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Allowed Eta-Decay Modes and Chiral Symmetry

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Abstract

Recently, the development of chiral perturbation theory has allowed the generation of rigorous low-energy theorems for various hadronic processes based only on the chiral invariance of the underlying QCD Lagrangian. Such techniques are highly developed and well-tested in the domain of pionic and kaonic reactions. In this note we point out that with the addition of a few additional and reasonable assumptions similar predictive power is available for processes involving the eta meson.

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1 Introduction

It has long been the holy grail for particle and nuclear knights to generate rigorous predictions from the Lagrangian of QCD

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}G_{\mu\nu}G^{\mu\nu} + \bar{q}(i\gamma_\mu D^\mu - m)q \quad (1)$$

where

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ D_\mu q &= (\partial_\mu - igA_\mu)q. \end{aligned} \quad (2)$$

However, despite the ease with which one can write this equation, because of its inherent nonlinearity and the large value of the coupling constant— $g^2/4\pi \sim 1$ —progress in this regard has been slow. One approach—lattice gauge theory—holds great promise, but is currently limited by the need for large computational facilities[1]. A second tack, that of perturbative QCD, exploits the feature of asymptotic freedom—the vanishing of the running coupling constant at high momentum transfer[2]. However, such predictions are valid only for the very highest energy processes. It is gratifying then to see that in recent years a third procedure has become available, that of chiral perturbation theory (χ PT) which exploits the chiral symmetry of QCD and allows rigorous predictive power in the case of low energy reactions. This technique, based on a suggestion due to Weinberg[3], was developed (at one loop level) during the last decade in an important series of papers by Gasser and Leutwyler and others[4]. The idea is based on the feature that the QCD Lagrangian—Eq. 1—possesses a global $SU(3)_L \times SU(3)_R$ (chiral) invariance in the limit of vanishing quark mass. Such invariance is manifested in the real world not in the conventional Wigner-Weyl fashion. Rather, it is spontaneously broken, resulting in eight light pseudoscalar Nambu-Goldstone bosons— π, K, η —which would be massless if the corresponding quark masses also vanished[5]. While the identification of this symmetry is apparent in terms of quark/gluon degrees of freedom, it is not so simple to understand the implications of chiral invariance in the arena of experimental meson/baryon interactions.

Early attempts in this direction were based on current algebra/PCAC methods[6], yielding relationships between processes differing in the number of pions, *e.g.*

$$\lim_{q \rightarrow 0} \langle B\pi_q^a | \mathcal{O} | A \rangle = \frac{-i}{F_\pi} \langle B | [F_5^a, \mathcal{O}] | A \rangle \quad (3)$$

where $F_\pi = 92.4$ MeV is the pion decay constant[7]. However, it was soon realized that the most succinct way to present these restrictions is in terms of an effective chiral Lagrangian, the simplest (two-derivative) form of which is, in the Goldstone sector[8],

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{\bar{F}^2}{4} \text{Tr} D_\mu U D^\mu U^\dagger + \frac{\bar{F}^2}{4} \text{Tr} 2B_0 m (U + U^\dagger) + \dots \quad (4)$$

where

$$U = \exp \left(\frac{i}{\bar{F}} \sum_{j=1}^8 \lambda_j \phi_j \right) \quad (5)$$

is a nonlinear function of the pseudoscalar fields, $m = (m_u, m_d, m_s)_{\text{diag}}$ is the quark mass matrix,

$$2B_0 = \frac{2m_K^2}{m_u + m_s} = \frac{2m_\pi^2}{m_u + m_d} \quad (6)$$

is a phenomenological constant, $D_\mu = \partial_\mu - i[V_\mu, \cdot] - i\{A_\mu, \cdot\}$ is the covariant derivative coupling to external fields V_μ, A_μ , and $\bar{F}$ is the pion decay constant in the limit of chiral symmetry. Although these are only two of an infinite number of terms, already at this level there exists predictive power—*e.g.*, tree level evaluation of $\mathcal{L}^{(2)}$ yields the familiar Weinberg predictions (at $\mathcal{O}(p^2, m^2)$) for S-wave $\pi - \pi$ scattering lengths[9]

$$a_0^0 = \frac{7m_\pi}{32\pi F_\pi^2} \quad a_0^2 = -\frac{m_\pi}{16\pi F_\pi^2} \quad (7)$$

which are approximately borne out experimentally. Loop diagrams, of course, produce terms of $\mathcal{O}(p^4, p^2 m^2, m^4)$ and contain divergences. However, just as in QED such infinities can be absorbed into renormalizing phenomenological chiral couplings, and the most general "four-derivative" Lagrangian has been given by Gasser and Leutwyler[3]

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{(4)} = & L_1(\text{Tr} D_\mu U D^\mu U^\dagger)^2 + L_2(\text{Tr} D_\mu U D_\nu U^\dagger)^2 \\ & + L_3 \text{Tr}(D_\mu U D^\mu U^\dagger)^2 + L_4 \text{Tr} D_\mu U D^\mu U^\dagger \text{Tr}(m(U + U^\dagger)) \\ & + L_5 \text{Tr} D_\mu U D^\mu U^\dagger m(U + U^\dagger) + L_6 (\text{Tr} m(U + U^\dagger))^2 \\ & + L_7 (\text{Tr} m(U - U^\dagger))^2 + L_8 \text{Tr}(mU mU + mU^\dagger mU^\dagger) \\ & + iL_9 (\text{Tr} F_{\mu\nu}^L D^\mu U D^\nu U^\dagger + \text{Tr} F_{\mu\nu}^R D^\mu U D^\nu U^\dagger) \\ & + L_{10} \text{Tr} F_{\mu\nu}^L U F^{R\mu\nu} U^\dagger \end{aligned} \quad (8)$$

where $F_{\mu\nu}^L, F_{\mu\nu}^R$ are external field strength tensors defined via

$$\begin{aligned} F_{\mu\nu}^{L,R} &= \partial_\mu F_\nu^{L,R} - \partial_\nu F_\mu^{L,R} - [F_\mu^{L,R}, F_\nu^{L,R}] \\ F_\mu^L &= V_\mu + A_\mu \quad F_\mu^R = V_\mu - A_\mu. \end{aligned} \quad (9)$$

Here the bare L_i coefficients are themselves unphysical and are related to empirical quantities $L_i^r(\mu)$ measured at scale μ via

$$L_i^r(\mu) = L_i + \frac{\Gamma_i}{32\pi^2} \left(\frac{1}{\epsilon} + \ln \frac{4\pi}{\mu^2} + 1 - \gamma \right), \quad (10)$$

where Γ_i are constants defined in ref. 4b and $\epsilon = 4 - d$ is the usual parameter arising in dimensional regularization, with d being the number of dimensions.

L_1^r	L_2^r	L_5^r	L_9^r	L_{10}^r
0.71 ± 0.28	2.01 ± 0.37	2.7 ± 0.3	7.7 ± 0.2	-5.2 ± 0.3

Table 1: Empirical values of Chiral Expansion Parameters($\times 10^{-3}$) with $\mu = m_\eta$.

Gasser and Leutwyler have obtained empirical values for the phenomenological constants $L_1^r(\mu), \dots, L_{10}^r(\mu)$, values for some of which are given in Table 1.

At the four-derivative level it is also necessary to include the contribution of the anomaly, which in the case of coupling to the photon field A_μ has the form[10]

$$\begin{aligned} & \Gamma_{\text{WZW}}(U, A_\mu) = \Gamma_{\text{WZW}}(U) \\ & + \frac{N_c}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} \int d^4x [e A_\mu \text{Tr}(Q(R_\nu R_\alpha R_\beta + L_\nu L_\alpha L_\beta)) \\ & - ie^2 F_{\mu\nu} A_\alpha \text{Tr}\left(Q^2(R_\beta + L_\beta) + \frac{1}{2}(QU^\dagger QU R_\beta + QUQU^\dagger L_\beta)\right)] \quad (11) \end{aligned}$$

where $R_\mu \equiv (\partial_\mu U^\dagger)U$, $L_\mu \equiv U\partial_\mu U^\dagger$ and Γ_{WZW} is independent of the photon field and will not be needed for our purposes. A corresponding form involving coupling to a general nonabelian gauge field can also be written, but is lengthy and will not be given here[11].

In a series of recent papers it has been conclusively demonstrated that this chiral effective action formalism provides a succinct and successful description of low energy electroweak interactions of pions and kaons[12]. Specifically, the reactions given in Table 2 are successfully described in terms of GL parameters $L_9(\mu), L_{10}(\mu)$. Clearly, there are far more reactions than parameters, and this overdetermination enables one to construct *required* relationships between experimental quantities, the empirical validity of which constitutes a strong test of the chiral formalism and indeed thereby of QCD itself[13]. Such tests are found to be well satisfied, with the possible exception of the relationship between the axial structure function in radiative pion beta decay— h_A —and the charged pion polarizability— α_E^+ [14]. However, recent work indicates that this may not be a problem and in any case a number of experimental efforts are underway to retest this critical stricture[15].

