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Optimal pricing in electrical networks over space and time

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An electrical system is modelled with a transmission network, customers, central generators, and independent generators. The system is subject to stochastic failures and stochastic demand parameters. Optimal spot prices are derived for the system. They vary stochastically with space and time, and depend on electrical load flow patterns. The price difference between two locations or two voltage levels, and the wheeling charge between them, will change magnitude and sometimes sign over time, as a function of events throughout the network. Current spatial pricing methods are significantly different from the spot-price-based methods derived here.

1. Introduction

■ An essential feature of modern electric power systems is their use of transmission and distribution networks to convey power among producers and users. These networks play a critical role in reducing costs and improving reliability, by allowing the effective "pooling" of diverse generators and customers. Their configuration and operation influence the optimal spatial and temporal patterns of production, transmission, and use of electricity. To the extent possible, without incurring uneconomically high transactions costs, the characteristics of the network should be reflected in the price of electricity at different locations and times.

A theory of electricity pricing that accurately reflects the underlying physical and engineering properties of electricity has not yet been developed. This article attempts to provide such a theory. The results we derive are quite different from spatial pricing results for conventional commodities. For example, optimal spatial prices of electricity depend

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critically on time; they change radically and stochastically as functions of demand and supply events across hundreds of miles. It is theoretically possible to have negative prices at some points for short periods, i.e., pay users to consume.

Four characteristics of electricity affect its optimal spatial pricing. First, for every unit injected at one end of a transmission line, less than one unit can be removed at the far end. The difference is transmission losses. Second, each transmission line has a maximum safe capacity. Third, an energy balance constraint must be observed at all times. The transmission system itself has little ability to buffer supply-demand imbalances. Fourth, electrical flows cannot be directly allocated among transmission lines. For example, if a generating plant is connected to two transmission lines, it cannot control how much of its output goes along each line. Of course, the flow along one line can be reduced to zero by disconnecting the line. But for alternating current systems, if the line is connected, the amount flowing over it cannot be directly controlled.

These characteristics result in spatial prices for electricity that differ qualitatively from prices for most other commodities.[1]

2. Model specification

■ For the most part, we start from the customary specification of the "peak load pricing" problem (Crew and Kleindorfer, 1979). We assume a single welfare-maximizing public utility, which owns and operates multiple generating plants, and sells to independent customers, who have stochastic demands. We extend the classical models in four ways. First, demand and supply are spatially located. Second, all electricity travels over a fixed network. Lines of this network are subject to stochastic outages and to fixed safe flow limits. Transmission losses are a nonlinear function of the amount carried. Third, both demands and generator availability are stochastic.[2] Fourth, we assume the utility can set and communicate prices instantly, and can set a different price for each customer location at each moment. We call this "spot pricing." Vickrey, who first proposed it (1971), called it "responsive pricing." With spot pricing, the utility can induce socially optimal behavior by each customer and avoid system overload without having to resort to collective or individual rationing schemes.

For simplicity we assume that demands are independent of past and future prices, and correspondingly for generating costs. Therefore, we can solve our model as a single-period "deterministic" model. The solution is contingent on the current state of nature; as that evolves, so do optimal spot prices. Hence, all utility decisions (prices, outputs) are stochastic.

□ **Generation.** The utility owns J generating units. Unit j has maximum output K_j, marginal generating cost λ_j, and availability $\tilde{a}_j(t)$ at time t.[3] Marginal generating costs

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[1] Several previous authors have examined how electricity prices should vary over space. Only Scherer (1977) used an approximately accurate model of transmission system limits and losses. He did not discuss why or how much prices varied over space. Dansby (1980) assumes a radial transmission system with all generation at the hub, which is a special case and gives results that do not generalize. There are other problems with his formulation.

Schuler and Hobbs (1981) and Uri (1976) effectively assume that transmission costs can be dealt with by constant "freight charges," i.e., some number of cents per kilowatt hour per mile transmitted. This turns out to be incorrect, as we shall show. A later model by Hobbs uses a linearized version of the correct equations, as in Scherer (1977).

[2] In fact, generator outages affect prices and control needs more than unanticipated demand fluctuations do. A single large generator's output can be more than 10% of a system's load and can fall to zero within a few seconds. Such events are larger and harder to predict than the stochastic component of demand variation. Chao (1983) models generation with independent generator outages.

[3] Tildes on a indicate exogenous random variables.

equal heat rate times fuel price, plus variable maintenance costs. For convenience, units are numbered in order of operating costs, i.e., $\lambda_1 \leq \lambda_2, \ldots \leq \lambda_J$. Unit availability is an exogenous stochastic random variable between 0 and 1 which places a limit on generator output. Let $Y_j(t)$ be output from unit j at t, a utility decision variable. Then it is constrained to satisfy:

$$0 \leq Y_j(t) \leq K_j\tilde{a}_j(t) \qquad \forall j. \tag{1}$$

□ **Demand.** Individual customers act independently. Their demands depend on time of day, weather, the price of electricity, the price of other inputs, and so on. A customer may be a firm, a household, or a neighboring utility. We shall model customers as price-taking expected profit-maximizing firms. Let F_i be the short-run value-added function for customer i's use of electricity. Thus, it is i's profit, minus the cost of all nonelectricity variable inputs. It depends on the customer's electricity use $D_i(t)$ and on the random "weather" variable $\underline{\tilde{w}}(t)$, which reflects exogenous economic and weather variables. Thus $F_i = F_i(D_i(t), \underline{\tilde{w}}(t))$, and the customer will choose $D_i(t)$ to maximize its profit:[4]

$$\text{profit for } i = F_i(D_i(t), \underline{\tilde{w}}(t)) - p_i(t)D_i(t) \tag{2}$$

$$\frac{\partial F_i(D_i(t), \underline{\tilde{w}}(t))}{\partial D_i(t)} = p_i(t), \tag{3}$$

which implies $D_i(t) = D_i(p_i(t), \underline{\tilde{w}}(t))$ because of customer profit maximization. Since $\underline{\tilde{w}}(t)$ is experienced by all customers, their demands will be correlated. $D_i(t)$ may be negative, i.e., a "customer" may in fact be a net producer of electricity. The optimal spot price is the same whether a "customer" is a net user or a net generator.

