our set of pairwise stable networks.[33]

Finally, our paper also connects with the literature on social influence, social learning and contagion (see, e.g., Jackson and Rogers (2007b), Jackson and Yariv (2007), Lopez-Pintado (2008), Young (2009)). The different approaches used in these studies to analyzing diffusion generally assume a connectivity distribution of the population, and/or a payoff function whose arguments include an individual's and her neighbors' choice of a certain behavior, and often rely on mean-field approximation theory to identify equilibria. Each individual faces the choice of adopting a certain behavior, such as buying a new product or not, and this behavior spreads as it is adopted. Our model differs in that it studies "information transmission", not "information adoption." Distinguishing between the two notions is important. Within our framework, an agent who receives information about, say a new product, communicates it to her friends, but we do not pose the receiver's choice problem. An agent who is infected with the AIDS virus infects his/her sexual partners; the latter do not make the choice of becoming infected, and the former may not even be aware of his/her HIV status (in this sense,

[32] Several other papers have studied link formation based on utility considerations (see, e.g., Snijers (2001), and Staudigl (2011)).

[33] Strict improvements are well justified in the absence of side payments (recall Aumann (1959) again). Jackson and Watts (2002) show that stochastic stability in their model of uniform mistakes yields the core in their marriage markets. In our setup, we get the core before resorting to stochastic stability, which helps to refine our answer quite substantially leading to either egalitarian or anti-egalitarian networks.

we are closer to the literature on epidemiological contagion (see, e.g., Pastor-Satorras and Vespignani (2000, 2001)). We also note, as remarked by Young (2009), that most papers on social diffusion assume, unlike we do, infinite populations and purely random decisions.

9 Concluding Remarks

We view our contribution as twofold. First, we have proposed a dynamic theory of fidelity networks. Under the general underlying assumption of discrimination against women and additional sociological assumptions, we have characterized static equilibria (pairwise stable networks), as well as long-run equilibria (stochastically stable networks). Second, the findings reveal that the configurations of long-run networks are such that the spread of any random infection would (weakly) affect women more than men.

In understanding the trend of increased vulnerability of women to HIV/AIDS, our analysis reveals that female discrimination may be a key factor. We remark that our findings on HIV/AIDS are a comparative statics exercise on the stable network long-run prediction for each case. That is, in societies in which the assumption underlying the first process prevails (i.e., relationships are harder to break the lower the number of partners of one's old partner), one should see in the long run only egalitarian networks, where the gender difference in contagion potential is zero. In contrast, in societies better described by the second process (i.e., relationships are harder to break the higher the number of partners of one's old partner), one should see in the long run only anti-egalitarian pairwise stable networks, where the gender difference in contagion potential always goes against women. But we do not model explicitly the dynamics of the spread of HIV/AIDS. To do so one should probably incorporate the agents' sexual behavior reactions to statistics about the disease, something that our model completely abstracts from.

Our analysis also sheds light on dating and *union formation* patterns in some societies. Imagine that women's optimal number of partners is 1. Then, in the first process, the model predicts a situation of "serial monogamy." Theorem 3 shows for this case that only monogamous networks are stable in the long run. But note that this notion of stability does not mean that if the process reaches a monogamous network, it will stay there, since people might still make mistakes or be tempted by other potential partners. Indeed, if a woman moves from her only partner to another one, creating a non-monogamous network, the latter network will transition to another monogamous network which is not necessarily the initial one, and so on. Serial monogamy, known to be more prevalent in western societies, is associated with high divorce rates (e.g., Schoen and Standish (2001) and Goldstein (1999) document that the divorce rate in the U.S. is above 40%). In contrast, under the second process, the prediction of the model is "polygyny", and then divorce rates may be low.[34] Consider the following example. There are 3 men and 3 women, $s_w^* = 1$ and $s_m^* = 3$. Theorem 4 tells us that the only stochastically stable network (up to permutations) is the one in which the first man is

[34] Strictly speaking, to cover polygyny in our model, we should assume that $\pi = 1$ for men, as a man does not hide from having more than one wife. In this case, instead of speaking of infidelity, c should be interpreted as a cost of disruption in the household. All our results apply to this model as well.

matched to all three women. Assume that the process reaches that network. If a woman moves from the first man to another man, then considering that networks evolve following the path of least resistance, it is easy to see that that woman will return to the first man (so, there is reconciliation and no divorce). The model may be suggesting union formation patterns in regions where polygyny coexists with low divorce rates.

