

1995

# Dispersion Relations and Effective Field Theory

John Donoghue

*University of Massachusetts Amherst*, [donoghue@physics.umass.edu](mailto:donoghue@physics.umass.edu)

Follow this and additional works at: [http://scholarworks.umass.edu/physics\\_faculty\\_pubs](http://scholarworks.umass.edu/physics_faculty_pubs)



Part of the [Physics Commons](#)

---

Donoghue, John, "Dispersion Relations and Effective Field Theory" (1995). *Physics Department Faculty Publication Series*. Paper 246.  
[http://scholarworks.umass.edu/physics\\_faculty\\_pubs/246](http://scholarworks.umass.edu/physics_faculty_pubs/246)

This Article is brought to you for free and open access by the Physics at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Physics Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact [scholarworks@library.umass.edu](mailto:scholarworks@library.umass.edu).

# Dispersion Relations and Effective Field Theory

John F. Donoghue

Department of Physics and Astronomy  
University of Massachusetts Amherst, MA 01003 U.S.A.

## Abstract

The techniques of dispersion relations match very well with those of effective field theory. I describe the techniques for using dispersion relations effectively, and give some pedagogical examples to illustrate the range of applications.

UMHEP-426, hep-ph/9607351 Lecture presented at the International School on Effective Field Theory, Almunecar, Spain, June 1995

Dispersive techniques were common in the 1960's, but have been less popular since then due to the triumph of field theory. Why should we bother to resurrect them at this time? Besides being a useful aspect of the toolkit for any theorist, I feel that they have a special utility in effective field theory. More particulars will be given below, but briefly stated they are useful in effective theories because they can accurately describe the low energy propagation of the light degrees of freedom as well as, and sometimes better than, perturbative Feynman diagrams. This can lead to new and/or better types of calculation as well as increasing our insight into the workings of effective field theory.

In this lecture, I briefly explore the methods, successes and limitations of combining dispersion relations with effective field theory. The goal is to first demonstrate that one can reproduce the low energy content of Feynman diagram calculations in an effective theory by simply using the lowest order vertices as the input to a dispersion relation. Then we address what is needed to improve on this lowest order calculation and how to do the matching of the dispersion relation with the effective field theory method. I will describe various examples to illustrate the method, and finally summarize the advantages and disadvantages of this approach.[1]

## 1 Dispersion relations - general

Scattering amplitudes and vertex functions will in general contain both real and imaginary parts.[2] The imaginary portions are due to the propagation of on-shell intermediate states. Causality implies certain properties for the analytic structure of the amplitudes that allows us to relate the real and imaginary parts. Such dispersion relations have the general form

$$Re f(s) = \frac{1}{\pi} P \int_0^\infty \frac{ds'}{s' - s} Im f(s') \quad (1)$$

With the identity

$$\frac{1}{x - x_0 - i\epsilon} = P \frac{1}{x - x_0} + i\pi\delta(x - x_0) \quad (2)$$

one can write the full amplitude as an integral over its imaginary part,

$$f(s) = \frac{1}{\pi} \int_0^\infty ds' \frac{Imf(s')}{s' - s - i\epsilon} \quad (3)$$

Notice that the dispersive integral involves all  $s'$ . In order to know  $f(s)$  at small  $s$ , we need to know  $Imf(s')$  also at large  $s'$ . We will see that subtractions can lessen the dependence on large  $s'$ , but the integral still runs over all  $s'$ . We in general need to know the properties of on-shell intermediate states.

Given a dispersion relation, one may also write a “subtracted” relation for  $(f(s) - f(0))/s$ , i.e.,

$$\frac{f(s) - f(0)}{s} = \frac{1}{\pi} \int \frac{ds'}{s' - s - i\epsilon} Im \left[ \frac{f(s') - f(0)}{s'} \right] \quad (4)$$

which, since  $Imf(0) = 0$ , is equivalent to

$$f(s) = f(0) + \frac{s}{\pi} \int \frac{ds'}{s'} \frac{Imf(s')}{s' - s - i\epsilon} \quad (5)$$

Subtractions may be needed if  $f(z) \neq 0$  as  $|z| \rightarrow \infty$ , as a good behavior at infinity is required for the derivation of the dispersion relation. However, even if subtractions are not required, it may still be desirable to perform them. This is especially true in effective field theories, where we are interested primarily in the low energy quantum effects, while we do not know how to calculate the higher energy physics. Generally the input to the dispersion relation,  $Imf(s)$ , is not well known at high energy. The subtracted dispersion integral weights lower energies more heavily and lessens the influence of the high energy region. The previous ignorance of the high energy effects of  $Imf(s)$  is reduced to a single number, the subtraction constant. We will see that these subtraction constants are equivalent to the parameters in the effective lagrangians. Further subtractions may be performed, with the introduction of further subtraction constants.

## 2 Calculating with Dispersion Relations

The pion form factor obeys dispersion relations. An unsubtracted form is

$$f_\pi(q^2) = \frac{1}{\pi} \int_{4m_\pi^2}^\infty ds' \frac{Imf_\pi(s')}{s' - q^2 - i\epsilon} \quad (6)$$

while with one subtraction the form is

$$f_\pi(q^2) = 1 + \frac{q^2}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{Imf_\pi(s')}{s' - q^2 - i\epsilon} \quad (7)$$

Here the subtraction constant has been fixed to unity by the normalization of the form factor. A twice subtracted form is

$$f_\pi(q^2) = 1 + cq^2 + \frac{q^4}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{Imf_\pi(s')}{s' - q^2 - i\epsilon} \quad (8)$$

where  $c$  is presently unknown.

The key step in a complete dispersion relation calculation is in specifying the input into the dispersion integral, i.e. the imaginary part of the function. In this case, the imaginary part is known experimentally up to about  $1.5GeV$ . The good news is that the physics of the imaginary part is relatively simple. It is well known that the largest effect in the pion form factor is the rho intermediate state. This of course has led to the technique of vector dominance. The simplicity of intermediate states in dispersion relations can help in their implementation. However before turning to phenomenology, let us see how this technique can be reconciled with a Feynman diagram approach.