In the pion and kaon arena then one finds strong evidence for the correctness and utility of chiral perturbative techniques, and it is an obvious next step to attempt to extend this success into the eta sector, which is the subject of this note. This examination of the eta system is important both as a theoretical exercise and because of the existence of high intensity sources of etas[16]. In the next section then we study the ability of the chirally inspired methods to make reliable predictions for eta decay processes. We examine only the "allowed" modes— $\eta \rightarrow 2\gamma, 3\pi, 2\pi\gamma, \pi 2\gamma, 3\pi\gamma$, *i.e.* those modes which can occur assuming only isospin violation or the anomaly, eschewing the temptation to

Reaction	Quantity	Theory	Experiment
$\gamma \rightarrow \pi^+ \pi^-$	$\langle r_\pi^2 \rangle (\text{fm}^2)$	0.44 ^a	0.44 ± 0.02
$\gamma \rightarrow K^+ K^-$	$\langle r_K^2 \rangle (\text{fm}^2)$	0.44	0.34 ± 0.05
$\pi^+ \rightarrow \pi^+ \nu_e \gamma$	$h_V (m_\pi^{-1})$	0.027	0.029 ± 0.017
	h_A/h_V	0.46 ^a	0.46 ± 0.08
$K^+ \rightarrow e^+ \nu_e \gamma$	$(h_V + h_A)(m_\pi^{-1})$	0.038	0.043 ± 0.003
$\pi^+ \rightarrow e^+ \nu_e e^+ e^-$	r_A/h_V	2.6	2.3 ± 0.6
$\gamma \pi^+ \rightarrow \gamma \pi^+$	$(\alpha_E + \beta_M)(10^{-4}(\text{fm}^3))$	0	1.4 ± 3.1
	$\alpha_E(10^{-4}(\text{fm}^3))$	2.8	6.8 ± 1.4
$\gamma \gamma \rightarrow \pi \pi$	$\alpha_E(10^{-4}(\text{fm}^3))$	2.8	2.2 ± 1.6
$K \rightarrow \pi e^+ \nu_e$	$\xi = f_-(0)/f_+(0)$	-0.13	-0.20 ± 0.08
	$\lambda_+(\text{fm}^2)$	0.067	0.065 ± 0.005
	$\lambda_0(\text{fm}^2)$	0.040	0.050 ± 0.012

Table 2: Chiral predictions and data in the radiative complex of transitions. The superscript a indicates that this parameter was used as input and is *not* a predicted quantity.

analyze “rare” processes such as $\eta \rightarrow 3\gamma$, as these have been well-discussed elsewhere[17]. Finally, we summarize our results in a concluding section III.

2 Eta Decay Processes

Strictly from a kinematic perspective, inclusion of the $\eta(547)$ as part of the chiral formalism should not be a problem, as the eta and kaon are roughly degenerate in mass, and as mentioned above the kaon (and pion) predictions obtained in this way are quite successful. Rather the real challenge involves mixing with $\eta'(958)$, which lies outside the simple chiral $SU(3)_L \times SU(3)_R$ framework. To lowest order things are simple—in the chiral limit the pseudoscalar mass spectrum would consist of a massless octet of Goldstone bosons plus a massive $SU(3)$ singlet (η_0). With the breaking of chiral invariance the octet pseudoscalar masses become nonzero and are related, at first order in chiral symmetry breaking, by the Gell-Mann-Okubo formula[18]

$$m_{\eta_8}^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2 \approx (0.57 \text{ GeV})^2 \quad (12)$$

where η_8 is the eighth member of the octet. At this same order in symmetry breaking the singlet η_0 will in general mix with η_8 producing the physical eigenstates η, η' given by

$$\begin{aligned} \eta &= \cos \theta \eta_8 - \sin \theta \eta_0 \\ \eta' &= \sin \theta \eta_8 + \cos \theta \eta_0. \end{aligned} \quad (13)$$

The mixing angle θ can be determined via diagonalization of the mass matrix (written in the η_8, η_0 basis)

$$m^2 = \begin{pmatrix} m_{\eta_8}^2 & m_{08}^2 \\ m_{08}^2 & m_{\eta_0}^2 \end{pmatrix} \quad (14)$$

Here $m_{\eta_8}^2$ is given in Eq. 12 while $m_{08}^2, m_{\eta_0}^2$, and θ are unknown. Diagonalizing and fitting these parameters with the two known masses yields the results

$$\theta = -9.4^\circ, \quad m_{08}^2 = -0.44m_K^2, \quad m_{\eta_0} = 0.95 \text{ GeV} \quad (15)$$

However, there is good reason not to trust this simple and lowest order analysis, since higher order chiral symmetry breaking terms can generate important modifications. For example, taking the leading log correction arising from Figure 1, we find[19]

$$\begin{aligned} m_{\eta_8}^2 &= \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2 - \frac{2}{3} \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{\mu^2} \\ &\approx (0.61 \text{ GeV})^2 \quad \text{for } \mu = 1 \text{ GeV}, \end{aligned} \quad (16)$$

for which diagonalization of the mass matrix yields

$$\theta \approx -19.5^\circ, \quad m_{08}^2 = -0.81m_K^2, \quad m_{\eta_0} = -0.90 \text{ GeV} \quad (17)$$

suggesting a doubling of the mixing angle. Of course, this is just an approximate

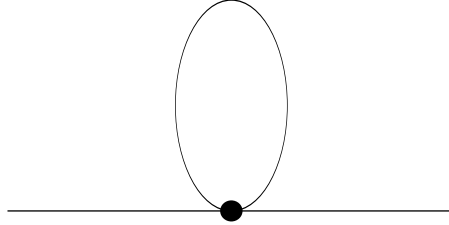


Figure 1: Mass and wavefunction renormalization diagram.

result. However, a full one loop calculation using χ PT yields essentially the same result[4], and consequently in our analysis below we shall use the value in Eq. 17, *i.e.*

$$\sin \theta \approx -\frac{1}{3} \quad \cos \theta \approx \frac{2\sqrt{2}}{3}. \quad (18)$$

It is also intriguing that this solution is consistent with the assumptions of simple U(3) invariance wherein η_8, η_0 have the same quark wavefunction, leading to

$$\frac{m_{08}^2}{m_K^2} \simeq \frac{2\sqrt{2}}{3} \left(\frac{\hat{m} - m_s}{\hat{m} + m_s} \right) \simeq -0.9 \quad (19)$$

At this same (one-loop) level of symmetry breaking there is generated a shift in the lowest order value of the pseudoscalar decay constant F_P . Thus one finds from the diagrams in Figure 2, in leading log approximation,

$$\begin{aligned} F_\pi &= \bar{F} \left[1 - \frac{1}{2} \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{\mu^2} \right] \approx 1.12 \bar{F} \\ F_{\eta_8} &= \bar{F} \left[1 - \frac{3}{2} \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{\mu^2} \right] \approx 1.25 F_\pi \quad \text{for } \mu \approx 1 \text{ GeV.} \end{aligned} \quad (20)$$

Once again these estimates are in excellent agreement with those given in the full one-loop analysis[4]. (We note in addition that the corresponding prediction

$$\frac{F_K}{F_\pi} = 1 - \frac{1}{4} \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{\mu^2} - \frac{3}{8} \frac{m_\eta^2}{(4\pi F_\pi)^2} \ln \frac{m_\eta^2}{\mu^2} \approx 1.22 \quad (21)$$

is quite consistent with the experimental value 1.22 ± 0.01)

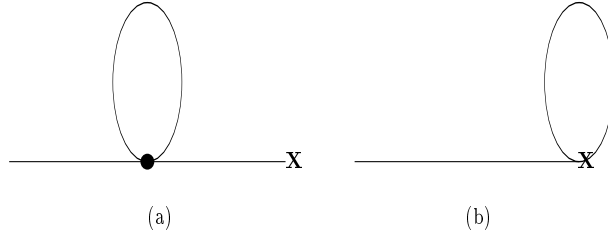


Figure 2: Loop diagrams leading to renormalization of the pseudoscalar decay constant. Here the symbol x indicates coupling to the axial current.

2.1 $\eta \rightarrow \gamma\gamma$

With this introductory material in hand we can now confront the subject of our report—that of eta decay. First consider the dominant two-photon decay mode, which to leading order arises due to the anomaly. In the analogous $\pi^0 \rightarrow \gamma\gamma$ case we find from Eq. 11

$$\text{Amp} \equiv F_{\pi\gamma\gamma}(0) \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu k_\nu \epsilon'_\alpha k'_\beta \quad \text{with} \quad F_{\pi\gamma\gamma}^{theo}(0) = \frac{N_c \alpha}{3\pi F_\pi} = 0.025 \text{ GeV}^{-1}. \quad (22)$$

General theorems guarantee that this result is not altered in higher orders of chiral symmetry breaking[20] and, using the experimental value[21]

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = (7.7 \pm 0.7) \text{ eV}, \quad (23)$$

we determine

$$F_{\pi\gamma\gamma}^{exp} = (0.0250 \pm 0.0005) \text{ GeV}^{-1} \quad (24)$$

in excellent agreement with the theoretically predicted value and eloquently confirming the value $N_c = 3$ as the number of colors. Strictly speaking, the prediction of the anomalous four-derivative Lagrangian Eq. 11 should be in terms of $\bar{F}$ rather than F_π . Indeed the difference between the value given in Eq. 22 and the strict four-derivative prediction involves terms of dimension six and is higher order in the chiral expansion. Clearly, however, our prediction for $F_{\pi\gamma\gamma}(0)$, which arises from what we shall term extended- χ PT, is in excellent agreement with experiment. Nevertheless, although very reasonable, this is *not* a firm prediction of χ PT itself.