□ **Transmission.** A complete characterization of the network at time t requires that we know the flows and losses along each line, and the net injections or withdrawals at each node. These are related by a number of equations, which are discussed in detail in the Appendix.

Let the flows along each line at time t be given by the vector

$$\underline{Z}(t) = \langle Z_1(t), \ldots, Z_K(t)\rangle.$$

Then total real power losses throughout the network are:[5]

$$L(t) = L(\underline{Z}(t)). \tag{4}$$

An electric power system has an *energy balance constraint:*

$$\sum_j Y_j(t) = \sum_i D_i(t) + L(t). \tag{5}$$

Attempting to violate this constraint, by either excessive or inadequate generation, will within seconds cause an uncontrolled blackout of the system. Generators will automatically disconnect from the network to avoid physical damage.

Other network constraints must also be observed. Flows in each line must not exceed design limits, or the line will fail:

$$Z_{min,k}\tilde{b}_k(t) \leq Z_k(t) \leq Z_{max,k}\tilde{b}_k(t), \tag{6}$$

where $\tilde{b}_k(t)$ is a stochastic 0–1 variable which is 0 when line k is out of operation,[6] and $Z_{min,k}$ and $Z_{max,k}$ are design limits for line k.

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[4] $\underline{\tilde{w}}(t)$ is a vector-valued exogenous stochastic variable.

[5] Directions of flows in $\underline{Z}$ are assigned arbitrarily. A negative $Z_k(t)$ indicates an actual flow direction opposite to the assigned direction.

[6] For AC electricity, $0 < b < 1$ is essentially impossible.

The power flows $\underline{Z}(t)$, in turn, depend on generation and demand at each node:

$$\underline{Z}(t) = \underline{Z}(\underline{Y}(t), \underline{D}(t), \underline{\tilde{b}}(t)), \tag{7}$$

where $\underline{Y}(t)$ and $\underline{D}(t)$ are the vectors of generation and demand augmented to have one element for each node, and $\underline{\tilde{b}}(t) = \langle \tilde{b}_1(t), \ldots, \tilde{b}_K(t) \rangle$. $\underline{Z}(t)$ also depends on network characteristics defined in the Appendix, including which nodes connect to which lines, and line impedances. $\underline{Z}(t)$ is determined by Kirchoff's laws, which are as follows. First, the algebraic sum of all line flows into each node is zero. Second, the algebraic sum of all voltage drops around any loop of a network equals zero. This leads to the inability to allocate flows, mentioned in Section 1 (Elgerd, 1971).

□ **Optimization.** We use the standard welfare criterion of maximizing consumers' plus producers' surplus, subject to the previously discussed constraints. These constraints depend on the utility's capital stock of generators and transmission lines and on the stochastic exogenous variables. For pricing and operational decisions, we maximize short-run welfare with a fixed capital stock. That is, at time t we wish to maximize, over generator output levels $Y_j(t)$ and customer prices $p_i(t)$,

$$\text{Max} \sum_i F_i(D_i(t), \underline{\tilde{w}}(t)) - \sum_j \lambda_j Y_j(t), \tag{8}$$

subject to constraint (1) for all generators j, inverse demand functions (3) for all customers i, and network constraints (4)–(7).[7] Bohn *et al.* (1983) derive optimal investment conditions for long-run expected welfare maximization.

3. Model solution

■ We now have a constrained optimization problem. Some of the dual variables will turn out to have interpretations as optimal prices; others will be the shadow values of additional capacity.

The Lagrangian to be maximized over all generation levels Y_j and over prices $p_i(t)$ is:

$$\begin{aligned}
&\sum_i F_i(D_i(p_i(t), \underline{\tilde{w}}(t)), \underline{\tilde{w}}(t)) && \text{[customer value added]}\\
&-\sum_j \lambda_j Y_j(t) && \text{[fuel costs]}\\
&+\theta(t)[\sum_j Y_j(t) - L(t) - \sum_i D_i(t)] && \text{[energy balance constraint]}\\
&-\sum_j \mu_j(t)[Y_j(t) - K_j \tilde{a}_j(t)] && \text{[unit capacity constraint]}\\
&-\sum_k (Z_k(t) - Z_{k,max})\eta_{k,max}(t) && \text{[transmission line constraints, one pair per line].}\\
&+\sum_k (Z_k(t) - Z_{k,min})\eta_{k,min}(t) &&
\end{aligned} \tag{9}$$

[7] Another constraint is that the line flow pattern $Z(t)$ will lead to voltage deviations at different points, which should not be allowed to exceed design limits. Reactive energy is also produced and affects losses and voltages. Finally, because of system stability problems, there may be constraints on total flows on all lines into a region. See Joskow and Schmalensee (1983, Chapter 4). We shall not discuss these constraints further, since they can be handled analogously to our treatment of equation (6).

Here $\theta(t)$ is the shadow price of another unit of demand at a "general" location. It will turn out to be the optimal spot price at one of the nodes which is arbitrarily chosen as the zero point for measurement. This node is termed the "swing bus" in power system parlance. The multiplier $\mu_j(t)$ is the shadow value of extra generating capacity of type j. It is zero except when generator j is fully loaded. $\eta_k(t)$ is the crucially important shadow value of additional transmission capacity:

$$\eta_k(t) = \eta_{k,max}(t) - \eta_{k,min}(t). \tag{10}$$

It is nonzero when line k is fully loaded. It is positive if the line is fully loaded in the "forward" direction; negative if the line is fully loaded in the "backwards" direction.