Finally, as already noted, a distinctive feature of fidelity networks in the real world is that *a priori*, individuals do not know their partners' other partners. In addition, they may or may not gain anything from being indirectly related to them. A natural extension of our analysis will be to consider the case in which an individual's well-being is affected by indirect "invisible" links and their consequent externalities. Our basic framework should be amenable to this and other realistic extensions, once incomplete information is incorporated to the analysis.

10 Proofs

Proof of Theorem 1.

Proof. $(1) \Longrightarrow (2)$: Let g be a pairwise stable network. It is straightforward that $\forall (m,w) \in M * W$, $0 \leq s_m \leq s_m^*$ and $0 \leq s_w \leq s_w^*$. In fact, if an agent has more than his/her optimal number of partners, he/she will be better off by unilaterally severing one link, thus implying that g is not pairwise stable, a contradiction.

Therefore, it only remains to show that $\forall w \in W$, $s_w = s_w^*$. By contradiction, suppose that there exists a woman w_0 with $s_{w_0} < s_w^*$. First, it should be clear that for every man m not matched with w_0, $s_m = s_m^*$. This is because, if at least one such man were matched with fewer women, that man and w_0 would improve by forming a new link, implying that g is not pairwise stable, which is a contradiction.

It then follows that the number of links coming from the men side is at least $(|M| - s_{w_0})s_m^*$, which is greater than or equal to $[|M| - (s_w^* - 1)]s_m^*$, which by Assumption A1 is greater than $|M|s_w^* = |W|s_w^*$, an upper bound on the number of links coming from the women side. Since the number of links coming from the men side must exactly equal the number of links coming from the women side, this is impossible. We conclude that $\forall w \in W$, $s_w = s_w^*$.

$(2) \Longrightarrow (1)$: Let g be a network. Assume that $\forall (m,w) \in M * W$, $0 \leq s_m \leq s_m^*$ and $s_w = s_w^*$, and let us show that g is pairwise stable. A man alone cannot improve by severing a link since he is at the upward sloping part of his utility function. He cannot form a new link with another woman since each woman has her optimal number of partners. And a woman cannot be part of any blocking move (either by herself or with a man) since she is at her peak. Therefore, g is a pairwise stable network. ■

Proof of Theorem 2.

Proof. Let g be a pairwise stable network. No agent can thus be part of a blocking move either by himself/herself or with another agent, implying that there is no improving path leading out of g. $\{g\}$ is therefore a recurrent class of P^0. Conversely, if g is not pairwise stable, it cannot be part of a recurrent class of P^0. First, it is clear that if g has some agents to the right of their peaks, unilateral severance of links will constitute an improving path out of g, leading to strict individual improvements that put every agent weakly to the left of their peaks. But then, if g is not pairwise stable, it must be the case that at least one woman has strictly less partners than at her peak, and since there must be at least one man with the same property, such a link will be formed in an improving path, never to return to g. This contradicts that g is part of a recurrent class of P^0. ■

Proof of Lemma 1.

Proof. Appealing to the characterization of pairwise stable networks in Theorem 1 and using the definition of egalitarian networks, the proof is easy and left to the reader. ■

Proof of Lemma 2.