The $\eta, \eta' \rightarrow \gamma\gamma$ couplings also arise from the anomalous component of the effective chiral Lagrangian and, in the extended χ PT approximation, have the values

$$\begin{aligned} F_{\eta\gamma\gamma}(0) &= \frac{F_{\pi\gamma\gamma}(0)}{\sqrt{3}} \left(\frac{F_\pi}{F_8} \cos \theta - 2\sqrt{2} \frac{F_\pi}{F_0} \sin \theta \right) \\ F_{\eta'\gamma\gamma}(0) &= \frac{F_{\pi\gamma\gamma}(0)}{\sqrt{3}} \left(\frac{F_\pi}{F_8} \sin \theta + 2\sqrt{2} \frac{F_\pi}{F_0} \cos \theta \right). \end{aligned} \quad (25)$$

Using the experimental numbers[24]

$$\Gamma(\eta \rightarrow \gamma\gamma) = (0.51 \pm 0.05) \text{ keV} \quad \Gamma(\eta' \rightarrow \gamma\gamma) = (4.7 \pm 0.7) \text{ keV} \quad (26)$$

we find

$$F_{\eta\gamma\gamma}(0) = 0.0249 \pm 0.0010 \text{ GeV}^{-1} \quad F_{\eta'\gamma\gamma}(0) = 0.0328 \pm 0.0024 \text{ GeV}^{-1} \quad (27)$$

In order to solve this system, we require an additional assumption since there are three unknowns— F_0, F_8, θ —but only two pieces of data—Eq. 27. The standard approach at this point has been to use the leading log prediction from one-loop chiral perturbation theory—

$$\frac{F_8}{F_\pi} = 1 - \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{\mu^2} + \frac{m_\pi^2}{(4\pi F_\pi)^2} \ln \frac{m_\pi^2}{\mu^2} \simeq 1.30 \quad \text{at } \mu \sim 1 \text{ GeV} \quad (28)$$

as input, and then to solve for F_0, θ , yielding

$$\theta \simeq -20^\circ, \quad \frac{F_0}{F_\pi} \approx 1.04 \quad (29)$$

It is intriguing that these results from two photon decay are quite compatible with those obtained from the one-loop analysis of the mass matrix—*i.e.*, $\theta \simeq -20^\circ$ and F_0/F_π consistent with the value of unity which one would obtain if the singlet state and the pion were to have the same quark wavefunction.

Closely related to the above modes are the associated Dalitz decays— $\pi, \eta, \eta' \rightarrow \gamma e^+ e^-$ —which have been studied at DESY[25]. These are traditionally parameterized in terms of a dipole slope parameter Λ_P such that

$$\frac{1}{\Gamma} \frac{d\Gamma}{ds} = \left(1 + \frac{s}{\Lambda_P^2}\right)^2 \quad \text{with } s = (p_+ + p_-)^2. \quad (30)$$

The experimentally obtained values

$$\Lambda_\pi = 0.75 \pm 0.03 \text{ GeV} \quad \Lambda_\eta = 0.84 \pm 0.06 \text{ GeV} \quad \Lambda_{\eta'} = 0.79 \pm 0.04 \text{ GeV} \quad (31)$$

are in reasonable agreement with the vector dominance predictions[26]

$$\begin{aligned} \Lambda_\pi^2 &= m_{\rho,\omega}^2 \approx (0.77 \text{ GeV})^2 \\ \Lambda_\eta^2 &= m_{\rho,\omega}^2 \left(\frac{3 \cos \theta - 6\sqrt{2} \sin \theta}{(5 - 2\xi^2) \cos \theta - (5 + \xi^2)\sqrt{2} \sin \theta} \right) \approx (0.75 \text{ GeV})^2 \\ \Lambda_{\eta'}^2 &= m_{\rho,\omega}^2 \left(\frac{3 \sin \theta + 6\sqrt{2} \cos \theta}{(5 - 2\xi^2) \sin \theta + (5 + \xi^2)\sqrt{2} \cos \theta} \right) \approx (0.83 \text{ GeV})^2 \\ \text{where } \xi^2 &= (m_{\rho,\omega}^2/m_\phi^2) \end{aligned} \quad (32)$$

These results are consistent with the observation that vector/axial dominance provides a remarkably successful representation of the values of the GL parameters $L_i^r(\mu)$ obtained empirically[27] and suggest an additional extension of our (already) extended χ PT to include an effective vector-dominated Lagrangian[28]

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{VPP}} + \mathcal{L}_{V\gamma} + \mathcal{L}_{\text{VVP}} + \mathcal{L}_{\text{PPP}\gamma} \quad (33)$$

with

$$\begin{aligned} \mathcal{L}_{\text{VPP}} &= \frac{-ig}{4} \text{Tr} V_\mu [\phi, \partial^\mu \phi] \\ \mathcal{L}_{V\gamma} &= 2eF_\pi^2 g A^\mu \left(\rho_\mu + \frac{1}{3} \omega_\mu - \frac{\sqrt{2}}{3} \phi_\mu \right) \\ \mathcal{L}_{\text{VVP}} &= -\frac{\sqrt{3}}{4} g_{VVP} \epsilon^{\mu\nu\alpha\beta} \text{Tr} (\partial_\mu V_\nu \partial_\alpha V_\beta \phi) \\ \mathcal{L}_{\text{PPP}\gamma} &= \frac{-ieN_c}{24\pi^2 F_\pi^3} \epsilon^{\mu\nu\alpha\beta} A_\mu \partial_\nu \pi^+ \partial_\alpha \pi^- \partial_\beta \pi^0 \end{aligned} \quad (34)$$

and

$$g_{VVP} = -\frac{3g^2}{8\pi^2 F_\pi}, \quad (35)$$

in order to understand the six-derivative component of the chiral expansion. Here g is given by the KSRF relation as[29]

$$g^2 = \frac{m_\rho^2}{2F_\pi^2}. \quad (36)$$

In this formalism then the amplitude for $\pi^0 \rightarrow \gamma\gamma$ is determined from the diagram in Figure 3 to be

$$F_{\pi\gamma\gamma}(0) = \frac{2e^2}{3m_\omega^2 m_\rho^2} (2eF_\pi g)^2 g_{\omega\rho\pi} = \frac{e^2}{4\pi^2 F_\pi} \quad (37)$$

as the value required by the anomaly.

With this background in hand, we can now discuss a second important decay mode of the eta—that of $\eta \rightarrow \pi\pi\gamma$ which also arises from the anomalous component of the Lagrangian.

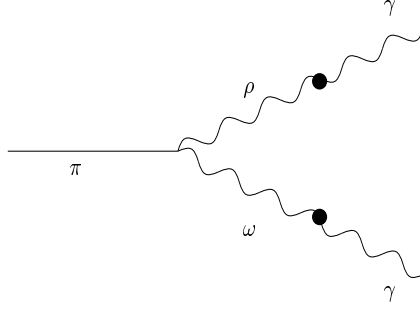


Figure 3: Vector dominance diagram responsible for $\pi^0 \rightarrow \gamma\gamma$.

2.2 $\eta \rightarrow \pi^+\pi^-\gamma$

In the previous section we performed an analysis of the QCD anomaly as manifested in the two photon decay of the pseudoscalar mesons and, by use of the one loop leading log value for F_8/F_π , were able to determine a solution for the η, η' mixing angle which is close to that found in the mass analysis together with a value for F_0/F_π which is near that which results from the assumption that the η_0 and pion have the same wavefunction. While this is somewhat satisfying, it is intriguing to inquire whether one can assess the mixing angle purely phenomenologically. We shall show below that this question can be answered in the affirmative, provided one utilizes the additional information available in the anomalous decays $\eta, \eta' \rightarrow \pi^+\pi^-\gamma$. Such processes involving a photon coupled to three pseudoscalar mesons involve the anomaly and, at zero four-momentum, are completely determined from Eq. 11. However, inclusion of higher order effects generates structure and study of such processes requires proper attention to issues of unitarity and final state interactions. Before considering η, η' decay, however, we consider first the closely related case of $\gamma \rightarrow \pi^+\pi^-\pi^0$. At zero four-momentum the anomaly requires[30]

$$\begin{aligned} \text{Amp}(3\pi - \gamma) &= A(s_{+-}, s_{+0}, s_{-0}) \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu p_{+\nu} p_{-\alpha} p_{0\beta} \\ \text{where } A(0, 0, 0) &= \frac{eN_c}{12\pi^2 F_\pi^3} = 9.7 \text{ GeV}^{-3} \quad \text{and} \quad s_{ij} = (p_i + p_j)^2 \end{aligned} \quad (38)$$

and one might suspect that vector dominance could reproduce this result directly, as in the case of $\pi^0 \rightarrow \gamma\gamma$ discussed above. However, this turns out *not*

to be the case. Rather, use of the diagram shown in Figure 4a yields

$$A(s, t, u) = \frac{eN_c}{24\pi^2 F_\pi^3} \left(\frac{m_\rho^2}{m_\rho^2 - s} + \frac{m_\rho^2}{m_\rho^2 - t} + \frac{m_\rho^2}{m_\rho^2 - u} \right) \quad (39)$$

which at zero four-momentum is 50% larger than the value given by the anomaly. The resolution of this problem is well-known and arises from the a direct $\gamma - 3\pi$ coupling—Figure 4b—given in $\mathcal{L}_{PPP\gamma}$ whose origin presumably is from unspecified high-momentum-scale processes[31] . Addition of this contribution to the $\gamma - 3\pi$ process yields an amplitude

$$A(s, t, u) = \frac{eN_c}{12\pi^2 F_\pi^3} \left[1 + \frac{1}{2} \left(\frac{s}{m_\rho^2 - s} + \frac{t}{m_\rho^2 - t} + \frac{u}{m_\rho^2 - u} \right) \right] \quad (40)$$

which has the structure required by vector dominance, but also agrees with the value required by the chiral anomaly at zero four-momentum. Inclusion of loop corrections modifies Eq. 40 slightly but does not change the general form.[32]

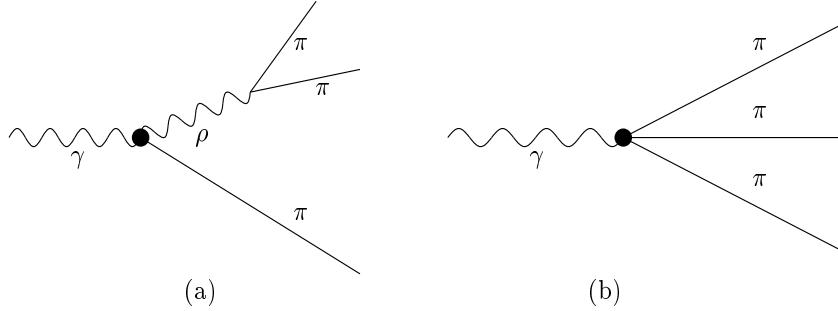


Figure 4: Vector dominance diagrams responsible for the anomalous process $\gamma\pi \rightarrow \pi\pi$.