All Feynman diagrams have the same analytic structure as the amplitude that they contribute to. They can therefore be rewritten as dispersion relations, most often with subtractions. Therefore the content of chiral loops can equally well be specified as a particular choice for  $Imf(s')$  in a dispersion integral. Let us see how this occurs for the pion form factor.

In chiral perturbation theory, a one loop calculation involves all the machinery of field theory including a full set of diagrams, regularization of divergent integrals, and renormalization of the parameters in the Lagrangian. This is a well functioning machine and beautifully extracts the predictions of the theory. However much of this effort is irrelevant for the real dynamical content of an effective theory. The effective theory is only able to predict the low energy behavior of the light degrees of freedom of the theory. All of the high energy effects associated with divergences, regularization and renormalization are irrelevant to the final answer. One needs to carry out the full field-theoretic procedure because otherwise the Feynman diagram machine breaks down and gives wrong answers, but the high energy portions of the calculation do not express the real physics of the effective theory.

In the case of the pion electromagnetic form factor, at lowest order  $O(E^2)$ , one predicts simply that  $f_\pi(q^2) = 1$ . The analysis to  $O(E^4)$  is described in detail in Ref [3]. The two important ingredients are the low energy constant  $L_9$  and the effect of loops. The tree level contribution involves the parameter  $L_9$ , which occurs in the general chiral Lagrangian at order  $E^4$  [4], plus other constants i.e.,

$$f_\pi^{(tree)}(q^2) = 1 + \frac{2L_9}{F_\pi^2}q^2 + \frac{8m_\pi^2}{F_\pi^2}(2L_4 + L_5) \quad (9)$$

Of the loop diagrams, Fig. 1b, c, that of Fig. 1b has no  $q^2$  dependence, contributing only a constant. Since we know that the pion form factor is absolutely normalized to unity at  $q^2 = 0$ , we know that all corrections which are independent of  $q^2$  must be canceled by the wavefunction renormalization constant along with constant terms in Eq. 9 and the diagram of Fig. 1c. Thus diagram 1b and the renormalization constant are dynamically irrelevant although in this method of calculation they enforce the constraint on the normalization of the formfactor.

However, Fig. 1c is more interesting because it also contains important dynamical information of the propagation of the two pion state, including the imaginary part of the amplitude due to on-shell intermediate states, and the result involves a nontrivial function of  $q^2$ ,

$$\Delta f_\pi^{(1c)}(q^2) = \frac{1}{16\pi^2 F_\pi^2} \left\{ \left( m_\pi^2 - \frac{1}{6}q^2 \right) \left[ \frac{2}{d-4} + \gamma - 1 - \ln 4\pi + \ln \frac{m_\pi^2}{\mu^2} \right] + \frac{1}{6} \left( q^2 - 4m_\pi^2 \right) H(q^2) - \frac{1}{18}q^2 \right\} \quad (10)$$

with

$$H(q^2) = 2 + \beta \left[ \ln \left( \frac{1-\beta}{1+\beta} \right) + i\pi\theta(q^2 - 4m_\pi^2) \right] \quad (11)$$

$$\beta = \sqrt{1 - \frac{4m_\pi^2}{q^2}}$$

Multiplying by the wavefunction renormalization constant and defining the renormalized value of  $L_9$

$$L_9^r = L_9 - \frac{1}{192\pi^2} \left[ \frac{2}{d-4} - \ln 4\pi + \gamma - 1 \right] \quad (12)$$

we get the final result

$$f_\pi(q^2) = 1 + \frac{2L_9^r}{F_\pi^2} q^2 + \frac{1}{96\pi^2 F_\pi^2} \left[ (q^2 - 4m_\pi^2)H(q^2) - q^2 \ln \frac{m_\pi^2}{\mu^2} - \frac{q^2}{3} \right] \quad (13)$$

All of the dynamical content of the effective theory was in the vertices and propagation of the two pion state in Fig. 1c. This represents low energy-long range propagation that the effective theory is capable of predicting.

Now let us get this same physics in a dispersion relation.[5] The key feature is the choice of what to use for the imaginary part of the amplitude. The one loop diagram, Fig. 1c, involves the  $\pi\pi$   $I = 1$  scattering amplitude, and the tree level  $\pi\pi \rightarrow \gamma$  vertex, so that

$$2(p_1 - p_2)_\mu \text{Im} f_\pi(s) = \int \frac{d^3 p'_1 d^3 p'_2}{(2\pi)^6 2E'_1 2E'_2} (2\pi)^4 \delta^4(s - p'_1 - p'_2) \langle \pi\pi | T | \pi\pi \rangle \langle \pi\pi | J_\mu | 0 \rangle \quad (14)$$

If we choose the forms for these amplitudes which is predicted at lowest order in chiral symmetry, we obtain

$$\text{Im} f_\pi(s) = \frac{1}{96\pi F_\pi^2} \frac{(s - 4m_\pi^2)^{\frac{3}{2}}}{\sqrt{s}} \theta(s - 4m_\pi^2) \quad (15)$$

We use this in the twice subtracted form, Eq. 8, and the dispersion integral can be exactly done using

$$\int_{4m^2}^{\infty} \frac{ds}{s^2} \left( 1 - \frac{4m^2}{s} \right)^{\frac{1}{2}} \left( \frac{a + bs}{s - q^2 - i\epsilon} \right) = \frac{(a + bq^2)}{q^4} H(q^2) - \frac{a}{6m^2 q^2} \quad (16)$$

to give

$$f_\pi(q^2) = 1 + cq^2 + \frac{1}{96\pi^2 F_\pi^2} \left[ (q^2 - 4m_\pi^2)H(q^2) + \frac{2}{3}q^2 \right] \quad (17)$$