Proof. Let g be a pairwise stable network. Pose $g^0 = g$. If g is egalitarian, then $\forall i \in M \cup W$, $s_i(g) = s_w^*$. Thus $L(g) = \sum_{i \in I^*(g)}(s_i(g) - s_w^*) = 0$, implying that the sequence searched for is (g). If g is non-egalitarian, then it is obvious that $L(g) > 0$ since from Lemma 1, at least one man has more than s_w^* partners. There exists a pair of men $(i_0, j_0) \in I(g) * J(g)$. Again by Lemma 1, since $s_{i_0}(g) \geq s_{j_0}(g) + 2$, there exists a woman k_0 such that $(i_0, k_0) \in g$ and $(j_0, k_0) \notin g$. Sever the link (i_0, k_0), and add the link (j_0, k_0); call the resulting network g^1. It is easy to check that g^1 is pairwise stable and that $L(g^1) = L(g) - 1$. Then, either g^1 is egalitarian and we are done, or not. That is, repeating the same operation $L(g) - 1$ more times induces a sequence $(g^1, \ldots, g^{L(g)})$ of pairwise stable networks. We have $L(g^{L(g)}) = L(g) - L(g) = 0$. Therefore, in the network $g^{L(g)}$, no man has more than s_w^* partners. But given that each woman has s_w^* partners in $g^{L(g)}$, that $|M| = |W|$, and that $\sum_{i \in M} s_i(g^{L(g)}) = \sum_{j \in W} s_j(g^{L(g)}) = s_w^*|W|$, it is necessarily the case that $\forall i \in M$, $s_i(g^{L(g)}) = s_w^*$. Thus $g^{L(g)}$ is pairwise stable and egalitarian. ■

Proof of Lemma 3.

Proof. Let g and g' be two distinct egalitarian pairwise stable networks. Pose $g^0 = g$. Pose $g' \setminus g = \{(m, w) : (m, w) \in g' \text{ and } (m, w) \notin g\}$. Since g and g' are different, $g' \setminus g$ is non-empty. Thus, there exists a pair (m_0, w_0) such that $(m_0, w_0) \in g'$ and $(m_0, w_0) \notin g$. Since g and g' are egalitarian, this implies that there exists a man m_0' such that $(m_0', w_0) \in g$ and $(m_0', w_0) \notin g'$. (In fact, if we assumed by contradiction that the latter statement were wrong, then it would mean that for any pair $(m_0', w_0) \in g$, then $(m_0', w_0) \in g'$; and since $(m_0, w_0) \in g'$ and $(m_0, w_0) \notin g$, this would imply that w_0 has more than s_w^* in the network g', contradicting the fact that g' is egalitarian and pairwise stable.)

Then, in g, add the link (m_0, w_0) and delete the link (m_0', w_0) (this is equivalent to woman w_0 severing her link with m_0' to form a new link with m_0), and call the resulting network g^1. In g^1, m_0 and m_0' have respectively $s_w^* + 1$ and $s_w^* - 1$ partners, and each woman has s_w^* partners as in g. Thus g^1 is pairwise stable, but it is not egalitarian. Also, note that g^1 is (one step) closer to g' than $g^0 = g$ (that is, $g' \setminus g^1 \subset g' \setminus g$).

We now want to construct g^2. Let $g^1(m_0) = \{w \in W : (m_0, w) \in g^1\}$. There exists a woman $w_0' \in g^1(m_0)$ such that $w_0' \neq w_0$, $(m_0', w_0') \notin g^1$ and $(m_0, w_0') \notin g'$ (in fact, since $|g^1(m_0)| = s_w^* + 1 > 1$ and $w_0 \in g^1(m_0)$, there exists $w_0' \in g^1(m_0)$ such that $w_0' \neq w_0$; now, if by contradiction, we assume that for any such w_0', $(m_0', w_0') \in g^1$, then it will turn out that $|g^1(m_0')| = s_w^*$, which is a contradiction since we know from the last paragraph that m_0' has exactly $s_w^* - 1$ partners in g^1; finally, if by contradiction, we assume that for any such w_0', $(m_0, w_0') \in g'$, then it will turn out that $g'(m_0) = g^1(m_0)$, implying that $|g'(m_0)| = s_w^* + 1$, thereby contradicting the fact that g' is egalitarian). Therefore, sever the link (m_0, w_0'), add the link (m_0', w_0'), and call the resulting network g^2. It is easy to check that in g^2, each man and each woman has exactly s_w^* partners. Thus g^2 is egalitarian and pairwise stable.

We also note that g^2 is at least 1 step closer to g' (in fact, since $(m_0, w_0') \notin g'$, severing this link in g^1 does not take us 1 step further from g'; also, if possible, one can choose w_0' in such a way that $(m_0', w_0') \in g'$, and in that case, g^2 will be 2 steps closer to g'; if not, g^2 will be 1 step closer to g').