□ **Solution.** The Lagrangian (9) can be differentiated to obtain the first-order conditions for the various generator outputs Y_j and customer electrical use D_i. A central utility cannot calculate the socially optimal $D_i(t)$ since it does not know $\underline{\tilde{w}}(t)$ or the value-added functions $F_i(D_i, \underline{\tilde{w}})$. But by setting prices $p_i(t)$ properly and relying on customer profit maximization, the utility can induce socially optimal behavior. The optimal price $p_i^*(t)$ for customer i at time t turns out to be:[8]

optimal spot price to i = [social cost of additional demand at the swing bus]

× [1 + incremental losses caused by i]

+ [transmission constraint terms, summed over lines]

$$p_i^*(t) = \theta(t)\left[1 + \frac{\partial L(t)}{\partial D_i(t)}\right] + \sum_k \frac{\partial Z_k(t)}{\partial D_i(t)}\eta_k(t). \tag{11}$$

This is the key equation of optimal spatial spot pricing for electricity, and it gives the value of energy at time t and location i. Most of the remainder of this article is devoted to expanding, explaining, and exploiting it. We shall assume for now that the utility has completely solved (9) conditional on the exogenous variables so that it has output levels for each of its generators, and prices leading to demands at each major customer node.

The pattern of demand and generation at each node plus the condition of the network uniquely determine flows $\underline{Z}(t)$ and $L(t)$. We shall show that (11) can be evaluated and interpreted as a function of $\underline{Z}(t)$.[9]

□ **Interpreting θ, the shadow price on demand.** The most important component of (11) is $\theta(t)$, which is the same for all customers. Define system lambda $\lambda(t)$ as the short-run marginal generating cost. Specifically, it is the cost of generating another kilowatt hour of electricity from a marginal unit, then getting it back to the swing bus despite losses and transmission constraints.

Then

$$\begin{array}{cccc} \theta(t) & = \lambda(t) & + & \mu(t) \\ \text{[shadow price]} & = \text{[marginal generating cost]} & + & \text{[curtailment premium].} \end{array} \tag{12}$$

Here $\mu(t)$ is the premium needed to curtail demand back to available supply, when rationing would otherwise be necessary. This is a generalization of Riordan's (1984,

[8] Derivations are in Caramanis, Bohn, and Schweppe (1982) and Bohn (1982, Chapter 3 Appendix).

[9] If some or all customers are not on spot prices, equation (11) can still be evaluated in terms of $Z(t)$. Demands of these customers are exogenous constraints which augment (9).

p. 110) statement of optimal pricing in the case of one generation technology with deterministic availability: ". . . [I]n excess capacity states, set price at short-run marginal cost; . . . otherwise, price to equate demand with capacity. . . ."

To show how $\theta(t)$ can be decomposed, and to define $\lambda(t)$ precisely, first consider the case where there is at least one marginal generating unit, i.e., there exists a unit m satisfying:

$$0 < Y_m(t) < \tilde{a}_m(t)K_m. \tag{13}$$

Then by complementary slackness, the shadow value of capacity, $\mu_m(t)$, is temporarily zero. But for $Y_m(t) > 0$, $\mu_m(t)$ can be zero only if the value of electricity at location m is exactly the cost of generating there:[10]

$$\lambda_m = p_m^*(t). \tag{13a}$$

Furthermore, since there is one marginal generating unit, there is some available generating capacity for the system as a whole; therefore, the shadow price θ on the energy balance constraint (5) does not contain any rationing premium, and (12) becomes $\theta(t) = \lambda(t)$. Using (13a) and setting $\theta(t) = \lambda(t)$ in equation (11), we get a precise definition of system marginal costs considering transmission losses and constraints:

$$\lambda(t) = \frac{\lambda_m + \sum_k \left[\frac{\partial Z_k(t)}{\partial Y_m(t)} \eta_k(t)\right]}{1 - \frac{\partial L(t)}{\partial Y_m(t)}}. \tag{14}$$

Here λ_m is the direct cost of incremental generation at m. The denominator is the losses due to generating at m instead of at the swing bus. The summation term is the effect of generation at m on ameliorating or aggravating binding transmission constraints.[11]

Consider now the other case in which all available units are fully used, i.e., there is no generating unit on the margin. Then all available units j have local price higher than cost:

$$p_j^*(t) = \lambda_j + \mu_j(t) > \lambda_j \qquad \text{for all } j \text{ with } \tilde{a}_j(t) > 0. \tag{13b}$$

We can still define $\lambda(t)$ by using equation (14) for the most expensive unit, unit J, substituted for m in the equation. But because of (13b) we need a positive curtailment premium, $\mu(t) > 0$, to make the equations balance.

Thus, when there is a generator on the margin, $\mu(t)$ is zero.[12] It becomes nonzero whenever rationing would otherwise be needed to avoid excess demand. Figure 1 illustrates this. Figure 1's "supply curve" is the envelope of the λ_j's of all available units, out to the limit $\sum_{j=1}^{J} \tilde{a}_j(t)K_j$.[13] Thereafter it rises vertically. The "instantaneous demand curve" measures $\sum_{i=1}^{I} D_i(p_i(\theta(t), \underline{\tilde{w}}(t)))$. That is, total demand depends on individual prices, which in turn depend on $\theta(t)$, the price at the swing bus, and other terms discussed below.

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[10] Prices in (11) can also be defined for generating nodes j with $+\partial L/\partial D_i$ replaced by $-\partial L/\partial Y_j$ since a positive demand is equivalent to negative generation, and similarly for the $\partial Z_k/\partial D_i$.