Comparing this with the chiral calculation, Eq. 7, leads to the identification of the subtraction constant

$$c = \frac{2L_9^{(r)}(\mu)}{F_\pi^2} - \frac{1}{96\pi^2 F_\pi^2} \left( \ln \frac{m_\pi^2}{\mu^2} + 1 \right) \quad (18)$$

We see that we can reproduce the content of the Feynman diagram approach simply by using the lowest order vertices and propagators in the dispersion integral. When it is phrased this way, it is clear that the content of chiral loop diagrams such as Fig. 1c and the content of a dispersive integral are similar. The chiral calculations uses a predicted approximation to  $Imf(s')$ , while a properly performed dispersion integral uses the real world data for  $Imf(s')$ . We also see that the chiral parameters ( $L_i$ ) play a similar role to the subtraction constants in dispersion relations.

The logic of effective theories is even clearer with dispersion relations than with Feynman diagrams, although both are adequate tools. When giving talks on chiral perturbation theory, one routinely encounters individuals who think that the use of effective Lagrangians in loop diagrams is not allowed. Their confusion generally centers on the fact that effective Lagrangians only predict the low energy behavior, yet loops involve an integration over all energies, which they mistakenly interpret as a prohibition on performing loop calculations. A dispersive analysis, such as is given above, is presumably more acceptable to this way of thinking since, when twice subtracted, it is only sensitive to the chiral amplitude at low energies. The equivalence of the dispersive treatment and the Feynman diagrams can then be used to demonstrate that the output of chiral loops is also the result of the low energy amplitude.

### 3 Relative Strengths

The above example does not display the full power of either chiral symmetry or dispersion relations. Both can bring new information to the calculation. Let us discuss these special features before we attempt to combine the best of the two methods.

The special feature of the chiral symmetry is that it provides relations between amplitudes with different numbers of pions. Most symmetries, such as isospin, relate amplitudes of states that belong to the same multiplets. For a dynamically broken symmetry with Goldstone bosons, the symmetry



transformation does not relate states within a multiplet, but rather it relates states with greater or fewer zero energy Goldstone bosons. Corrections to this limit can be given in an expansion in the energy. There exist various reduced matrix elements which are not predicted by the symmetry and which therefore must be measured. These are the parameters in the chiral Lagrangian. The outputs of chiral perturbation theory are relations between amplitudes, order by order in the expansion in  $E$ ,  $m_q$ . At any given order, these relations form low energy theorems of QCD. In the above example, chiral symmetry is more powerful than dispersion relations when it comes to the subtraction constant, because an extended chiral analysis can also tell us that the same constant  $L_9$  appears in other reactions. For example,  $L_9$  also contributes to radiative pion decay,  $\pi \rightarrow e\nu\gamma$ , and can be independently measured in that process.[4]

On the other hand, dispersion relations can ultimately do a better job on the low energy dynamical effects, as captured in the dispersive integral. The one loop chiral analysis is equivalent to the lowest order vertices and propagators, while a dispersive treatment can use instead the full answer given by Nature. In this particular example, there is in fact quite a substantial difference, because there is a resonance in the  $I=1$  channel of  $\pi\pi$  interactions. In Fig. 2, I display the integrand of the dispersion relation as a function of energy  $E = \sqrt{s}$ . Specifically, for the twice subtracted relation

$$f_\pi(q^2) = 1 + cq^2 + \frac{q^4}{\pi} \int_{2m_\pi}^{\infty} dE N(E) \frac{E^2}{E^2 - q^2 - i\epsilon} \quad (19)$$

with

$$N(E) = \frac{(2E)Imf(E)}{E^6} \quad (20)$$

Fig 2 plots  $N(E)$ , whose integral determines the low  $q^2$  behavior of the dispersive integral. The dashed line is the lowest order chiral amplitude that is the integrand discussed in the previous section, equivalent to the one loop answer. In contrast, the solid line is the full answer from a fit to the experimental imaginary part. We see that the results agree at very low energy, but that at moderate energies the resonance is much larger than the lowest order chiral result. This is not a problem of principle because it describes physics that would appear in the chiral expansion at higher order.

However it does indicate that the dispersive analysis can be used to obtain a more accurate answer than can the one loop chiral analysis.

## 4 Matching Conditions

We have seen that chiral symmetry can provide a more extensive analysis of the subtraction constants, while dispersion relations are capable of yielding more accurate information on the intermediate states. This suggests that it may sometimes be advantageous to combine the best of both techniques. To do this we need to be able to tie the two formalisms together in the most accurate way.

In dispersion relations involving subtraction constants we need a precise identification of them. Chiral perturbation theory provides these constants. The key is to reformulate chiral calculations as dispersion relations, order by order. An important point is that the matching is different at order  $E^2$  [6,7] and at order  $E^4$  [5,8].

At order  $E^2$  one needs to reproduce only the tree level chiral results, which do not involve imaginary parts. Thus we only need to ensure that the normalization at low energy is correct. The dispersion integral will then produce new effects at order  $E^4$  which are equivalent to the prediction of the low energy constants at order  $E^4$ , i.e., of the  $L_i$ . This procedure will be more sensitive to high energy effects because one will be using a dispersion integral with at most one subtraction.

At order  $E^4$  one knows more about the low energy structure so one can use a dispersion relation with an extra subtraction. The low energy constants  $L_i$  are no longer predicted, but are inputs to fix the subtraction constants [The dispersion integral then produces new effects at order  $E^6$  and higher]. To match at this order one must reproduce the one loop chiral calculation. Therefore the inputs to the dispersive integral must involve the lowest order vertices, and will only have free propagations of the intermediate state, i.e., the same inputs that go into the Feynman diagram calculation.