If $g^2 = g'$, we are done; if not, repeat the same operation as previously by replacing g^0 with g^2. That will induce g^3 and g^4, and will take us at least one step closer to g'. In general, since $|g' \setminus g|$ is finite, repeating this operation a finite number of times (at most $\left\lceil \frac{|g' \setminus g|}{2} \right\rceil$ times) induces a finite sequence of pairwise stable networks $(g^0, g^1, \ldots, g^{2k})$ that ends at $g^{2k} = g'$ and satisfying that for any t such that $0 \leq t \leq k$, g^{2t} is egalitarian. ■

Proof of Theorem 3.

Proof. The proof is divided in two steps, as follows:

Step 1: Let g be a non-egalitarian pairwise stable network. We shall show that g is not stochastically stable. It suffices to show that there exists a network g' such that $r(g') < r(g)$.

Call $T(g)$ the g-tree on which the calculation of $r(g)$ is based. There exists a pair of men $(i_0, j_0) \in I(g) * J(g)$. Since from Lemma 1, $s_{i_0}(g) \geq s_{j_0}(g) + 2$, there exists a woman k_0 such that $(i_0, k_0) \in g$ and $(j_0, k_0) \notin g$. Sever the link (i_0, k_0), and add the link (j_0, k_0), and call the resulting network g^1.

Consider now the tree $T(g)$. Let $S(g^1, T(g))$ be the successor of g^1 in the tree. Now, in $T(g)$, delete the edge $(g^1, S(g^1, T(g)))$ that leads away from g^1 and add the edge (g, g^1). This results in a g^1-tree that we denote by $T(g^1)$.

Since $T(g^1)$ is not necessarily optimal for g^1, we have $r(g^1) \leq r(g) - r(g^1, S(g^1, T(g))) + r(g, g^1)$. Because $\forall i \in I(g^1)$, $s_i(g) \leq s_{i_0}(g)$, we have $r(g^1, S(g^1, T(g))) \geq f(\frac{1}{s_{i_0}(g)}) = r(g, g^1)$. This is because the cheapest way of getting away from g^1 (which is pairwise stable) is for a pair of a man and a woman to undertake an action that benefits one of them and leaves the other indifferent; such an action is taken with probability at least equal to $\varepsilon^{f(\frac{1}{s_{i_0}(g)})}$. This implies that $r(g^1) \leq r(g)$.

If g^1 is egalitarian, then $r(g^1, S(g^1, T(g))) = f(\frac{1}{s_w^*}) > r(g, g^1)$, implying $r(g^1) < r(g)$. If g^1 is non-egalitarian, repeat the same operation $L(g) - 1$ more times. From lemma 2, that will induce a sequence of pairwise stable networks $(g^1, \ldots, g^{L(g)})$ where $g^{L(g)}$ is an egalitarian network. The induced sequence of g^ℓ-trees, $1 \leq \ell \leq L(g)$, $(T(g^1), \ldots, T(g^{L(g)}))$ will be such that for any $\ell \in \{2, \ldots, L(g)\}$, $r(g^\ell) \leq r(g^{\ell-1})$ with $r(g^{L(g)}) < r(g^{L(g)-1})$. This obviously implies $r(g^{L(g)}) < r(g)$, and therefore, g is not stochastically stable.

Recall that in any perturbed finite Markov process the set of stochastically stable states is always non-empty. Step 1 has therefore established that the set of stochastically stable networks of the perturbed process P_1^{ε} is a non-empty subset of the set of egalitarian pairwise stable networks.

Step 2: We shall next show that the set of stochastically stable networks of P_1^{ε} coincides with the set of egalitarian pairwise stable networks. It suffices to show that all egalitarian pairwise stable networks have the same stochastic potential.

Let g and g' be any two egalitarian pairwise stable networks, and $r(g)$ and $r(g')$ their respective stochastic potentials. Call $T(g)$ the g-tree on which the calculation of $r(g)$ is based. From Lemma 3, we know that there exists a finite sequence of pairwise stable networks $(g^0, g^1, \ldots, g^{2k})$ such that $g^0 = g$, $g^{2k} = g'$, and for any t such that $0 \leq t \leq k$, g^{2t} is egalitarian.