The $\gamma - 3\pi$ reaction has been studied experimentally via pion pair production by the pion in the nuclear Coulomb field and yields a number[33]

$$A(0, 0, 0)_{\text{exp}} = 12.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3} \quad (41)$$

in apparent disagreement with Eq. 38 and suggesting the value $N_c \approx 4!$ This value was obtained, however, assuming no energy dependence of the amplitude and is reduced to

$$A(0, 0, 0)_{\text{exp}} = 11.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3} \quad (42)$$

if Eq. 40 is utilized, but is still too large. The most likely conclusion is that this an experimental problem associated with this difficult-to-measure process, but it has recently been pointed out by Ametller, Knecht, and Talavera that an important electromagnetic effect—the photon exchange diagram connecting

$\pi^0\gamma\gamma^*$ and $\pi^+\pi^-\gamma$ vertices—can reduce this number by another 1×10^{-3} or so[36]. In any case a new high-precision experiment is clearly called for, and this has been accomplished at JLab using the CLAS detector and the reaction $\gamma p \rightarrow \pi^+\pi^-n$. Such a measurement has been also been proposed at DaΦne[34].

The JLab experiment is currently being analyzed and this high statistics measurement will require an equally careful theoretical analysis in order to produce the desired extrapolation from the rho resonance region, where most of the data has been obtained, to the zero four-momentum point where the anomaly stricture obtains. This issue has been approached in a number of authors:

- i) Holstein has used a simple matching of the one loop chiral correction to the rho dominance form[37].
- ii) Truong has utilized a unitarization based upon the Omnes-Muskhelishvili method[38].
- iii) Hannah unitarizes the amplitude using the inverse amplitude procedure[39].

Regardless of the method used, the results are similar and are somewhat robust. The resulting value of the anomaly obtained in a preliminary analysis of the CLAS measurement are consistent, within a significant uncertainty, with the expected number—three. However, a definitive value awaits further analysis.

Having warmed up on the $\gamma - 3\pi$ process, it is now straightforward to construct the analogous $\eta \rightarrow \pi^+\pi^-\gamma$ amplitude. Using the extended χ PT assumption we find[41]

$$\text{Amp}(\eta \rightarrow \pi^+\pi^-\gamma) = B(s_{+-}, s_{+\gamma}, s_{-\gamma})\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu^*p_+\nu p_- \alpha k_{\gamma\beta} \quad (43)$$

with the anomaly stricture yielding

$$B_\eta(0, 0, 0) = \frac{eN_c}{12\sqrt{3}\pi^2 F_\pi^3} \left(\frac{F_\pi}{F_8} \cos \theta - \sqrt{2} \frac{F_\pi}{F_0} \sin \theta \right) \quad (44)$$

However, the physical region for the decay— $4m_\pi^2 \leq s_{\pi\pi} \leq m_\eta^2$ —is far from the zero-momentum point which is constrained by the anomaly. One indication of this fact is that the decay rate obtained via neglect of momentum dependence— $\Gamma_{\eta \rightarrow \pi\pi\gamma}^{(0)} = 35.7$ eV—is significantly different from the experimental value— $\Gamma_{\eta \rightarrow \pi\pi\gamma}^{expt} = 64 \pm 6$ eV. For the η' channel the situation is, of course, much worse. The experimental value— $\Gamma_{\eta' \rightarrow \pi\pi\gamma}^{expt} = 61 \pm 5$ keV—is a factor of twenty larger than the value $\Gamma_{\eta' \rightarrow \pi^+\pi^-\gamma}^{(0)} = 3$ KeV obtained via use of the simple anomaly prediction

$$B_{\eta'}(0, 0, 0) = \frac{eN_c}{12\sqrt{3}\pi^2 F_\pi^3} \left(\frac{F_\pi}{F_8} \sin \theta + \sqrt{2} \frac{F_\pi}{F_0} \cos \theta \right) \quad (45)$$

Thus proper inclusion of momentum dependence is essential. The $\eta \rightarrow \pi^+\pi^-\gamma$ spectrum was measured in the experiment of Gormley et al.[41] and was found

to be approximately fit in terms of a pure (width-modified) ρ -dominated matrix element. This result is *not* in agreement, however, with the simple vector dominance prediction—*cf.* Eq. 50—which would require

$$|B(s, t, u)|_{\text{theo}}^2 \sim 1 + 3 \frac{s}{m_\rho^2} + \dots \quad (46)$$

and corresponds instead to

$$|B(s, t, u)|_{\text{exp}}^2 \sim 1 + 2 \frac{s}{m_\rho^2} + \dots \quad (47)$$

Thus a careful look at the unitarization procedure is called for.

We begin by noting that a one-loop chiral perturbation theory calculation gives

$$\begin{aligned} B_\eta^{1\text{-loop}}(s, s_{\pi\pi}) &= B_\eta(0, 0) \left[1 + \frac{1}{32\pi^2 F_\pi^2} \left((-4m_\pi^2 + \frac{1}{3}s_{\pi\pi}) \ln \frac{m_\pi^2}{m_\rho^2} \right. \right. \\ &\quad \left. \left. + \frac{4}{3}F(s_{\pi\pi}) - \frac{20}{3}m_\pi^2 + \frac{3}{2m_\rho^2}s_{\pi\pi} \right] \end{aligned} \quad (48)$$

where

$$F(s) = \begin{cases} (1 - \frac{s}{4m_\pi^2}) \sqrt{\frac{s-4m_\pi^2}{s}} \ln \frac{1 + \sqrt{\frac{s-4m_\pi^2}{s}}}{-1 + \sqrt{\frac{s-4m_\pi^2}{s}}} - 2 & s > 4m_\pi^2 \\ 2(1 - \frac{s}{4m_\pi^2}) \sqrt{\frac{4m_\pi^2-s}{s}} \tan^{-1} \sqrt{\frac{s}{4m_\pi^2-s}} - 2 & s \leq 4m_\pi^2 \end{cases} \quad (49)$$

while the vector dominance picture (*cf.* Figure 5) yields

$$B_\eta(s_{\pi\pi}) = B_\eta(0, 0, 0) \left(1 + \frac{3}{2} \frac{s_{\pi\pi}}{m_\rho^2 - s_{\pi\pi}} \right) \quad (50)$$

Certainly, in order to treat the decay of the η' , one *must* include unitarity effects via final state interactions. One very obvious approach is simply to include the (energy-dependent) width of the rho-meson in the propagator in the vector-dominance form Eq. 44 via

$$\frac{s_{\pi\pi}}{m_\rho^2 - s_{\pi\pi}} \rightarrow \frac{s_{\pi\pi}}{m_\rho^2 - s_{\pi\pi} - im_\rho \Gamma_\rho(s_{\pi\pi})} \quad (51)$$

This use of vector width-modified vector dominance already makes an important difference from the simple anomaly—tree level—results (especially in the case of the η'), changing the predicted decay widths from the values 35 eV and 3 KeV quoted above to the much more realistic numbers

$$\Gamma_{\eta-\pi\pi\gamma}^{\text{theo-VM}} = 62.3 \text{ eV}, \quad \Gamma_{\eta'-\pi\pi\gamma}^{\text{theo-VM}} = 67.5 \text{ KeV} \quad (52)$$

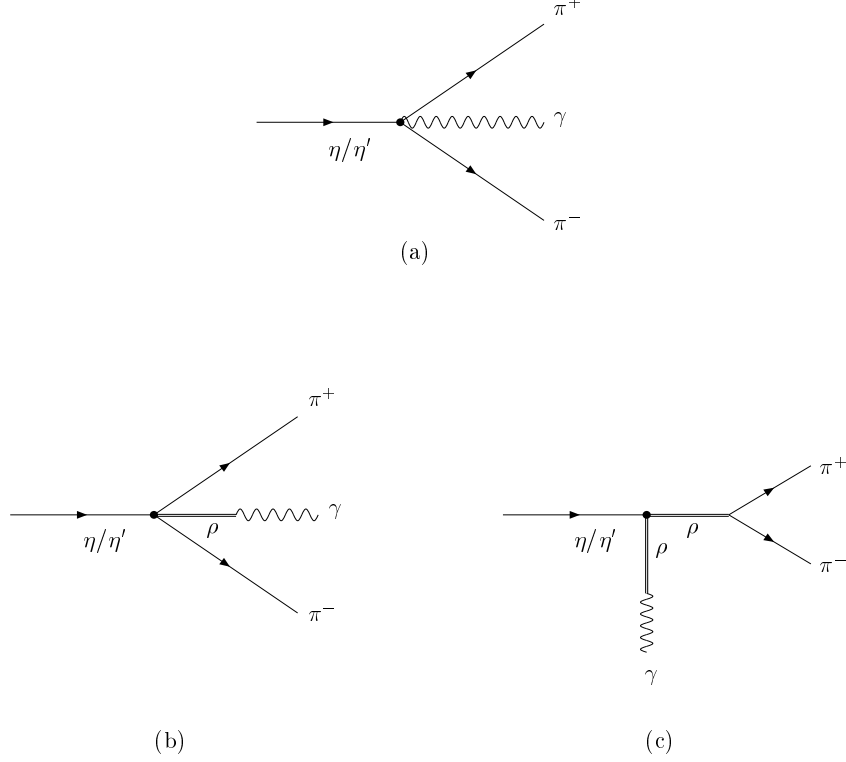


Figure 5: Shown are contact (a) and VMD (b,c) contributions to $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$ decay.

if the parameters

$$F_8/F_\pi = 1.3, \quad F_0/F_\pi = 1.04, \quad \theta = -20^\circ \quad (53)$$

are employed. However, this procedure does not match onto the one-loop chiral form in the low energy limit.