To actually carry this out, we need three steps. First, the chiral calculation needs to be carried out to the given order. Then we reformulate the problem as a dispersion relation, requiring that the dispersion relation give the same result when treated to the same order. This fixes the subtraction constants in terms of chiral parameters. Finally we need to use a representa-

tion of the full imaginary part which is compatible with the low energy chiral constraints. This procedure will exactly reproduce the chiral calculation to the order that it is valid, yet add more physics at higher orders in the energy expansion.

The example of the pion formfactor that was used above illustrates the matching technique at  $O(E^4)$ . We determined the subtraction constant in a way which was accurate to  $O(E^4)$  by a comparison with a direct chiral calculation, and then to complete the calculation we feed the experimental imaginary part, given in Fig 2, into the dispersion integral. The use of the experimental  $Imf_\pi(s)$  then generates the full  $f_\pi(q^2)$  at all  $q^2$ . In principle, the only inaccuracy in this calculation is that we have given the subtraction constant  $c$  by an expression which is exact only to order  $E^4$ . There can be corrections to this by extra factors of  $m_\pi^2$  or  $m_\pi^2 \ln m_\pi^2$ .

Let us also briefly consider the same quantity matched at  $O(E^2)$ , using Eq. 7. Now the only matching is the simple constraint  $f_\pi(0) = 1$ , and the effect of the dispersive integral starts at  $q^2$ . This leads to a prediction of the low energy constant

$$2L_9^r(\mu) + \frac{1}{96\pi^2} \left( \ln \frac{m_\pi^2}{\mu^2} + 1 \right) = F_\pi^2 \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} Imf_\pi(s') \quad (21)$$

Note that the lowest order form of  $Imf_\pi(s)$  cannot be inserted in the once subtracted dispersion integral, as the result diverges. The lowest order form for  $Imf_\pi(s)$  is not valid at high energies, but the twice subtracted integral used above was not sensitive to this. The use of the real data for  $Imf_\pi(s')$  leads to a successful prediction of  $L_9^r$  in terms of the mass of the rho meson.

As a side point, let me note that the dispersive treatment also gives us the answer to a previously puzzling feature in the application of vector dominance to the prediction of the chiral parameters.[9,10] The renormalized chiral parameters are scale dependent and it was never clear at what scale they were supposed to equal the prediction of vector dominance. A commonly used but ad-hoc answer is that the predictions were valid at a scale equal to the mass of the rho meson[10]. The dispersive treatment shows the correct result. Rather than predicting the constant  $L_9^r(\mu)$  itself, one is predicting the physical scale-invariant combination given by the dispersive sum rule. This has no scale ambiguities, and is directly physically relevant.

Once one has provided an accurate matching of the two techniques, one

can use the dispersive integral to extend the calculation beyond the range of the chiral result. Thus in the best of all worlds (full data on  $Imf(s)$ , many related reactions) the two techniques form a powerful combination which allows rigorous results at all energies. Chiral perturbation theory provides the subtraction constants from symmetry relations and dispersion relations allows the extrapolation to higher energy.

## 5 Example: The Weinberg sum rules and some relatives

The simplest amplitudes are two point functions, and within QCD the simplest of these are the particular combination of vector and axial vector currents.

$$\pi_V^{\mu\nu}(q^2) - \pi_A^{\mu\nu}(q^2) \equiv i \int d^4x e^{iq \cdot x} \langle 0 | T [V^\mu(x)V^\nu(0) - A^\mu(x)A^\nu(0)] | 0 \rangle \quad (22)$$

This combination is analytic in the complex  $q^2$  plane, except for a pole at  $q^2 = m_\pi^2$  and a cut for  $q^2 > 4m_\pi^2$ . The vector current is conserved. The axial current is conserved in the  $m_q \rightarrow 0$  limit, but with a Goldstone boson. If we define scalar function by

$$\begin{aligned} \pi_V^{\mu\nu}(q^2) &= (q^\mu q^\nu - g^{\mu\nu} q^2) \pi_V(q^2) \\ \pi_A^{\mu\nu}(q^2) &= (q^\mu q^\nu - g^{\mu\nu} q^2) \pi_A(q^2) - q^\mu q^\nu \pi_A^{(0)}(q^2) \end{aligned} \quad (23)$$

we can prove the dispersion relations

$$\pi_V(q^2) - \pi_A(q^2) = \frac{F_\pi^2}{q^2} + \int_{4m_\pi^2}^{\infty} ds' \frac{\rho_V(s') - \rho_A(s')}{s' - q^2 - i\epsilon} \quad (24)$$

with the imaginary parts conventionally named via

$$\rho_{V/A}(s) = \frac{1}{\pi} Im \pi_{V/A}(s) \quad (25)$$

What is known theoretically about these amplitudes? At low energy, chiral perturbation theory predicts the form[4]

$$\begin{aligned}
\pi_V^{\mu\nu}(q^2) - \pi_A^{\mu\nu}(q) &= \left[ \frac{q_\mu q_\nu}{q^2 - m_\pi^2 + i\epsilon} - g_{\mu\nu} \right] F_\pi^2 \tag{26} \\
&+ (q_\mu q_\nu - g_{\mu\nu} q^2) \left[ \frac{1}{48\pi^2} \left( 1 - \frac{4m_\pi^2}{q^2} \right) H(q^2) - 4L_{10}^r \right. \\
&\quad \left. - \frac{1}{48\pi^2} \left( \ln \frac{m_\pi^2}{\mu^2} + \frac{1}{3} \right) \right] \\
\rho_V(s) &= \frac{1}{48\pi^2} \left[ 1 - \frac{4m_\pi^2}{s} \right]^{\frac{3}{2}} \theta(s - 4m_\pi^2) + O(s) \\
\rho_A(s) &= \frac{s}{96(4\pi F_\pi)^2} + O(s^2)
\end{aligned}$$

Here  $L_{10}^r$  is a low energy constant measured in radiative pion decay,  $\pi \rightarrow e\nu\gamma$ .