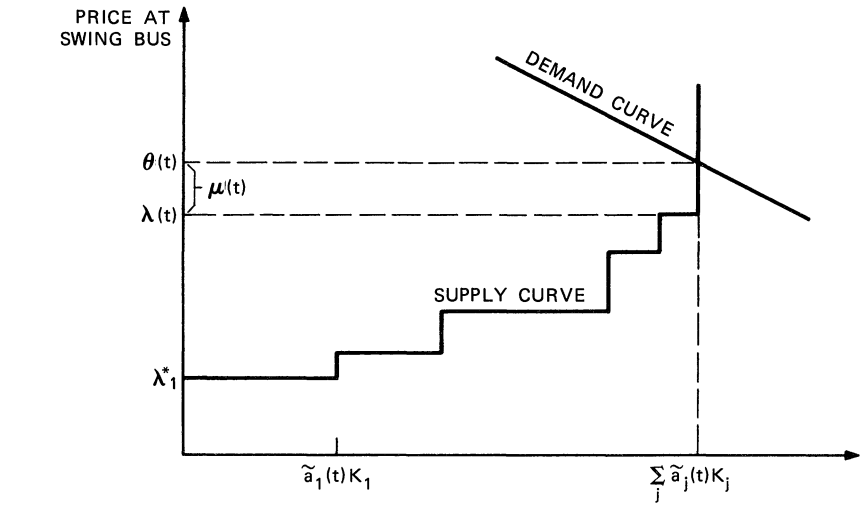
[11] Note that in the nonspatial case (14) is just $\lambda(t) = \lambda_m$, a well-known definition. Common power system practice is to consider losses but not transmission constraints: $\lambda(t) = \lambda_m/(1 - \partial L/\partial Y_m)$.

[12] In the model we are using, μ may also be nonzero when $\lambda_{m+1} - \lambda_m$ is large, putting a vertical step in the system supply function. In that case $\mu(t)$ will be proportional to $\mu_m(t)$, the capacity constraint multiplier on the last generator in use. In reality, marginal costs λ_j are upward sloping in Y_j, and few systems have such vertical steps.

[13] Even though we have assumed that each generator has a constant marginal cost λ_j, because of losses the system supply curve is not exactly piecewise linear.

FIGURE 1

DETERMINATION OF $\theta(t)$



*NOTE THAT GENERATING COST λ_1 MUST BE ADJUSTED AS IN EQUATION (14).

The optimal $\theta(t)$ is found at the intersection of the supply and demand curves. It is misleading, however, to think of θ as "the market equilibrium price" in a spatial public utility market. First, there is no single correct price since different prices are optimal at different locations. Second, the only "equilibrium" is a dynamic one. As $\underline{\tilde{w}}$, $\underline{\tilde{b}}(t)$, and especially the unit availabilities $\tilde{a}_j(t)$ evolve, $\theta(t)$ will change rapidly and stochastically. Figure 2 shows nonspatial $\lambda(t)$ for a medium-sized Midwestern utility over one month. Both the pattern and the level of prices changed radically week to week. Observe that any conventional time-of-day price, with three levels and set a year in advance, would track $\lambda(t)$ very poorly.

□ **The effect of losses on optimal spatial price differences.** The shadow price θ is usually the largest component of optimal spot prices in equation (11), and it is the same at all points. But to get the price at each point, θ must be multiplied by a term involving incremental transmission losses, $\partial L(t)/\partial D_i(t)$. This term may be greater or less than one. It depends on the location and voltage level of the customer. Customers at locations that cause a larger system loss when they increase their demands are charged a higher price, and conversely for customers or generators that cause a smaller loss. We shall now examine this in detail.

From equation (11), the difference between optimal spot prices at two locations at one time is:

$$p_i^*(t) - p_j^*(t) = \theta(t)\left[\frac{\partial L(t)}{\partial D_i(t)} - \frac{\partial L(t)}{\partial D_j(t)}\right] + \sum_k \left[\frac{\partial Z_k(t)}{\partial D_i(t)} - \frac{\partial Z_k(t)}{\partial D_j(t)}\right][\eta_k(t)]. \qquad (15)$$

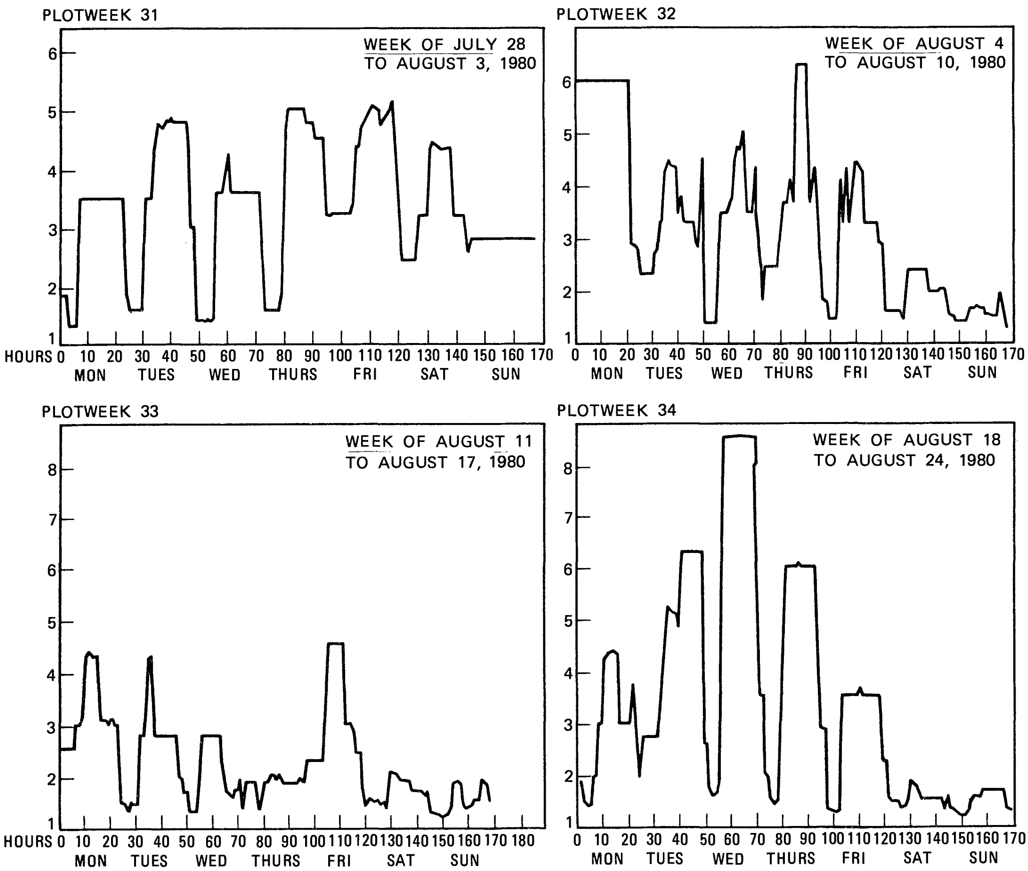
The first term is the difference in incremental losses caused by the two customers, times the current value of $\theta(t)$. The second term is the differential effect of each customer's incremental demand on transmission capacity limits throughout the network. It is zero if no transmission constraints are currently taut. We shall now discuss the incremental effect of demand at different points on total system losses, $\partial L/\partial D_i$.