Construct g^1 from g as in the proof of Lemma 3, and consider the g-tree $T(g)$. In it, delete the edge $(g^1, S(g^1, T(g)))$ that leads away from g^1 and add the edge (g, g^1). This results in a g^1-tree that we denote by $T(g^1)$. Note that $r(g^1, S(g^1, T(g))) \geq f(\frac{1}{s_w^*+1})$ and $r(g, g^1) = f(\frac{1}{s_w^*})$.

Next, construct g^2 from g^1 as in the proof of Lemma 3, and consider the g^1-tree $T(g^1)$. In it, delete the edge $(g^2, S(g^2, T(g^1)))$ and add the edge (g^1, g^2). This results in a g^2-tree that we denote by $T(g^2)$. We have $r(g^2, S(g^2, T(g^1))) = f(\frac{1}{s_w^*})$ and $r(g^1, g^2) = f(\frac{1}{s_w^*+1})$.

Therefore, noting that $T(g^2)$ is not necessarily optimal as a g^2-tree, we have that $r(g^2) \leq r(g) - r(g^1, S(g^1, T(g))) + r(g, g^1) - r(g^2, S(g^2, T(g^1))) + r(g^1, g^2) = r(g) - r(g^1, S(g^1, T(g))) + f(\frac{1}{s_w^*+1}) \leq r(g)$ since $r(g^1, S(g^1, T(g))) \geq f(\frac{1}{s_w^*+1})$. This establishes that $r(g^2) \leq r(g)$, and by symmetry, going back from g^2 to g, that $r(g) \leq r(g^2)$. Therefore, $r(g) = r(g^2)$.

If $g' = g^2$, then we have shown that $r(g') = r(g)$. If $g' \neq g^2$, repeat the same exercise as previously, constructing g^{ℓ} from $g^{\ell-1}$ as in Lemma 3, until g' is obtained. This induces a sequence of $g^t - trees$ $(T(g), T(g^1), T(g^2), T(g^3), \ldots, T(g^{2k}) = T(g'))$ satisfying that for any t such that $1 \leq t \leq k$, $r(g^{2t}) \leq r(g^{2(t-1)})$. This implies $r(g') \leq r(g)$. By symmetry, going back in the opposite direction, we also have $r(g) \leq r(g')$, thus implying $r(g) = r(g')$, which completes the proof. ■

Proof of Theorem 4.

Proof. The proof is again organized in two steps, as follows:

Step 1: Let g be a pairwise stable network that is not anti-egalitarian. We shall show that g is not stochastically stable. It suffices to show that there exists a network g' such that $r(g') < r(g)$.

Consider $T(g)$, the g-tree on which the calculation of $r(g)$ is based. We claim that, if g^{λ} and $g^{\lambda+1}$ are two pairwise stable networks such that for some m, m', w, $g^{\lambda} \setminus g^{\lambda+1} = \{(m, w)\}$ and $g^{\lambda+1} \setminus g^{\lambda} = \{(m', w)\}$, the underlying transition does not involve non-pairwise stable networks: if it did, at least one agent directly involved in it would decrease his or her utility, which implies that the resistance of such a transition would exceed 1, whereas the resistance of the direct transition between the two (being adjacent) is strictly less than 1. A simple induction argument shows that this is still true even if two pairwise stable networks are not adjacent (by constructing a path going from one to the other consisting of direct transitions between pairs of adjacent networks).

Therefore, in any transition described in $T(g)$, only pairwise stable networks are visited. By Theorem 1, we know that each pairwise stable network contains exactly the same number of links, i.e., $s_w^*|W|$. It follows that each transition described in the tree involves a woman w who severs a link with a man m and replaces it with another link with man m'. Specifically, the pair (m', w) is offered the opportunity to revise their situation, and as a result, woman w severs (m, w) and gets matched with m'.

But then, in describing the transition between any two pairwise networks in $T(g)$, one can, without loss of generality, list the transitions that are required going through each individual woman. That is, starting with the woman with the lowest index who has a different set of men to which she is matched in the two networks, one can describe the required severance/creation of links that takes her from her configuration of men in the original network to the one in the final network, and one can proceed like these with each such woman until the full transition is complete.