At high energy, perturbative QCD may be used to analyze the two point function. In the chiral limit,  $m_q = 0$ , which will be used for the rest of this section, the operator product expansion can be used to show that the difference  $\pi_V - \pi_A$  falls as  $\frac{1}{q^6}$  and  $\rho_V(s) - \rho_A(s) \sim \frac{1}{s^3}$ . In terms of four quark operators, which are here evaluated in the vacuum saturation approximation[11], one has

$$\begin{aligned}
\pi_V(q^2) - \pi_A(q^2) &= \frac{32\pi}{9} \frac{\langle \sqrt{\alpha_s} \bar{q}q \rangle_0^2}{q^6} \left\{ 1 + \frac{\alpha_s(q^2)}{4\pi} \left[ \frac{247}{12} + \ln \frac{\mu^2}{-q^2} \right] \right\} \tag{27} \\
\rho_V(s) \rightarrow \rho_A(s) &\rightarrow \frac{1}{8\pi^2} \left[ 1 + \frac{\alpha_s(s)}{\pi} \right], \quad s \rightarrow \infty \\
\rho_V(s) - \rho_A(s) &\sim \frac{8}{9} \frac{\alpha_s \langle \sqrt{\alpha_s} \bar{q}q \rangle_\infty^2}{s^3}
\end{aligned}$$

We see that  $\pi_V - \pi_A$  and  $\rho_V - \rho_A$  are very well behaved at large  $q^2, s$ .

We can combine up this information to get a set of sum rules. The requirement that, as  $q^2 \rightarrow \infty$ , there is no  $\frac{1}{q^2}$  term in the dispersion relation Eq.24, requires

$$F_\pi^2 = \int_0^\infty ds (\rho_V(s) - \rho_A(s)) \tag{28}$$

while the absence of  $\frac{1}{q^4}$  implies

$$0 = \int_0^\infty ds s (\rho_V(s) - \rho_A(s)) \quad (29)$$

These are the Weinberg sum rules[12], the second of which is only true in the  $m_q \rightarrow 0$  limit. At low energy, expansion of the dispersion integral and chiral results in powers of  $q^2$  imply[13,4]

$$-4\bar{L}_{10} = \int_{4m_\pi^2}^\infty \frac{ds}{s} (\rho_V(s) - \rho_A(s)) \quad (30)$$

with

$$\begin{aligned} \bar{L}_{10} &= L_{10}^r(\mu) + \frac{1}{192\pi^2} \left[ \ln \frac{m_\pi^2}{\mu^2} + 1 \right] \\ &= (-0.7 \pm 0.03) \times 10^{-2} \text{ (Expt : } \pi \rightarrow e\nu\gamma \text{)} \end{aligned} \quad (31)$$

Here I have given the sum rule for finite  $m_\pi^2$  since there is a behavior proportional to  $\ln m_\pi^2$  at the low energy end of the integral. These sum rules illustrate one of the uses of chiral dispersion relations, which is the prediction/calculation of low energy constants (here  $F_\pi$  and  $L_{10}$ ). When these constraints are satisfied we have an accurate matching of the two descriptions, valid to  $O(E^4)$ .

Another use of chiral dispersion relations is in extending the reach of calculations and even opening up the possibility of entirely new types of calculations. Consider the Compton amplitude  $\gamma\pi \rightarrow \gamma\pi$ . In the soft pion limit, chiral symmetry relates this to the vacuum polarization tensors

$$\begin{aligned} &\lim_{p \rightarrow 0} \langle \pi^+(p) | T(V^\mu(x)V^\nu(0)) | \pi^+(p) \rangle \\ &= -\frac{1}{F_\pi^2} \langle 0 | T(V^\mu(x)V^\nu(0) - A^\mu(x)A^\nu(0)) | 0 \rangle \\ &= -\frac{1}{F_\pi^2} [\pi_V^{\mu\nu}(x) - \pi_A^{\mu\nu}(x)] \end{aligned} \quad (32)$$

If one takes the Compton amplitude and ties together the two electromagnetic currents with a photon propagator, one obtains the pion electromagnetic mass shift.[14] Clearly the chiral representation, Eq. 26, would be

inadequate to calculate this, as the photon loop integral goes over all values of  $q^2$ . However, after some algebra plus the application of the Weinberg sum rules, the dispersive representation allows one to write this as [14]

$$m_{\pi^+}^2 - m_{\pi^0}^2 = -\frac{3e^2}{16\pi^2 F_\pi^2} \int_0^\infty ds s \ln s [\rho_V(s) - \rho_A(s)] \quad (33)$$

which is an exact relation in the chiral limit. Note that here chiral symmetry was used to relate different amplitudes in Eq. 32 and to provide low energy constraints, as in the Weinberg sum rules, while dispersion relations were needed to provide a predictive framework for the intermediate energy region.

In a similar way, one can calculate reliably a new weak nonleptonic matrix element.[15] Consider the hypothetical weak Hamiltonian

$$H_V = \frac{g_2^2}{8} \int d^4x i D_F^{\mu\nu}(x, M_w) T \left( \bar{d}(x) \gamma_\mu u(x) \bar{u}(0) \gamma_\nu s(0) \right) \quad (34)$$

Up to some KM factors, this would be the usual weak Hamiltonian if the vector currents were replaced by  $\gamma_\mu(1 + \gamma_5)$ . In the chiral limit, we have another chiral sum rule

$$\langle \pi^- | H_V | K^- \rangle = \frac{3iG_F}{32\pi^2 \sqrt{2} F_\pi^2} A \quad (35)$$

with

$$A = M_w^2 \int_0^\infty ds \frac{s^2 \ln(s/M_w^2)}{s - M_w^2 + i\epsilon} [\rho_V(s) - \rho_A(s)] \quad (36)$$

which is exact in the chiral limit.