The key approximation for losses is a modification of equation (4):[14]

$$L(t) \cong \underline{Z}'(t)R\underline{Z}(t), \qquad (16)$$

[14] See the Appendix.

FIGURE 2
ACTUAL SYSTEM LAMBDA, (¢/kwh)



where

$\underline{Z}(t)$ = vector of real power flows in each line k at time t, in kw,
$\underline{R}$ = diagonal matrix of line resistances.

Thus losses are an approximately quadratic function of line flows. This nonlinearity is fundamental to our results. Note that the incremental loss in a line is proportional to its flow times its resistance:

$$\frac{\partial L_k}{\partial Z_k} = 2Z_k R_k, \tag{17}$$

where R_k is the kth diagonal element of $\underline{R}$.

Hence, the price difference is:

$$p_i^*(t) - p_j^*(t) = 2\theta(t)\underline{Z}'(t)\underline{R}\left[\frac{\partial \underline{Z}(t)}{\partial D_i} - \frac{\partial \underline{Z}(t)}{\partial D_j}\right] + \text{term involving } \eta(t). \tag{18}$$

Since flows $\underline{Z}(t)$ depend on the level and spatial pattern of demand at t, and so does $\theta(t)$, we see that *both absolute and proportional price differences depend on the level and spatial pattern of demand at* t. When i and j are directly connected by line k, equation (18) can be transformed to

$$p_i^*(t) - p_j^*(t) \cong 2\theta(t)R_k Z_k(t) + \text{term involving } \eta(t), \tag{19}$$

where $Z_k(t)$ is the flow from j to i. If i and j are connected by several lines, (19) holds for each line individually.

The proof of (19) in the two-bus case is straightforward: $\underline{Z}$ and $\underline{R}$ in (18) are scalars $Z_k(t)$ and R_k, respectively, and the bracketed term is unity,[15] giving (19). The general network case requires the further assumption that the resistance/inductance ratio is constant across all transmission lines. The proof is developed in the Appendix.

Note from (16) that total losses in line k are approximately $R_k Z_k(t)$. Define the average percentage line loss at time t as the total loss divided by the amount of power flowing through the line, $R_k Z_k(t)$.[16] Typically it is a few percent in a moderately loaded line, and up to 20% in a heavily loaded line. Hence, when $\eta(t) = 0$

$$p_i^*(t) - p_j^*(t) \cong 2\theta(t) \times [\text{average percentage line loss in line } k \text{ at time } t]. \qquad (20)$$

From (19) and (20) we have the following results for general networks when transmission constraints are not binding:

Result 1. The percentage price difference between two locations at time t equals approximately twice the average percentage loss at t over the line connecting them, provided that transmission constraints are not binding.

Result 2. The absolute and percentage price difference between two points are approximately proportional to the flow over the line connecting them, provided that transmission constraints are not binding. If the flow is nonzero, the price at the "upstream" point is lower than at the "downstream" point.

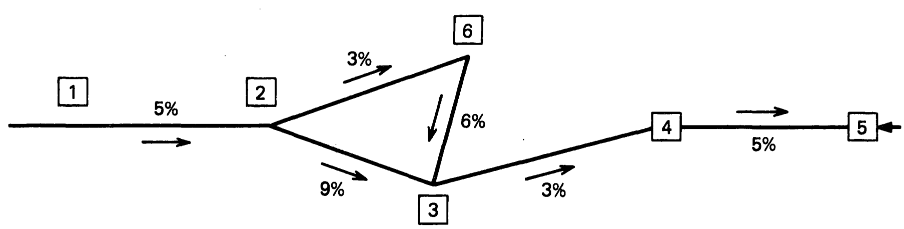
Result 3. The absolute level of price differences among all points in the network when transmission constraints are not binding is *ceteris paribus* proportional to $\theta(t)$, which is greater than or equal to $\lambda(t)$. Therefore, the absolute level of price differences is stochastic.

These results are different from those for conventional commodities, which have transportation costs that depend only on distance and are constant over time. For example, applying Result 3 to Figure 2 suggests that optimal spatial prices between two points of that utility vary more than 2:1 on most days and more than 5:1 on occasion.[17]

An example clarifies spatial prices. Suppose that at noon on a particular day the average line losses in a network are as shown in Figure 3. As long as no transmission line is at its capacity limit, approximate optimal percentage price differences between any

FIGURE 3

HYPOTHETICAL AVERAGE FRACTIONAL LOSSES AND DIRECTION OF FLOWS



[15] Define one bus as the "swing bus," i.e., the source of extra generation when demand increases. Then an increase in demand at the swing bus has no effect on Z. An increase at the other bus increases flow Z by the amount of increased demand.

[16] This terminology is awkward but necessary. There are four possible ways of calculating line losses by using one concept from each of the following pairs: average versus marginal, and percentage versus absolute. Furthermore, all four fluctuate over time. This terminology distinguishes them.