In order to determine a form for the final state interactions which matches onto both the one-loop chiral correction *and* to the vector dominance result in the appropriate limits, we postulate an N/D structure

$$B_{\eta-\pi\pi\gamma}(s, s_{\pi\pi}) = B_{\eta-\pi\pi\gamma}(0, 0) \left[1 - c + c \frac{1 + as_{\pi\pi}}{D_1(s_{\pi\pi})} \right] \quad (54)$$

where $D_1(s)$ is the Omnes function and is defined in terms of the p-wave $\pi\pi$ phase shifts via

$$D_1(s) = \exp \left(-\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds' \delta(s')}{s'(s' - s - i\epsilon)} \right) \quad (55)$$

and a, c are free parameters to be determined. In order to reproduce the coefficient of the $F(s_{\pi\pi})$ function, which contains the rho width, we require $c = 1$. On the other hand, matching onto the VMD result at $\mathcal{O}(p^6)$ can be achieved by the choice $a = 1/2m_\rho^2$. Thus in the case of the η the form is completely determined. Since the η' spectrum is closely related and is dominated by the presence of the rho we shall postulate an identical form for the η' case. Using these forms we can then calculate the decay widths assuming the theoretical values for the anomaly. Using the parameters given in Eq. 53 one finds, for example,

$$\begin{aligned} i) D_1^{\text{exp}}(s) & \quad \Gamma_{\eta-\pi\pi\gamma} = 65.7 \text{ eV}, \quad \Gamma_{\eta'-\pi\pi\gamma} = 66.2 \text{ KeV} \\ ii) D_1^{\text{anal}}(s) & \quad \Gamma_{\eta-\pi\pi\gamma} = 69.7 \text{ eV}, \quad \Gamma_{\eta'-\pi\pi\gamma} = 77.8 \text{ KeV} \end{aligned} \quad (56)$$

There is a tendency then for the numbers obtained via the analytic form of the Omnes function to be somewhat too high.

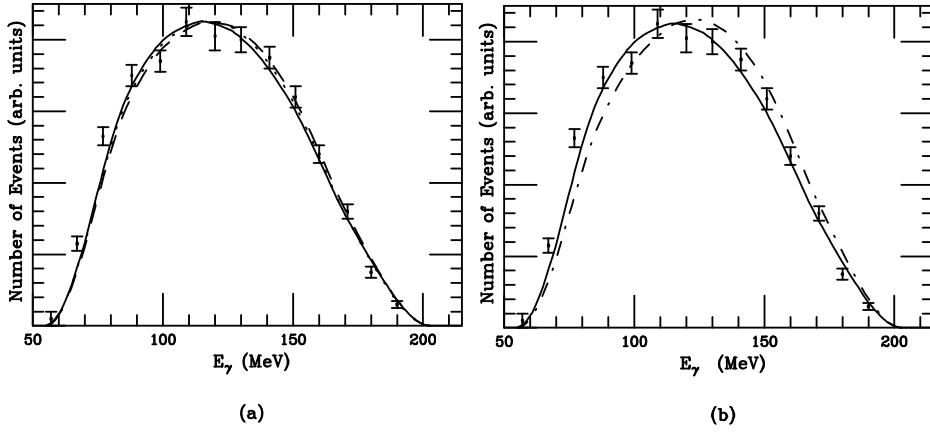


Figure 6: Shown is the photon spectrum in $\eta \rightarrow \pi^+ \pi^- \gamma$ from Gormley et al.[41] as well as various theoretical fits. In the Figure 6a, the dashed line represents the (width-modified) VMD model. The (hardly visible) dotted line and the solid line represent the final state interaction ansatz Eq.54 with use of the analytic and experimental version of the Omnes function respectively. Figure 6b shows the experimental Omnes function result (solid line) compared with the one-loop result (dotdash line).

We can also compare the predicted spectra with the corresponding experimentally determined values. As shown in Figure 6, we observe that the experimental spectra are well fit in the η case in terms of both the N/D or the VMD forms, but that the one-loop chiral expression does not provide an adequate representation of the data. In the case of the corresponding η' decay the results are shown in Figure 7, wherein we observe that either the unitarized VMD or the use of N/D_1^{exp} provides a reasonable fit to the data (we get $\chi^2/\text{dof}=32/17$ and $20/17$, respectively), while the use of the analytic form for the Omnes

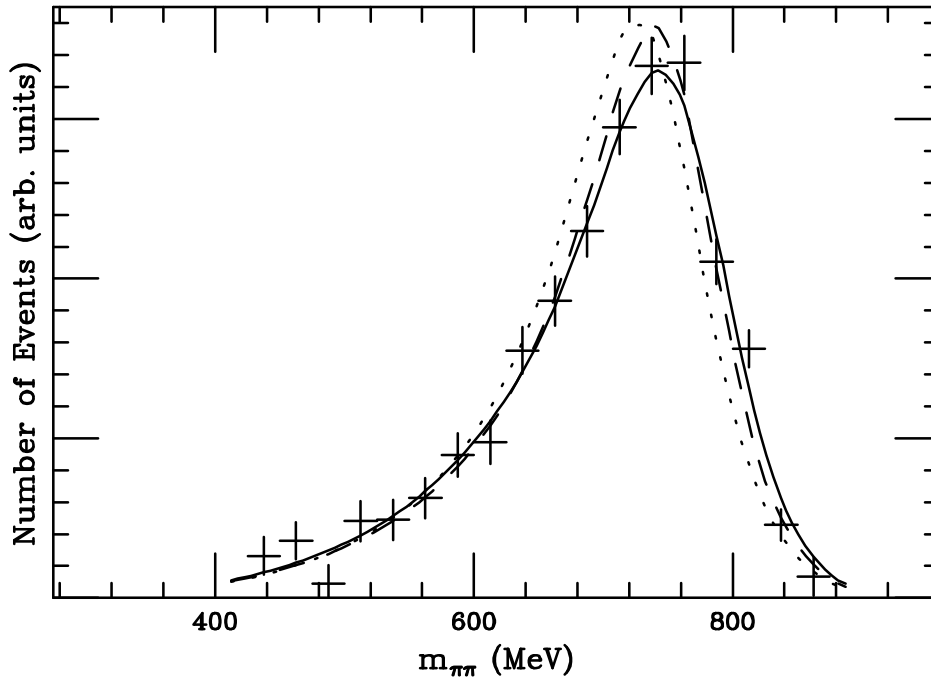


Figure 7: Shown is the photon spectrum in $\eta' \rightarrow \pi^+\pi^-\gamma$ from Abele et al.[42] as well as various theoretical fits. As in Figure 6a, the dashed line represents the (width-modified) VMD model. The dotted and solid lines represent the final state interaction ansatz Eq.54 with use of the analytic and experimental version of the Omnes function, respectively. Here the curves have been normalized to the same number of events.

function yields a predicted spectrum ($\chi^2/\text{dof}=104/17$) which is slightly too low on the high energy end. However, for both η and η' we see that our simple ansatz—Eq.54—provides a very satisfactory representation of the decay spectrum. Our conclusion in the previous section was that if the mixing angle and pseudoscalar coupling constants were assigned values consistent with present theoretical and experimental leanings, then the predicted widths and spectra of both $\eta, \eta \rightarrow \pi^+\pi^-\gamma$ are basically consistent with experimental values. Our goal in this section is to go the other way, however. That is, using the assumed N/D forms for the decay amplitude, and treating the pseudoscalar decay constants F_8, F_0 as well as the $\eta - \eta'$ mixing angle θ as free parameters, we wish to inquire as to how well they can be constrained purely from the experimental data on $\eta, \eta' \rightarrow \gamma\gamma$ and $\eta, \eta' \rightarrow \pi^+\pi^-\gamma$ decays, with reasonable assumptions made about the final state interaction effects in these two channels.

On theoretical grounds, one is somewhat more confident about the extraction of the threshold amplitude in the case of the lower energy $\eta \rightarrow \pi^+\pi^-\gamma$ system. Indeed, in this case the physical region extends only slightly into the tail of

	F_8/F_π	F_0/F_π	θ
VMD	1.28 ± 0.24	1.07 ± 0.48	$-20.3^\circ \pm 9.0^\circ$
N/D_1^{anal}	1.49 ± 0.29	1.02 ± 0.42	$-22.6^\circ \pm 9.6^\circ$
N/D_1^{exp}	1.37 ± 0.26	1.02 ± 0.45	$-21.2^\circ \pm 9.3^\circ$

Table 3: Values of the renormalized pseudoscalar coupling constants and the $\eta - \eta'$ mixing angle using the $\eta, \eta' - \gamma\gamma$ and $\eta - \pi\pi\gamma$ amplitudes in a three parameter fit.