Consider then the network g, and recall it is not anti-egalitarian. We propose the following algorithm. Without loss of generality, label the men so that $s_{m_1}(g) \geq s_{m_2}(g) \geq \ldots \geq s_{m_{|M|}}(g)$. Let m be the lowest index such that $s_m(g) < s_m^*$. If there exists w who is matched in g to $m' > m$, sever the link (m', w) and replace it with (m, w). Call the resulting network g^1. We can have two cases. Either g^1 is anti-egalitarian, or not. If it is, let $g' = g^1$. If not, repeat the same step. Note how this algorithm always ends after a finite number of steps, say k, in a network $g' = g^k$ that is anti-egalitarian.

Consider the g-tree $T(g)$, and without loss of generality (as the first paragraphs of the proof showed), suppose that the transition $g' = g^k \to g^{k-1} \to \ldots \to g^1 \to g^0 = g$ constitutes a path of directed links in $T(g)$. Change the direction of this path and consider the transition $g = g^0 \to g^1 \to \ldots \to g^{k-1} \to g^k = g'$. It is obvious that the rest of edges of $T(g)$, along with these new edges (in which the only change introduced is the direction change of previous links in $T(g)$), constitute a g'-tree, which we call $T(g')$.

We claim that $r(g') < r(g)$. Indeed, $r(g')$ is no greater than the resistance of $T(g')$, which is equal to $r(g) + \sum_{\alpha=0}^{k-1}[r(g^\alpha, g^{\alpha+1}) - r(g^{\alpha+1}, g^\alpha)]$. And note that, by construction of the algorithm described, each bracketed term is negative. Indeed, in the transition $g^\alpha \to g^{\alpha+1}$, let m' be the man who loses a link in favor of man m. We know that $s_{m'}(g^\alpha) < s_m(g^{\alpha+1})$, and therefore, $r(g^\alpha, g^{\alpha+1}) = f(s_{m'}(g^\alpha)) < f(s_m(g^{\alpha+1})) = r(g^{\alpha+1}, g^\alpha)$.

We have therefore established that, if g is pairwise stable but it is not anti-egalitarian, it is not stochastically stable in the perturbed process P_2^ε. Given that the set of stochastically stable networks is non-empty, we just proved that this set is a non-empty subset of the set of pairwise stable and anti-egalitarian networks.

Step 2: We shall now prove that the set of stochastically stable networks of P_2^ε coincides with the set of pairwise stable and anti-egalitarian networks. It suffices to prove that all of them have the same stochastic potential.

Let g and g' be any two such networks. Assume for simplicity that, in each of them, exactly $\frac{s_w^*}{s_m^*}|M|$ men are matched with s_m^* each. Obviously, this must hold for both g and g'.[35]

It is easy to see that there must exist $m, m' \in M, m \neq m'$ and $w, w' \in W, w \neq w'$ such that $(m, w) \in g \setminus g'$ and $(m', w') \in g' \setminus g$. We propose the following algorithm that transforms g into g'. For each such pair of links, we describe the following steps:

- First, woman w severs her link to man m and gets matched to man m_0, where $s_{m_0}(g) = 0$ –we know such a man exists in g.
- Second, woman w' severs her link to man m' and gets matched to man m.
- And third, woman w severs her link to man m_0 and gets matched to man m'.

And to go back, travel the same steps in reverse.

Consider now an optimal g'-tree, and call it $T(g')$. In it, focus on the collection of directed edges connecting g to g'. By arguments similar to those at the beginning of Step 1 of this proof, one can argue that the transition outlined in the previous algorithm must be part of any optimal tree. (We know that transitions in optimal trees do not go through non-pairwise stable networks. In addition, a resistance of $f(s_m^*)$ must be paid every time a link with a man matched to his optimal number is broken, and aside from that, a resistance of $f(1)$ that comes from breaking a link with a man who was unmatched in g and remains unmatched in g' is the smallest possible positive resistance in this perturbed process.)

Thus, without loss of generality, let the directed path from g to g' in $T(g')$ be the set of transitions outlined. Now, change the direction of the edges in this path, and let that be the only change introduced to the directed edges of $T(g')$. Observe that the result is a g-tree, which we call $T(g)$.

We will now argue that the stochastic potentials of g and g' are the same:

$r(g) = r(g') + \sum_{\beta=0}^{k-1}[r(g^\beta, g^{\beta+1}) - r(g^{\beta+1}, g^\beta)] = r(g')$ because $\sum_{\beta=0}^{k-1}[r(g^\beta, g^{\beta+1}) - r(g^{\beta+1}, g^\beta)] = 0$. This can be easily established, by induction on the number of links that are different between g and g'.