[17] These ratios are conservative since in most lines $Z_k(t)$ and $\eta_k(t)$ are positively correlated with $\theta(t)$.

pair of points may be read directly from the average percentage losses in Figure 3. If we take node 6 as the swing bus, then p_6 (noon) = θ^* (noon) and

$$p_1^* \text{ (noon)} = .84\theta \qquad p_4^* = 1.18\theta$$

$$p_2^* = .94\theta \qquad p_5^* = 1.28\theta$$

$$p_3^* = 1.12\theta \qquad p_6^* = 1.0\theta.$$

In this example, average losses of about 5% on each line give a price difference of 44% between ends of the network, which is surprisingly large.[18]

□ **Pricing by voltage level.** An immediate application of Result 1 is to prices for different voltages. Voltage stepdown transformers have electrical losses analogous to those of transmission lines, and the low voltage distribution system usually has high percentage losses. Thus, a kilowatt hour provided at a high voltage costs less to generate than one provided at a low voltage from the same substation. Present practice is to charge according to the *average* stepdown and distribution loss. Equation (20) shows, however, that the optimal price difference is twice the average loss percentage. Similar analysis can be used for the value of buyback energy provided at different voltages and locations, a topic hotly argued in discussions of cogeneration.

□ **Transmission capacity limits.** So far we have discussed only transmission losses in equation (15). What happens if some transmission lines are at full capacity at time t, so that the $\eta_k(t)$ are not all zero? The optimal prices at all points may change as a result. The amount of price increase for customer i depends on $(\partial Z_k)/(\partial D_i)$, i.e., on how much power flow in the critical line is affected by i's demand. Even if the customer is not directly connected to the taut line, this derivative may be nonzero. Ponrajah (1984) works out a 24-node numerical example of this effect.

For example, suppose in Figure 3 that line 5, connecting points 3 and 6, would be overloaded if only losses were considered in setting prices. Then $\eta_5(t)$ must rise as much as necessary to cause demand and generation patterns to readjust. The change in η_5 affects every price in this network, by amount $\eta_5\ \partial Z_5(t)/\partial D_i(t)$. Hence, $p_3(t)$, $p_4(t)$, and $p_5(t)$ will all rise. Other prices also change, proportionally to the marginal impact of demand at those points on Z_5. The shadow price $\eta_5(t)$ must be determined empirically, as it depends on the responsiveness of demand at different points to price changes.

Result 4. If two points are connected by any line that *is* fully loaded, the price difference between them is larger than that given by Result 1. In effect, the two points become separate markets for electricity, with their own supply and demand curves.

□ **Line flows.** It is clear that spatial pricing depends on the pattern of power flows through the network. The flows $\underline{Z}(t)$ are in turn determined by the spatial pattern of demand and generation and the network configuration. Essentially, Kirchoff's laws say that Nature allocates flows so as to minimize transmission losses.[19]

An approximation to the solution (see the Appendix) is:

$$\underline{Z}(t) \cong [\underline{H}][\underline{Y}(t) - \underline{D}(t)], \tag{21}$$

[18] Note that examining losses and optimal price differences along any two paths connecting two nodes gives the same result. In Figure 3 the difference between nodes 2 and 3 is .18θ, whether measured 2 → 6 → 3, or directly 2 → 3. This is so because flows in electrical networks adjust appropriately. The choice of swing bus also does not matter in calculating p_j^*'s.

[19] This "minimize loss" formulation holds for the DC load flow problem; we have not been able to prove it for the AC load flow, but it is a useful way to think about flows.

where $\underline{Y}(t) - \underline{D}(t)$ is the vector of net generation minus use at each node. The matrix $\underline{H}$ is called the "transfer admittance matrix," and it gives the effect of net generation at each node on flows in each line. It is a familiar matrix to power systems engineers. It is not a sparse matrix; that is, demands at one point affect flows throughout the network. It depends on the network configuration at the moment, $\underline{b}(t)$, and on time-invariant characteristics of transmission lines (resistance, inductance, connectance).

From equation (21), we see that changes in demands, in network structure due to line outages, or in generation pattern will lead to different flows $\underline{Z}(t)$. A change in $D_i(t)$ at one location may affect $Z_k(t)$ for distant lines.

Furthermore, any of these events will change prices throughout the network. Combining (21) with optimal prices in (11) and with losses in (16) gives in terms of flows (see the Appendix):

$$\underline{p}^*(t) \cong \theta(t)[1 - 2\underline{Z}^T(t)\underline{RH}] - \underline{H'_\eta}(t), \tag{11a}$$

where $\underline{p}^*(t)$ is the vector of optimal spot prices at each location, and

$$\underline{p}^*(t) \cong \theta(t)[1 - 2\underline{B}(t)\cdot(\underline{Y}(t) - \underline{D}(t))] - \underline{H'_\eta}(t), \tag{11b}$$

where $\underline{B}$ is a matrix that depends only on the current network interconnections and time-invariant line characteristics. Equations (11b) or (11a) therefore give quick ways to calculate approximately optimal spot prices. The matrix $\underline{B}$, as well as $\underline{H}$, is nonsparse. Therefore:

Result 5. The price difference between any two points in the network will depend on demands throughout the network. The price differences will change over time, both in absolute amount (dollars per kilowatt hour) and as a percentage of $\theta(t)$.