	F_8/F_π	F_0/F_π	θ
VMD	1.28 ± 0.20	1.07 ± 0.04	$-20.8^\circ \pm 3.2^\circ$
N/D_1^{anal}	1.48 ± 0.24	1.09 ± 0.03	$-24.0^\circ \pm 3.0^\circ$
N/D_1^{exp}	1.38 ± 0.22	1.06 ± 0.03	$-22.0^\circ \pm 3.3^\circ$

Table 4: Values of the renormalized pseudoscalar coupling constants and of the $\eta - \eta'$ mixing angle obtained from a maximum likelihood analysis using the $\eta, \eta' - \gamma\gamma$ and $\eta, \eta' - \pi\pi\gamma$ amplitudes.

the rho unlike the related η' decay wherein the spectrum extends completely over the resonance so that there exists considerable sensitivity to details of the shape. Thus a first approach might be to utilize only the two-photon decays together with the $\eta \rightarrow \pi^+\pi^-\gamma$ width in order to determine the three desired parameters. In this fashion one finds the results shown in Table 1. We observe that the results are in agreement, both with each other and with the chiral symmetry expectations— $F_8/F_\pi \sim 1.3$, $F_0/F_\pi \sim 1$, and $\theta \sim -20^\circ$. However, the uncertainties obtained in this way are uncomfortably high.

In order to ameliorate this problem, we have also done a maximum likelihood fit including the $\eta' - \pi\pi\gamma$ decay rate, yielding the results shown in Table 2. We observe that the central values stay fixed but that the error bars are somewhat reduced. The conclusions are the same, however—substantial renormalization for $F_8 \sim 1.3F_\pi$, almost none for $F_0 \sim F_\pi$, and a mixing angle $\theta \sim -20^\circ$. These numbers appear nearly invariant, regardless of the approach.

An interesting aside here is the recent observation by Bär and Wiese that the $\pi^0 \rightarrow \gamma\gamma$ reaction alone does not verify the three color hypothesis—in a careful analysis the N_c -dependence of the Wess-Zumino-Witten term is completely canceled by the N_c -dependent part of a Goldstone-Wilczek term, and that it is only the $\eta \rightarrow \pi^+\pi^-\gamma$ measurement which truly confirms the result that $N_c = 3$ [43].

Having above confirmed the basic correctness of the predictions of the anomaly (and thereby of this important cornerstone of QCD) we move now to the important three pion decay of the eta, which occurs independent of the anomaly and which rather probes the conventional two- and four-derivative piece of the chiral Lagrangian.

2.3 $\eta \rightarrow \pi\pi\pi$

The decay of the isoscalar eta to the predominantly I=1 final state of the three pion system occurs primarily on account of the d-u quark mass difference[44], and the result arising from lowest order chiral perturbation theory is well-known[45]

$$\text{Amp}(\eta_8 \rightarrow \pi^a \pi^b \pi^c) = \delta^{ab} \delta^{c3} C(s_{ab}, s_{ac}, s_{bc}) + \text{Permutations} \quad (57)$$

where

$$C(s, t, u) = -\frac{B_0(m_d - m_u)}{3\sqrt{3}F_\pi^2} \left[1 + \frac{3(s - s_0)}{m_\eta^2 - m_\pi^2} \right] \quad (58)$$

and we have defined

$$s_{ab} = (p_a + p_b)^2 \quad \text{and} \quad s_0 = \frac{1}{3}(m_\eta^2 + m_{\pi^+}^2 + m_{\pi^-}^2 + m_{\pi^0}^2). \quad (59)$$

Equivalently, we can write the prefactor of Eq. 58 in a form which respects the reparameterization invariance of Kaplan and Manohar[46]

$$\frac{B_0(m_d - m_u)}{3\sqrt{3}F_\pi^2} = -\frac{1}{Q^2} \frac{m_K^2}{m_\pi^2} (m_K^2 - m_\pi^2) \quad (60)$$

where

$$Q^2 = \frac{m_s - \hat{m}}{m_d - m_u} \frac{m_s + \hat{m}}{m_d + m_u} \quad (61)$$

and $\hat{m} = \frac{1}{2}(m_d + m_u)$ is the average u, d quark mass.

Thus the decay $\eta \rightarrow 3\pi$ can be used in order to determine the quantity Q^2 . Alternatively, if Q^2 is given from some other process then the $\eta \rightarrow 3\pi$ amplitude is completely determined. The standard approach to such a determination is to utilize the pseudoscalar meson masses via the relation

$$Q^2 = \frac{m_K^2}{m_\pi^2} \frac{m_K^2 - m_\pi^2}{(m_{K^0}^2 - m_{K^+}^2)_{\text{QCD}}} \quad (62)$$

where $(m_{K^0}^2 - m_{K^+}^2)_{\text{QCD}}$ is the nonelectromagnetic component of the K^0, K^+ mass difference—

$$(m_{K^0}^2 - m_{K^+}^2)_{\text{QCD}} = (m_{K^0}^2 - m_{K^+}^2)_{\text{expt}} - (m_{K^0}^2 - m_{K^+}^2)_{\text{em}} \quad (63)$$

In order to evaluate the right hand side of Eq. 63 one generally uses Dashen's theorem, which guarantees the identity of the electromagnetic piece of the kaon and pion electromagnetic mass shifts in the chiral symmetric limit[47]

$$(m_{\pi^+}^2 - m_{\pi^0}^2) = (m_{K^+}^2 - m_{K^0}^2)_{\text{EM}}. \quad (64)$$

This simple assumption gives then

$$Q_{\text{Dashen}}^2 = \frac{m_K^2}{m_\pi^2} \frac{m_K^2 - m_\pi^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - m_{\pi^0}^2} = 24.1 \quad (65)$$

and results in a prediction

$$\Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0) = 66 \text{ eV} \quad (66)$$

in strong contradiction to the experimental result

$$\Gamma^{exp}(\eta \rightarrow \pi^+ \pi^- \pi^0) = 281 \pm 28 \text{ eV}. \quad (67)$$

At first sight this would appear to be a rather strong and irreparable violation of a lowest order chiral prediction and therefore not salvagable by the expected $\mathcal{O}(m_\eta^2/(4\pi F_\pi^2)^2) \sim 30\%$ corrections from higher order effects. However, this is not at all the case. In fact the one-loop and counterterm contributions were calculated by Gasser and Leutwyler and were found to enhance the lowest order prediction by a factor 2.6, yielding[48]

$$\Gamma^{theo}(\eta \rightarrow \pi^+ \pi^- \pi^0) \approx 167 \pm 50 \text{ eV}, \quad (68)$$

which is a significant improvement, but still somewhat too low. The origin of such a large correction lies primarily with η_8, η_0 mixing which generates a factor $(\cos \theta - \sqrt{2} \sin \theta)^2 \sim 2$ leaving the expected 30% corrections due to conventional higher order loop and counterterm contributions. However, recent work has indicated that Eq. 68 is probably a considerable underestimate due a significant violation of Dashen's theorem. Indeed, the Dashen requirement was derived in the limit of chiral symmetry— $m_\pi^2 = m_K^2 = 0$ —and plausible estimates of chiral breaking effects have yielded the estimate[49]

$$Q_{\chi\text{-broken}}^2 \approx 0.8 Q_{\text{Dashen}}^2, \quad i.e. \quad Q_{\chi\text{-broken}} \sim 21.7 \quad (69)$$

which corresponds to an additional $\sim 40\%$ enhancement of the chiral calculation Eq. 68, *i.e.* $\Gamma^{theo}(\eta \rightarrow \pi^+ \pi^- \pi^0) \sim 240 \text{ eV}$, and puts the result in the right ballpark. There are at least two possible sources for the existence of any remaining discrepancy. One is the fact that the estimate for the size of Dashen's theorem violation is just that—an estimate. It is possible that the size of the violation is more significant than that given in Eq. 69, leading to an even larger value for $\Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0)$. A second possibility is that the simple one loop estimate given in ref. 4 is not sufficient to include the full impact of final state interaction effects. This has been demonstrated in other processes where the $I=0$ S-wave $\pi - \pi$ plays an important role, as it does here[50]. Indeed the closely related $K \rightarrow 3\pi$ reaction is one such case[51].

In order to decide which—if either—possibility obtains it is necessary to make careful spectral shape measurements in addition to simple lifetime numbers. Also it is necessary to confront such results with precise theoretical calculations. Phenomenologically, we expand the decay amplitude about the center of the Dalitz plot as

$$C(s, t, u) \equiv \alpha [1 + \beta y + \gamma y^2 + \delta x^2 + \dots] \quad (70)$$

where

$$y = \frac{3(s - s_0)}{2m_\eta \Delta_\eta} \quad \text{and} \quad x = \frac{\sqrt{3}(t - u)}{2m_\eta \Delta_\eta}. \quad (71)$$

where $\Delta_\eta = m_\eta - 2m_{\pi^\pm} - m_\pi^0$ is the Q-value. These parameters have been determined experimentally to be

$$\begin{aligned} \text{Layter et al. [52]} : \beta &= 0.54 \pm 0.007 & \gamma &= 0.017 \pm 0.014 & \delta &= 0.023 \pm 0.016 \\ \text{Gormley et al. [53]} : \beta &= 0.585 \pm 0.010 & \gamma &= 0.105 \pm 0.015 & \delta &= 0.03 \pm 0.02 \\ \text{Amsler et al. [54]} : \beta &= 0.470 \pm 0.075 & \gamma &= 0.055 \pm 0.135 & & \end{aligned} \quad (72)$$

to be compared to the one-loop chiral prediction

$$\beta = 0.665 \quad \gamma = 0.21 \quad \delta = 0.04 \quad (73)$$

Clearly there is general (though certainly not excellent) agreement, suggesting the importance of higher order scattering contributions.