Indeed, suppose that g and g' differ in the smallest possible number of links, which is two, i.e., there exist $m \neq m'$ and $w \neq w'$ such that $g \setminus g' = \{(m, w)\}$ and $g' \setminus g = \{(m', w')\}$. Consider the transition $g \to g'$ in $T(g')$. By our previous arguments, such a transition is as follows:

- First, woman w severs her link to man m and gets matched to man m^0, where $s_{m_0}(g) = 0$ –we know such a man exists in g; the resistance of this step is $f(s_m^*)$.
- Second, woman w' severs her link to man m' and gets matched to man m; again, the resistance of this step is $f(s_m^*)$.

[35] If, instead, the number $\frac{s_w^*}{s_m^*}|M|$ is not an integer, and one man is matched to the remaining women, the argument is the same, but the notation is slightly more complicated. Again, in this case, both g and g' have the same structure of having only one man matched to the remaining women.

- And third, woman w severs her link to man m_0 and gets matched to man m'; the resistance of this step being $f(1)$.

The resistance of the whole transition is thus $2f(s_m^*) + f(1)$. But notice that travelling the same steps backwards takes us back from g' to g, with exactly the same resistance.

If g and g' differ by more links (note this must always be an even number), we use the fact that the path going from g to g' and the same path travelled in the opposite direction are "mirror images" of one another. Thus, since the cheapeast transition must always involve establishing links with unmatched men –like m_0 in the previous paragraph– (because $f(1)$ is the smallest resistance to be added to the $f(s_m^*)$ terms, which must be always there), a replication of the argument detailed in the previous paragraph establishes that the total resistance of travelling from g to g' is exactly the same as the one travelling backwards on the same path. This completes the proof. ■

Proof of Theorem 5.

Proof. Assume A1.

(1) The proof follows from the fact that in any egalitarian pairwise stable network g, there is an equal number of men and women in each component of g, from which it follows that $\mathcal{F}(g) = 0$.

(2) First remark that in any anti-egalitarian pairwise stable network g, the number of women exceeds the number of men for all non-isolated components, with strict inequality for some of them. Let us now show that it follows that $\mathcal{F}(g) < 0$. Let $[(|M_i|, |W_i|)]_{i \in I_k}$ be the bipartite component vector of g, of which the first ℓ components are non-isolated and the remaining $k - \ell$ are isolated (men). It obviously follows that $\sum_{i \in I_k} |M_i| = \sum_{i \in I_\ell} |M_i| + k - \ell$ and $\sum_{i \in I_k} |M_i| = \sum_{i \in I_\ell} |W_i|$ (given that no woman is isolated), which in turn implies $\sum_{i \in I_\ell}(|M_i| - |W_i|) = -(k - \ell) < 0$. Remark that each non-isolated component vector $(|M_i|, |W_i|)$ is such that $|M_i| + |W_i| = n_i \geq 2$ since it contains at least one man and one woman. Hence, we have the following:

$$\begin{array}{rcl}
\mathcal{F}(g) & = & \frac{2}{n^2} \sum_{i \in I_k}(|M_i|^2 - |W_i|^2) \\
& = & \frac{2}{n^2}\{\sum_{i \in I_\ell}(|M_i|^2 - |W_i|^2) + \sum_{\ell+1 \leq i \leq k}(|M_i|^2 - |W_i|^2) \\
& = & \frac{2}{n^2}\{\sum_{i \in I_\ell}(|M_i| - |W_i|)(|M_i| + |W_i|) + \sum_{\ell+1 \leq i \leq k}(1^2 - 0^2) \\
& = & \frac{2}{n^2}\{\sum_{i \in I_\ell}(|M_i| - |W_i|)n_i + k - \ell \\
& \leq & \frac{2}{n^2}\{2\sum_{i \in I_\ell}(|M_i| - |W_i|) + k - \ell\} \\
& < & 0.
\end{array}$$

■

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Figure 1: Percent of adults (15+) living with HIV/AIDS who are female, 1990-2007 (UNAIDS (2008))