For example, consider the simple network of Figure 4. The power flow $Z_1(t)$ is approximately the sum of demands $D_3(t)$ and $D_4(t)$. Then $p_3^*(t)$ depends on both demands. Define node 1 as the swing bus: $p_1^*(t) = \theta(t)$. From (19) we get

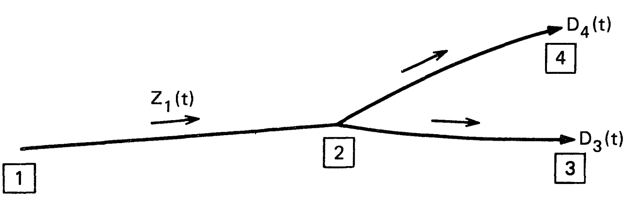
$$p_3^*(t) \cong \theta(t)[1 + 2(R_{12} + R_{23})D_3(t) + 2R_{12}D_4(t)],$$

where R_{ij} = resistance of the line from i to j. Hence, customer 3 pays a higher price and a higher location premium when either it or customer 4 demands more, even if the cost of generation $\theta(t)$ remains constant. If either location's demand rises to the point that $Z_1(t)$ reaches Z_1^{max}, then both locations will see prices rise still further, by $\eta_1(t)$.

It is quite possible that the sign of $p_3^*(t) - p_1^*(t)$ will change over the course of a day, as the direction and magnitude of power flows change owing to cyclic demand and to changes in optimal dispatch patterns. In fact, it is conceivable that the absolute price at a particular point could be negative.[20]

FIGURE 4

A SIMPLE NETWORK



[20] This can happen if transmission line k is heavily loaded, and demand at some point will reduce its load by causing flows $Z_k(t)$ to change. This will be reflected in the term involving η_k in equation (11), which can take on positive or negative values. If it is negative enough, the absolute price at some location can be negative.

□ **Optimal dispatch.** In the nonspatial case, optimal $Y_j(t)$ are found by dispatching units in order of marginal costs λ_j until $\sum_{j=1}^{m} \tilde{a}_j(t)K_j$ equals or exceeds total demand. The mth unit is not fully loaded and determines $\lambda(t)$.

In the spatial problem, this solution is inaccurate. It may be that $\lambda_5 < \lambda_6$, but that unit 5 will have a greater effect on system losses than unit 6, i.e., $\partial L/\partial Y_5 > \partial L/\partial Y_6$. Which unit should be dispatched if more generation is needed?

The optimal solution of (9) has the property that unit j should be operated if it would be optimal for a private owner facing spot pricing to do so, i.e., if $p_j^*(t) > \lambda_j$.[21] The $\underline{p}^*(t)$ can thus be found by iterating. Each iteration goes from trial $\underline{p}^*(t)$ to $\underline{Y}(t) - \underline{D}(t)$ to $\underline{Z}(t)$[22] to a new guess for $\underline{p}^*(t)$.[23] This method is equivalent to present dispatching algorithms.[24] Small generators can be controlled by telling them their spot price rather than their generation level. The same method can be used to help coordinate neighboring utilities that are not jointly dispatched.[25]

4. Market definition and spatial competition

■ Various proposals have been advanced to allow some form of competition in portions of the electric power industry. How competitive would the results be? In large part this depends on the legal and institutional arrangements specified in the deregulation proposal (Joskow and Schmalensee, 1983; Bohn *et al.,* 1984).

But, in any case, the preceding analysis sheds light on how to define the relevant market when attempting to analyze the likely extent of competition. We show here that the relevant market definition will be quite fluid. At some times a generator may have no effective competitors and thus considerable power to affect prices. At other times, the same generator may find itself competing with generators hundreds of miles away. The stronger the transmission system, the more effective competition will be.

A full analysis of competition requires modelling reactions of each generating firm to the price it is paid and to the prices and outputs of competitors. Rather than create and solve a full oligopolistic model, we look at one issue only: what is the initial effect of one generator's output decisions on the prices seen by itself and by competitors, before competitors' reactions are factored into the analysis? Specifically how do spatial issues influence these effects?

We assume a central administrator follows equation (11) at all times.[26] Suppose generator j is considering changing its output Y_j. How much impact will that have on its price $p_j^*(t)$, and on the price paid to competitor i, assuming i does not respond?[27]

To elaborate these interactions without using an involved model, we consider competing generators with outputs $\{Y_i\}$ located at each of N network nodes, but all demand D located in one node. The energy balance equation (demand equals total

[21] Seeming indeterminancy for marginal units with $p_m^*(t) = \lambda_m$ is avoided by the fact that the true short-run marginal cost curve for generators is U-shaped.

[22] A more accurate version of equation (21) is used.

[23] Equation (11a) or (11b) is used at some nodes, while more accurate versions are used at nodes that are on the margin.

[24] Solving a large system can require solution of several thousand simultaneous equations, using a special computer-implemented algorithm.

[25] Price-based control has advantages, but is complex for large generators. At a minimum, spinning reserve pricing would also be needed; this is the subject of research.

[26] Since (11) is derived by assuming pure price-taking behavior, a central coordinator with enough information might wish to alter (11) to influence oligopolists to be more competitive. (Luh *et al.,* 1982.) Even in this framework spatial issues will still affect the outcome.

[27] Calculating the total impact would require imbedding the following results in a model of response by all competitors.

generation minus losses) removes one degree of freedom and allows expression of network losses L and line flows $\underline{Z}$ as a function of power injections in all but one arbitrarily selected node, the swing bus. Assigning the index 1 to the demand node and selecting it as the swing bus, we have:

$$L = L(Y_2, Y_3, \ldots, Y_N); \qquad D = \sum_{i=1}^{N} Y_i - L; \qquad p_1^*(t) = \theta(t) = \frac{\partial F}{\partial D};$$

$$\frac{\partial \theta}{\partial Y_i} = \frac{\partial^2 F}{\partial D^2}\left(1 - \frac{\partial L}{\partial Y_i}\right),$$

where F is the aggregate value-added function of consumers at node 1. For a Cournot competitive world where in the short run only demand responds to a generator's change of output, while all other generators maintain constant output levels, equation (11) can be differentiated to give:

$$\frac{\partial p_i^*}{\partial Y_j} = -\theta(t)\frac{\partial^2 L}{\partial Y_i \partial Y_j} + \left(1 - \frac{\partial L}{\partial Y_i}\right)\left(1 - \frac{\partial L}{\partial Y_j}\right)\frac{\partial^2 F}{\partial D^2} - \sum_k \frac{\partial^2 Z_k}{\partial Y_i \partial Y_j}\eta_k - \sum \frac{\partial Z_k}{\partial Y_i}\frac{\partial \eta_k}{\partial Y_j}. \qquad (22)$$

Equation (22) shows that different generators do not have the same control over their own prices, i.e., $\partial p_i^*/\partial Y_i \neq \partial p_j^*/\partial Y_j$. But when transmission capacity is not binding, $\underline{\eta} = 0$, equation (22) becomes symmetric in i, j, thereby implying that $\partial p_i^*/\partial Y_j = \partial p_j^*/\partial Y_i$, and firms have equal effects on each other.