These have been examined by two Swiss collaborations using dispersion relation treatments in order to address the problem of higher order three-body scattering effects. The calculation of Anisovich and Leutwyler quotes only the integrated decay rate which is in agreement with experiment if the value $Q = 22.7 \pm 0.8$ is chosen[55]—consistent with the Dashen theorem violation calculated in [49]. Similarly the integrated decay rate found in the Khuri-Treiman calculation by Kambor, Wiesendanger, and Wyler agrees with the experimental rate if the value $Q = 22.4 \pm 0.9$ is used[56]. However, these authors also quote values for the spectral shape

$$\begin{aligned} \beta &= 0.58 & \gamma &= 0.12 & \delta &= 0.045 \\ \beta &= 0.58 & \gamma &= 0.115 & \delta &= 0.05 \end{aligned} \quad (74)$$

where the two sets of numbers correspond to different ways of determining the experimental input. Obviously the slope parameter β is in general agreement with experiment. However, the situation is more complex for the quadratic components. In particular the calculated values for γ, δ are in good agreement with the measurement of Gormley et al. (or Amsler et al.) but *not* with that of Layter et al. However, experimental uncertainties are significant and a new high statistics determination of the spectral shape is needed.

Additional information is available by studying the neutral decay mode $\eta \rightarrow 3\pi^0$, for which Bose symmetry determines the decay amplitude to be

$$\text{Amp}(\eta \rightarrow 3\pi^0) = N^{000} |1 + \epsilon(x^2 + y^2)_{\text{sym}}|^2 \quad (75)$$

where

$$(x^2 + y^2)_{\text{sym}} = \frac{1}{3} \sum_{i=1}^3 (x_i^2 + y_i^2) = 2y_{\text{sym}}^2 \quad (76)$$

Here the dispersive calculation by Kambor et al. predicts

$$\epsilon = -0.028, \quad \text{or} \quad \epsilon = -0.014 \quad (77)$$

depending on how the experimental input is handled. On the experimental side the only measurement of the energy dependence until recently had been of

limited accuracy

$$\begin{aligned}
\text{Amsler et al. [58]} : \epsilon &= -0.044 \pm 0.046 \\
\text{Baglin et al. [59]} : \epsilon &= -0.64 \pm 0.74 \\
\text{Abele et al. [60]} : \epsilon &= -0.104 \pm 0.04
\end{aligned} \tag{78}$$

which is consistent with (but with large experimental uncertainty) the dispersive calculation. However, recently a new result of unprecedented precision has been announced from the Crystal Ball group at BNL[57]

$$\epsilon = -0.062 \pm 0.006 \pm 0.004 \tag{79}$$

Clearly this number is significantly larger than expected from the Khuri-Treiman calculation, suggesting that new dynamical input is involved. This is not unexpected. Indeed the calculation of Kambor et al. utilized the effective chiral Lagrangian at $\mathcal{O}(p^4)$ is input. Clearly from the agreement with experiment this is the main effect, but one also expects contributions from pieces of the chiral Lagrangian of $\mathcal{O}(p^6)$ such as

$$\mathcal{L}^{(6)} \sim \frac{F_\pi^2}{\Lambda_\chi^2} \text{tr}[(\chi U^\dagger + U \chi^\dagger) D^\mu U D_\mu U^\dagger] \text{tr}(D^\nu U D_\nu U^\dagger) \tag{80}$$

where $\chi = 2B_0 m$, with B_0 being a constant and m is the quark mass matrix, $\Lambda_\chi \sim 4\pi F_\pi \sim 1$ GeV is the chiral scale. The coefficients of such terms are unconstrained by the strictures of chiral invariance and are experimentally undetermined at present, since they arise at two-loop order. Nevertheless, their presence can lead to peices of the decay amplitude of the form

$$\begin{aligned}
A &\sim \mathcal{A}_1 k \cdot q_c a_a \cdot q_b + \mathcal{A}_2 (k \cdot a_a q_b \cdot q_c + k \cdot q_b q_a \cdot q_c) \\
&\simeq m_\eta^4 \left(\frac{\mathcal{A}_1}{18} + \frac{\mathcal{A}_2}{9} \right) \left[1 - \frac{Q_\eta^2}{m_\eta^2} (x^2 + y^2) \right] \\
&+ \frac{1}{12} m_\eta^4 (\mathcal{A}_1 - \mathcal{A}_2) \left[\frac{2Q_\eta}{3m_\eta} y - \frac{8}{27} \frac{Q_\eta^2}{m_\eta^2} (y^2 - x^2) \right]
\end{aligned} \tag{81}$$

If such a dynamical component is present then its size should be set by chiral scaling arguments

$$\mathcal{A}_I \sim \frac{m_d - m_u}{\Lambda_\chi^2 F_\pi^2}$$

and an isospin relation

$$\gamma^{+-0}(\text{dyn}) + \delta^{+-0}(\text{dyn}) = \epsilon^{000}(\text{dyn}) \tag{82}$$

must exist between the quadratic parameters for the charged and neutral channels. Here the symbol *dyn* indicates the dynamical (*i.e.*, non-rescattering component of the coefficient in question and is found by subtracting the theoretical

value obtained from the Khuri-Treiman calculation from the experimental quantity. In this way we find, using the Gormley numbers for experimental input,

$$\gamma^{+-0}(\text{dyn}) = -0.025 \pm 0.015 \quad \delta^{+-0}(\text{dyn}) = -0.02 \pm 0.02 \quad (83)$$

and

$$\epsilon^{000}(\text{dyn}) = -0.034 \pm 0.007 \quad \text{or} \quad -0.048 \pm 0.007 \quad (84)$$

depending on the dynamical input chosen. The comparison

$$\gamma^{+-0}(\text{dyn}) + \delta^{+-0}(\text{dyn}) = -0.045 \pm 0.03 \quad (85)$$

vs.

$$\epsilon^{000}(\text{dyn}) = -0.034 \pm 0.007 \quad \text{or} \quad -0.045 \pm 0.007 \quad (86)$$

is obviously satisfactory within errors but again cries out for a high precision measurement of the $\eta \rightarrow \pi^+\pi^-\pi^0$ spectrum, such as would be possible using WASA.

Of course, there is one additional test which we can use. Since according to isotopic spin invariance the $3\pi^0$ amplitude at the center of the Dalitz plot must be a factor of three larger than the corresponding $\pi^+\pi^-\pi^0$ number, the total decay rates should differ by the factor

$$\frac{\Gamma^{(0)}(000)}{\Gamma^{(0)}(+ - 0)} = \frac{3^2}{3!} = 1.5 \quad (87)$$

When rescattering corrections are included, the prediction becomes

$$\frac{\Gamma(000)}{\Gamma(+ - 0)} = \begin{cases} 1.43 & \text{one loop} \\ 1.41 \pm 0.03 & \text{Khuri-Treiman} \end{cases} \quad (88)$$

Both numbers are consistent with the value quoted by the Particle Data Group[62]

$$\left(\frac{\Gamma(000)}{\Gamma(+ - 0)} \right)^{\text{exp}} = 1.404 \pm 0.034 \quad (89)$$

and are in good agreement with the recent Crystal Ball measurement[54]

$$\left(\frac{\Gamma(000)}{\Gamma(+ - 0)} \right)^{\text{exp}} = 1.44 \pm 0.09 \pm 0.01 \quad (90)$$

Clearly there is plenty of challenge in the three-pion sector for an eta facility, but there is also interest in examining the remaining radiative modes $\eta \rightarrow \pi^0\gamma\gamma, 3\pi\gamma$.

2.4 $\eta \rightarrow \pi^0\gamma\gamma$

For the decay $\eta \rightarrow \pi^0\gamma\gamma$ chiral symmetry does not play a important role, but vector dominance *does*. This can be seen from the feature that there exists

no contribution at all to this process from the tree level two-derivative Lagrangian. Rather the lowest order chiral contribution arises at one-loop level— $\eta \rightarrow 3\pi, K\bar{K}\pi \rightarrow \pi^0\gamma\gamma$. However, the 3π intermediate intermediate state is suppressed by the factor $m_d - m_u$ while the contribution from $K\bar{K}\pi$ is suppressed kinematically. To see this, we define the general decay amplitude

$$\begin{aligned} \text{Amp}(\eta \rightarrow \pi^0\gamma\gamma) &= D(s, t, u) [\epsilon_1 \cdot \epsilon_2 q_1 \cdot q_2 - \epsilon_1 \cdot q_2 \epsilon_2 \cdot q_1] \\ &- E(s, t, u) [-\epsilon_1 \cdot \epsilon_2 p \cdot q_1 p \cdot q_2 - \epsilon_1 \cdot p \epsilon_2 \cdot p q_1 \cdot q_2 \\ &+ \epsilon_1 \cdot q_2 \epsilon_2 \cdot p p \cdot q_1 + \epsilon_1 \cdot p \epsilon_2 \cdot q_1 p \cdot q_2] \end{aligned} \quad (91)$$