Gene Golowich and I have provided a phenomenological analysis of these sum rules.[16] The physics of the spectral functions  $\rho_{V,A}$  is basically simple. At intermediate energies they are measured in  $\tau$  decay and  $e^+e^-$  annihilation, and the largest features are the  $\rho$  and  $a_1$  resonances, with very much smaller  $4\pi, 5\pi$  etc. contributions. At low energy this can be merged smoothly to chiral predictions and at high energy  $\rho_V - \rho_A$  vanishes rapidly and we matched the data to QCD around  $s = 5GeV^2$ . There are some experimental uncertainties, but these can in principle be reduced in the future.

The  $L_{10}$  sum rule works well with very little uncertainty as it is sensitive to the lowest energy contributions. The Weinberg sum rules and that for

$\Delta m_\pi^2$  work within the experimental uncertainties. We have proceeded by imposing them exactly on our  $\rho_V - \rho_A$ , which requires only minor adjustments within the allowed error bars. That this is possible is a nontrivial test of the theoretical framework. Finally the weak matrix element is predicted ( $A = -0.062 \pm 0.017 \text{GeV}^2$ ). This can perhaps be compared with future lattice calculations.

## 6 The Elastic Approximation and the Omnes Problem

The pion formfactor and the Weinberg sum rules are particularly powerful because we have a direct measurement of the required imaginary parts. In many other cases, we do not have this luxury. Nevertheless, much of the dynamics of the intermediate states follows from the behavior of  $\pi\pi$  scattering, about which we know a good deal. This allows us to predict the behavior of the imaginary part of the desired amplitude, with the modest additional assumption that the only important intermediate state is the  $\pi\pi$  channel, i.e. the elastic approximation.

Consider some two particle amplitude  $f(s)$  of a given isospin and angular momentum which is analytic in complex  $s$  plane except for a cut above two particle threshold  $s_0 = 4m^2$ . The inelastic thresholds are somewhat higher, for example  $s_{inel} = 16m^2$ . In the elastic region, Watson's theorem tells us that the phase of the amplitude is that of the corresponding two particle scattering amplitude

$$f(s) = e^{i\delta(s)} |f(s)| \quad (37)$$

In practice inelasticities do not play a significant role in low energy pion physics up to 1 GeV ( $K\bar{K}$  threshold), and one often assumes an approximation of keeping only the elastic channel. While probably reasonable, it is important to realize that the elastic approximation relies on more than just Watson's theorem and produces more than just the phase of the amplitude.

The Omnes problem[17] is the mathematical exercise of finding functions which are analytic except for a cut  $4m^2 < s < \infty$ , which are real when  $s$  is real and less than  $4m^2$  and for which  $f(s)e^{-i\delta(s)}$  is real when  $s$  is real and greater than  $4m^2$ . The solution is given by



$$\begin{aligned}
f(s) &= P(s)D^{-1}(s) \\
D^{-1}(s) &\equiv \exp\left\{\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t}\frac{\delta(t)}{t-s-i\epsilon}\right\}
\end{aligned}
\tag{38}$$

as long as

$$\lim_{s\rightarrow\infty}\delta(s) = \text{finite}; \quad \lim_{s\rightarrow\infty}\frac{|f(s)|}{s} \rightarrow 0
\tag{39}$$

In the above  $P(s)$  is a polynomial in  $s$ , and  $D^{-1}(s)$  is called the Omnes function.

Note that this is not exactly the right problem for QCD. The assumption that  $f(s)e^{-i\delta(s)}$  is real above the cut implies that the reaction is elastic at *all* energies. Once inelastic channels open up, the quantity  $f(s)e^{-i\delta(s)}$  rapidly deviates from being real. In QCD, once one is above 1 GeV, the inelastic channels open rapidly and become quite numerous, leading to perturbative QCD behavior at precociously low energies. It is not known how to provide a general solution to the QCD type problem (although the form of the solution to the two channel problem is also known), nor is it known how much of an effect the multiple inelasticities of QCD have on the Omnes function.

## 7 Example: $\gamma\gamma \rightarrow \pi\pi$

The reaction  $\gamma\gamma \rightarrow \pi^+\pi^-$  and  $\gamma\gamma \rightarrow \pi^0\pi^0$  are of interest in the development of chiral theory because  $\gamma\gamma \rightarrow \pi^0\pi^0$  first arises as a pure loop effect as there are not tree level contributions at  $O(E^2)$  or  $O(E^4)$ . For these reactions, we have both a one-loop [18] and two loop [19] chiral analysis as well as dispersive treatments [20,8] and experimental data. This makes these reactions excellent illustrations of chiral techniques and of the ties with dispersion relations.

The  $\gamma\gamma \rightarrow \pi\pi$  matrix elements can be decomposed into isospin amplitudes

$$\begin{aligned}
f^{+-}(s) &= \frac{1}{3}[2f_0(s) + f_2(s)] \\
f^{00}(s) &= \frac{2}{3}[f_0(s) - f_2(s)]
\end{aligned}
\tag{40}$$

The dominant partial waves at low energy are the S waves and these are predicted in a one loop chiral analysis to be

$$f_I^{chiral}(s) = \frac{1-\beta^2}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{(1-\beta^2)}{4\pi} t_I^{CA}(s) \ln^2 \frac{\beta+1}{\beta-1} - \frac{1}{\pi} t_I^{CA}(s) + \frac{2}{F_\pi^2} (L_9^r + L_{10}^r) s \quad (41)$$

where

$$\beta = \sqrt{1 - \frac{4m_\pi^2}{s}} \quad (42)$$

and  $t_I^{CA}(s)$  are the lowest order  $\pi\pi$  scattering amplitudes

$$t_0^{CA} = \frac{2s - m_\pi}{32\pi F_\pi^2}; t_2^{CA} = -\frac{(s - sm_\pi^2)}{32\pi F_\pi^2} \quad (43)$$