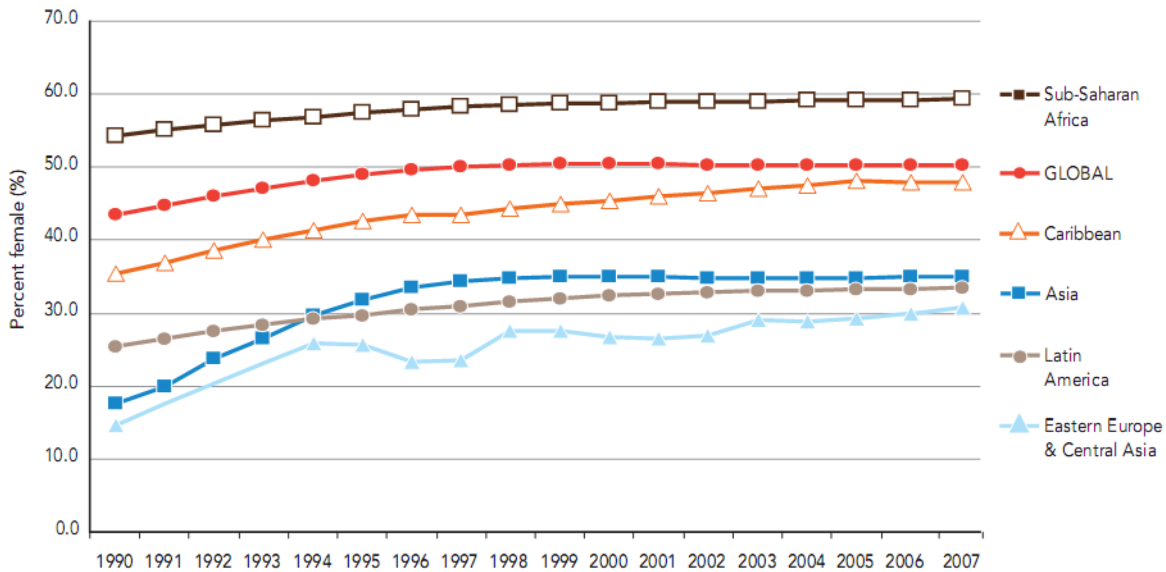

Figure 2-1

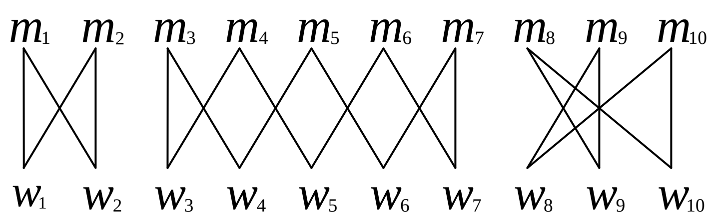

Figure 2-2

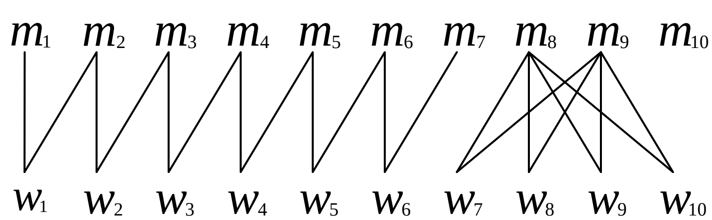

Figure 2-3

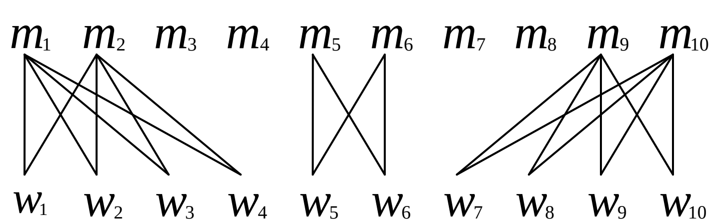

Figure 3-1

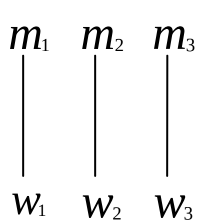

Figure 3-2

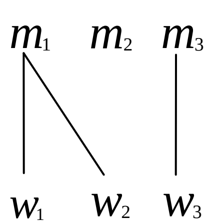

Figure 3-3