Then from the one-pion-loop contributions from $\mathcal{L}^{(2)}$ we find

$$D_\pi(s, t, u) = \frac{\sqrt{2}\alpha}{\pi} T(s) F(s, m_\pi^2), \quad E_\pi(s, t, u) = 0 \quad (92)$$

where $T(s)$ is the lowest order $\eta \rightarrow 3\pi$ amplitude given in Eq. 58 and

$$sF(s, m_\pi^2) = 1 + \frac{4m_\pi^2}{s} \ln^2 \left(\frac{\beta(s) + 1}{\beta(s) - 1} \right) \quad \text{with} \quad \beta(s) = \sqrt{\frac{s - 4m_\pi^2}{s}} \quad (93)$$

with a similar expression obtaining for the kaon loop contribution. Calculation of the associated rate yields[63]

$$\Gamma_{\text{loop}}(\eta \rightarrow \pi^0\gamma\gamma) \approx 4 \times 10^{-3} \text{ eV} \quad \text{vs.} \quad \Gamma_{\text{exp}}(\eta \rightarrow \pi^0\gamma\gamma) = 0.84 \pm 0.18 \text{ eV}. \quad (94)$$

Thus the one-loop chiral contribution plays a very minor role. Consider, however, the vector dominance diagram shown in Figure 8, for which

$$\begin{aligned} D(s, t, u) &= \frac{2\sqrt{3}}{9} g_{\omega\rho\pi}^2 \left(\frac{2eF_\pi^2 g}{m_V^2} \right)^2 \left[\frac{p \cdot q_2 - m_\eta^2}{m_V^2 - t} + \frac{p \cdot q_1 - m_\eta^2}{m_V^2 - u} \right] \\ &\times \left(\frac{F_\pi}{F_8} \cos \theta - \sqrt{2} \frac{F_\pi}{F_0} \sin \theta \right) \\ E(s, t, u) &= -\frac{2\sqrt{3}}{9} g_{\omega\rho\pi}^2 \left(\frac{2eF_\pi^2 g}{m_V^2} \right) \left[\frac{1}{m_V^2 - t} + \frac{1}{m_V^2 - u} \right] \\ &\times \left(\frac{F_\pi}{F_8} \cos \theta - \sqrt{2} \frac{F_\pi}{F_0} \sin \theta \right) \end{aligned} \quad (95)$$

and yielding for the decay rate

$$\Gamma_{\text{VD}}(\eta \rightarrow \pi^0\gamma\gamma) = 0.31 \text{ eV} \quad (96)$$

Inclusion of other higher order effects such as the contribution from a pair of anomalous terms— $\pi\pi\pi\gamma$ and $\eta\pi\pi\gamma$ —coupled via a pion loop increases this estimate to about half the experimental result, but a considerable discrepancy remains and should be the focus of future experimental as well as theoretical work.

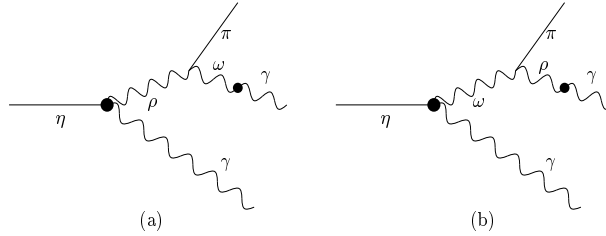


Figure 8: Vector dominance diagram responsible for $\eta^0 \rightarrow \pi^0 \gamma \gamma$.

A new number from the Brookhaven experiment was first announced at this meeting by Nefkens

$$\Gamma^{exp}(\eta \rightarrow \pi \gamma \gamma) = (0.38 \pm 0.11) \text{ eV} \quad (97)$$

is about a factor of two smaller than the Particle Data Group value. The problem with the previous measurements is probably associated with eliminating the background from other much more probably neutral modes such as $\eta \rightarrow 3\pi^0$ and confirmation using WASA would clearly be welcome. In this regard, spectral shape measurements could be helpful, although this will be difficult, since this is a low branching ratio— $\sim 7 \times 10^{-4}$ —experiment.

2.5 $\eta \rightarrow \pi \pi \pi \gamma$

The final mode which we shall mention in this report is $\eta \rightarrow 3\pi \gamma$, for which on the experimental side there exists at present only an upper bound[62]

$$\frac{\Gamma(\eta \rightarrow 3\pi \gamma)}{\Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0)}|_{\text{exp}} < 0.0024. \quad (98)$$

Ordinarily the dominant component of a radiative mode such as this is due to the the inner bremsstrahlung process, for which the matrix element is

$$\begin{aligned} \text{Amp}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) &\simeq \text{Amp}(\eta \rightarrow \pi^+ \pi^- \pi^0) \\ &\times i e \epsilon^\mu \left[\frac{(2p_+ + k)_\mu}{(p_+ + k)^2 - m_\pi^2} - \frac{(2p_- + k)_\mu}{(p_- + k)^2 - m_\pi^2} \right] \end{aligned} \quad (99)$$

and it is experimentally difficult to distinguish any direct photon emission. However, an exception occurs when the non-radiative process is suppressed in some fashion, such as occurs, *e.g.*, in the cases of $\pi^+ \rightarrow e^+ \nu_e \gamma$ and $K_L \rightarrow \pi^+ \pi^- \gamma$, wherein the nonradiative process is small because of helicity suppression and CP violation respectively[64]. One might have anticipated the same enhancement mechanism to apply in our case since the nonradiative reaction $\eta \rightarrow \pi^+ \pi^- \pi^0$ takes place only due to the relatively small u-d quark mass difference. For example, the direct emission associated with the vector-dominance diagrams shown in

Figure 9 could be expected to play an important role. However, a careful analysis by D’Ambrosio et al. has shown that this unfortunately does not appear to be the case[65]. The contribution from this direct—vector-dominated—mechanism is found to have the form

$$A_{\text{direct}}^\mu (\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = \frac{e64h_V\theta_V}{3\sqrt{3}M_V^2 F_\pi^4} [p_\eta \cdot p_0 g_{+-}^\mu + p_\eta \cdot p_+ g_{-0}^\mu + p_\eta \cdot p_- g_{0+}^\mu] \quad (100)$$

where

$$G_{ij}^\mu = k \cdot p_i p_j^\mu - k \cdot p_j p_i^\mu \quad (101)$$

and the combination of Bose symmetry and gauge invariance results in a suppression that makes such pieces even smaller than the pion loop component, which arise to $\mathcal{O}(p^4)$ and which are proportional to $m_u - m_d$. In ref. [65] it is estimated that the direct emission (DE) component is only a few percent addition to the inner bremsstrahlung (IB), even at relatively large photon energies

$$[(\Gamma_{\text{IB+DE}} - \Gamma_{\text{IB}})/\Gamma_{\text{IB}}]_{E_\gamma > 90 \text{ MeV}} \simeq 3.5 \times 10^{-2} \quad (102)$$

which appears to be too small to make this a realistic experimental goal.

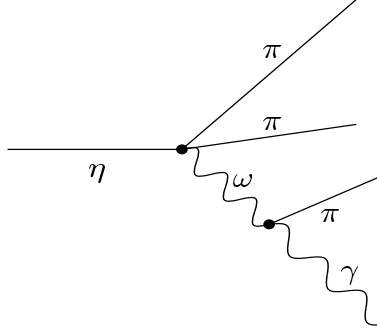


Figure 9: Vector dominance diagrams responsible for $\eta^0 \rightarrow \pi^+ \pi^- \pi^0 \gamma$.

3 Conclusions

We have examined the "non-rare" decay modes of the eta meson— $\eta \rightarrow \gamma\gamma, \pi^+ \pi^- \gamma, 3\pi, \pi^0 \gamma\gamma, 3\pi\gamma$ —in light of current theoretical knowledge and within the general framework of chiral symmetry. While there exist no *striking* discrepancies observed with respect to any of these predictions, we clearly identified problem areas wherein additional experimental scrutiny, such as would be possible at CELSIUS, would add significantly to our understanding. These conclusions can be summarized succinctly as follows:

a : $\eta \rightarrow \gamma\gamma$ —a resolution of the discrepancy between the rates measured via the Primakoff effect[23] and via QED production[22] is essential to future progress in understanding the eta system in general.

b : $\eta \rightarrow \pi^+\pi^-\gamma$ —a careful spectral shape measurement would be useful in order check the spectral shape with that predicted on fairly solid theoretical grounds from the anomalous sector of QCD.

c : $\eta \rightarrow 3\pi$ —a precise spectral shape measurement is called for in order to determine whether the existing disagreement between experimental findings and (what should be) solid theoretical predictions based on chiral perturbation theory are cause by an inaccurate value for the d-u quark mass difference or are due to the importance of higher order final state interaction effects.

d : $\eta \rightarrow \pi^0\gamma\gamma$ —a precision measurement of the Dalitz plot distribution is suggested in order to learn the origin of the existing disagreement between the experimental rate and that predicted from vector dominance.

e : $\eta \rightarrow 3\pi\gamma$ —a determination of an actual rate instead of the existing upper bound would be of interest, but theoretical indications are that it will be difficult to detect other than the inner bremsstrahlung component.

Clearly there is lots of interesting physics here and a marriage between precise and solid new experimental data and careful and well-motivated theoretical analysis would, I predict, be a happy one, leading to the offspring of a new degree of understanding of an important component of low energy phenomenology.

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$$D_1(s) = 1 - \frac{s}{m_\rho^2} - \frac{s}{96\pi^2 F_\pi^2} \ln \frac{m_\rho^2}{m_\pi^2} - \frac{m_\pi^2}{24\pi^2 F_\pi^2} F(s) \quad (103)$$
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