The dispersion relation has been derived by Morgan and Pennington[20], in terms of an amplitude  $p_I(s)$  which has the same left-hand singularity structure as  $f_I(s)$  but which is real for  $s > 0$ . Then  $[f_I(s) - p_I(s)]D_I(s)$  satisfies a twice subtracted dispersion relation and we have

$$f_I(s) = D_I^{-1}(s) \left[ (c_I + d_I s) + p_I(s) D_I(s) - \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{p_I(s') Im D_I(s')}{s' - s - i\epsilon} \right] \quad (44)$$

with two subtraction constants per channel  $c_I, d_I$ . As a prelude to the matching we note that Low's theorem requires that  $f_I(s)$  be the Born scattering amplitude at low energies. Therefore

$$p_I(s) = f_I^{Born}(s) + O(s) = \frac{1-\beta^2}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) + O(s) \quad (45)$$

This is the  $O(E^2)$  result. To proceed to order  $E^4$  we note that the leading piece of  $Im D_I(s)$  is also known, i.e.,

$$Im D_I(s) = -\beta t_I^{CA}(s) \quad (46)$$

as this is the lowest order  $\pi\pi$  scattering amplitude. Using this, the dispersive integral can be done exactly, leading to

$$f_I(s) = D^{-1}(s) \left[ c_I + s \left( d_I - \frac{t_I^{CA}(0)}{12\pi m_\pi^2} \right) + D_I(s) \frac{1-\beta^2}{2\beta} \ln \left( \frac{1+\beta}{1-\beta} \right) \right. \\ \left. - \frac{1}{4\pi} (1-\beta^2) t_I^{CA}(s) \ln^2 \left( \frac{\beta+1}{\beta-1} \right) \right] + \dots \quad (47)$$

A comparison of this with the  $O(E^4)$  chiral results then indicates that this procedure has reproduced all of the one loop results, as long as we choose the subtraction constants as[8]

$$c_I = 0 ; d_I = \frac{2}{F_\pi^2} (L_9^r + L_{10}^r) + \frac{t_I^{CA}(0)}{12\pi m_\pi^2} \quad (48)$$

Again we see that the dynamical content of the one loop chiral calculation is also contained in the dispersive treatment when the imaginary part is taken to be the lowest order scattering amplitude. However, chiral symmetry also predicts the subtraction constants, which in this case are known from measurements in radiative pion decay.

Having identified the subtraction constants one can add the ingredients to complete the calculation. The most important at threshold is the use of the real world  $D_I^{-1}(s)$ [7]. This change alone produces a significant effect in the amplitude even near threshold in the neutral case. The second step is a better determination of  $p_I(s)$  which includes the  $O(E^4)$  chiral corrections to it as well as the  $\rho, \omega, A1$  poles which are known (from  $\rho \rightarrow \pi\gamma$  etc. data) to occur in the Compton amplitude. Figure 3 shows the data for the reaction  $\gamma\gamma \rightarrow \pi^0\pi^0$  along with the one-loop chiral prediction (dashed line) and the modification obtained by the dispersive treatment (solid line). The one-loop chiral result is of the right rough size, its slope is low at threshold and it grows unphysically at high energy. Near threshold the difference in the two calculations comes almost exclusively from the rescattering corrections generated through the dispersion relation. The change is sizable even at low energy, since the rescattering is in the  $I = 0, J = 0$  channel. The Omnes function alone has brought the threshold region into better agreement with the data. It has also tamed the high energy growth. The final result (with

no free parameters) matches the data very well, and also gives the charged channel correctly.

Bellucci, Gasser and Sainio[19] have performed the enormously difficult two loop calculation. [In fact, technically they employ dispersive methods to do portions of this.] At two loop order, new low energy constants appear, which are not measured in any other process. Therefore the authors have to step outside of pure chiral perturbation theory in order to model these constants, using vector meson dominance. Much like the dispersive work described above, these constants play little role in the threshold region, but are important for the shape of the amplitude for moderate energy. It is very interesting that their results look very similar to the dispersive treatment described above.

Both of these methods have potential limitations. In principle, the only limitation of the dispersive treatment is the fact that it can miss  $O(E^6)$  terms in the subtraction constants  $c_I, d_I$ . These would be corrections to results given above by factors of  $m_\pi^2$  or  $m_\pi^2 \ln m_\pi^2$ . In practice we also need to model the higher order terms in  $p_I(s)$ . As for the two loop chiral result, its only limitation in principle is the fact that it misses higher order dependence in the energy. By construction it is valid to order  $E^6$ , but does not contain higher order  $s$  dependence, and so would be expected to fall apart soon after the  $E^6$  dependence became important. In practice, this approach also needs to do some modeling in order to estimate the unknown low energy constants. The fact that the results agree so well with each other and with the data indicates that these limitations are not very important at these energies. Both capture the important physics, and do so in a reasonably controlled fashion. There is of course a significant practical advantage to the dispersive approach—it is far easier!

## 8 Other uses of dispersion relations in effective field theories

The examples given above have all been in the context of chiral perturbation theory. We have a detailed knowledge of the behavior of chiral amplitudes at low energy as well as measurements of various reactions which can be used in dispersive integrals. However, other effective field theories can also benefit

from the dispersive method of analysis.

The main obstacle to many applications of dispersion relations is the lack of an experimental measurement of the imaginary part of the amplitude. This will of course preclude a rigorous model independent calculation, but one may still be able to use approximations or modeling to make progress on the problem. Dispersive calculations open up new methods of approximation.

An example is in the electroweak corrections due to TeV scale physics. Peshkin and Takeuchi have defined a variable  $S$  which describes some of the low energy effects on  $W$  and  $Z$  propagation due to new physics, and have given a dispersive sum rule for it[21]. The sum rule is equivalent to the  $L_{10}$  sum rule, Eq. 30, where the vacuum polarization functions are those for the  $W$  and  $Z$  self-energies. These sum rules have been applied extensively to Technicolor theories.[22] One potential new application is to the effect of a heavy Higgs boson. While this has been calculated in detail in perturbation theory[23], as the Higgs mass gets heavier, perturbation theory ultimately becomes inapplicable because the theory becomes strongly coupled. However in the strongly coupled regime we still know various features of the full amplitudes, such as threshold behavior and unitarity bounds, even if we cannot calculate precisely. Preliminary analysis suggests that that one can use the dispersive sum rule plus the general behavior of the amplitude to argue that the perturbative estimate is not far wrong[24].

There have also been a set of interesting applications of dispersive techniques to the Heavy Quark Effective Theory[25]. It is not possible to summarize all the types of applications that dispersion relations have in effective field theory. However, the main point of this lecture is that these techniques are well adapted to the type of questions that we ask in effective field theories and can sometimes be useful in extracting features that are not visible in the usual Feynman diagram methods.

## 9 Summary

We have seen how dispersion relations can add power to effective field theory. At its best it uses more physics input. It can match all perturbative effects to whichever order that they are known, and can be used to replace the modeling of unknown physics by using data instead of models. However there are some limitations, coming both from incomplete data and from the

fact that we can only determine the subtraction constants to a given order in the energy expansion.

The technology for combining these techniques is now developed. This involves first knowing the chiral analysis of the amplitude to a given order in the energy expansion. One also needs a dispersion relation for the amplitude in question. The number of subtractions is determined partially by the high energy behavior of the amplitude, but the use of more subtractions than are required can help in the matching with the effective field theory chiral result. The matching occurs order by order in the energy expansion. When it can be done, it is preferable to perform the matching at  $O(E^4)$  because the resulting dispersive treatment is less sensitive to what happens at high energy since a twice subtracted dispersion relation can be used. Finally, the real world data has to be found to use in the dispersive integral. Often, in chiral perturbation theory, the use of the elastic approximation is made for this, allowing the use of known  $\pi\pi$  scattering data.

The output of these efforts can be several. Most commonly, these techniques are used to extend the range and accuracy of the chiral calculations, by getting around the limitation of the energy expansion. The method can be used to predict unknown chiral coefficients, as was shown for the case of  $L_{10}$ . We can use these techniques to remove or reduce the model dependence of some result. Finally, dispersive techniques allow us to perform completely new types of calculations, such as the hadronic matrix elements of Section 5.

**Acknowledgements.** I would like to thank Juerg Gasser, Barry Holstein and Heiri Leutwyler for sharing their insights into dispersion relations and chiral symmetry. In addition, I thank the organizers, lecturers and students for making this school so enjoyable and productive.

#### References.

- 1) An earlier version of the present lecture has appeared in *Chiral Dynamics of Hadrons and Nuclei*, ed by D.P. Min and M. Rho (Seoul National University Press, Seoul, 1995), p87 (hep-ph/9506205).
- 2) R. Kronig, J. Op. Soc. Am. **12**, 547 (1926); H. A. Kramers, Atti Cong. Int. Fisici Como (1927).  
G. Barton, *Dispersion Techniques in Field Theory*, (Benjamin, NY, 1965).
- 3) J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model*, (Cambridge University Press, Cambridge, 1992).
- 4) J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465(1985).
- 5) T. N. Truong, Phys. Rev. Lett. **61**,2526 (1988).

- 6) As far as I know, the first attempt to use dispersion relations to improve the predictions of chiral perturbation theory occurred in Ref. 7.
- 7) J. F. Donoghue, J. Gasser and H. Leutwyler, Nucl. Phys. **B343**, 341(1990).
- 8) J. F. Donoghue and B. R. Holstein, Phys. Rev. **D48**, 137 (1993).
- 9) J. F. Donoghue, C. Ramirez and G. Valencia, Phys.Rev.**D39**,1947,1989.
- 10) G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nucl.Phys. **B321**, 311,1989.
- 11) M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979).
- L. V. Lanin, V. P. Spiridonov, and K. G. Chetyrkin, Sov. J. Nucl. Phys. **44**, 892 (1986).
- 12) S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
- 13) T. Das, V. Mathur, and S. Okubo, Phys. Rev. Lett. **19**, 859 (1967).
- 14) T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Lett. **18**, 759 (1967).
- 15) J. F. Donoghue and E. Golowich, Phys. Lett. **315**, 406 (1993).
- 16) J. F. Donoghue and E. Golowich, Phys. Rev. **D49**, 1513 (1994).
- 17) R. Omnes, Nuovo Cim. **8**, 1244(1958).
- 18) J. Bijnens and F. Cornet, Nucl. Phys. **B296**, 557 (1988).
- J. F. Donoghue, B. R. Holstein and Y. C. Lin, Phys. Rev. **D37**, 2423 (1988).
- 19) S. Bellucci, J. Gasser and M. Sainio, Nucl. Phys. **B423**,801994).
- 20) D. Morgan and M. R. Pennington, Phys. Lett. **B272**, 134 (1991).
- 21) M. E. Peskin and T. Takeuchi, Phys.Rev.Lett.**65**, 964(1990.) 22) M. E. Peskin and T. Takeuchi, Phys.Rev.**D46**, 381(1992). B. Holdom, J. Terning, Phys.Lett. **B247**, 88(1990).
- 23) M. J. Herrero, E. Ruiz Morales, Nucl.Phys.**B437**, 319(1995).
- M. J. Herrero, E. Ruiz Morales, Nucl.Phys.**B418**, 431(1994).
- A. C. Longhitano, Phys.Rev.**D22**, 1166(1980).
- T. Appelquist, C. Bernard, Phys.Rev.**D22**, 200(1980).
- 24) J. F. Donoghue, work in preparation
- 25) C. G. Boyd, B. Grinstein, R. F. Lebed, UCSD-PTH-95-15, hep-ph/9508242,
- C. G. Boyd, B. Grinstein, R. F. Lebed, Phys.Lett.**B353**, 306-312 (1995).