To exploit equation (22) further, consider a network consisting of two nodes connected by a single transmission line with resistance R. Suppose that the flow $Z(t)$ is from node 2 to node 1, with generators in both nodes, but all demand in node 1 only. Using the approximation for losses of equation (17), noting that $Z(t) \cong Y_2(t)$, and assuming that demand is characterized by instantaneous price elasticity of response $\epsilon(t)$, we obtain the following with equation (22) for $\underline{\eta} = 0$:

$$\frac{\partial p_1^*(t)}{\partial Y_1(t)} = \frac{\partial \theta(t)}{\partial Y_1(t)} = \frac{\partial^2 F}{\partial D^2(t)} = \frac{\theta(t)}{D(t)\epsilon(t)} \qquad (23)$$

$$\frac{\partial p_2^*(t)}{\partial Y_2(t)} = -2\theta(t)R + (1 - 2RY_2(t))^2\frac{\theta(t)}{D(t)\epsilon(t)} \qquad (24)$$

$$\frac{\partial p_1^*(t)}{\partial Y_2(t)} = \frac{\partial p_2^*(t)}{\partial Y_1(t)} = (1 - 2RY_2(t))\frac{\theta(t)}{D(t)\epsilon(t)}. \qquad (25)$$

Since $\epsilon(t)$ is a negative quantity, equations (23) and (24) imply that the "downstream" firms (node 1) have more control over their own price than upstream firms, i.e., $\left|\frac{\partial p_1^*}{\partial Y_1}\right| > \left|\frac{\partial p_2^*}{\partial Y_2}\right|$ if

$$2RD(t)|\epsilon(t)| < 1 - (1 - 2RY_2(t))^2. \qquad (26)$$

From equations (22) to (26) we see that:

(1) Firms in the two nodes have different abilities to influence their own prices. Downstream firms have more control than upstream firms when demand is relatively unresponsive to price changes. As the strength of the network increases (R decreases), the importance of location decreases.
(2) The ability to influence prices changes over time as $Y_1(t)$, $Y_2(t)$, $D(t)$, and $\epsilon(t)$ change.
(3) If the transmission line connecting the nodes is fully loaded, then η will adjust as necessary to keep the flow at that level. Even if firm 2 has more generating capacity and

$p_1^*(t) \geqslant \lambda_2$, $p_2^*(t)$ will be just the level to encourage the feasible flow, and firm 2 will not have an incentive to increase output further. Firm 1 can adjust its output over a range without affecting $p_2(t)$. Each city therefore becomes its own market "island," with (temporarily) no competitive interaction.
(4) Firms always have an equal ability to affect each other's price.

These results mean that the market in each city will on occasion be much less competitive than it was a few hours earlier. In the worst case, there might be only one generator on the margin in the downstream city with an effectively noncompetitive market. How often this will happen depends on the strength of the transmission system, the location, ownership and type of generators, and the space-time pattern of demand.

In conclusion, because of losses and transmission constraints it is not enough to show that on the basis of "average" demand patterns, a region of the country will be reasonably competitive under deregulation. Different demand patterns and generator availability scenarios must be considered, each of which will show different ability by generators to affect their own prices. In some areas the transmission system will need strengthening before deregulation can lead to approximately competitive behavior by generators. See also Schmalensee and Golub (1984).

5. Conclusions

■ Practical application of spatial spot prices at some levels would not be a major departure from present practice. Power control centers already collect and process much of the necessary information, and use it for real time dispatch of their generators. Spatial prices at major network nodes are developed as an intermediate calculation.[28]

Because of transactions costs, it is not desirable to use spatial spot prices for all customers and independent generators. Some partial spot pricing will be advantageous for some large customers and independent generators. Three possible arrangements are:[29]

(1) Hourly spot pricing. Prices are recalculated once an hour, and do not change until the next hour. Telephone or other methods can be used to communicate the prices to customers/generators. For most customers prices are differentiated by region and voltage levels. For interutility sales and purchases from large independent generators, location-specific prices would be worth calculating.
(2) Twenty-four hour update prices. Prices are recalculated once a day, with a different price for each hour.
(3) Monthly time-of-day pricing. Prices are recalculated once a month for the following month. They can be communicated by newspaper advertisements or by mail. Current time-of-day metering technology can be used.

All of these prices can coexist. The formulas for the underlying optimal spot prices remain the same. The actual prices that should be charged to customers not on full spot prices are approximately equal to the expected value of optimal spot prices plus a covariance term. Selecting which customer should be on which rate cannot be done perfectly because of adverse selection problems, but heuristic classification schemes will probably give good results. See Bohn (1982).

The methodology we have developed here, of pricing both losses and transmission constraints at each point in a network in real time, is applicable to other commodities. Natural gas appears to fit quite well, although the interesting price dynamics will have a period of weeks instead of hours. Similar models may also be possible for other

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[28] Transmission constraints are generally not incorporated into the prices, however. Thus, more software development would be necessary.

[29] These are discussed in more detail in Caramanis, Bohn, and Schweppe (1982).

commodities, such as network-interconnected information facilities with spatially distributed users (Agnew, 1973), and long distance data transmission.[30]

Appendix

■ **Derivation of electric network relationships.** In the following subsections we derive: (1) losses on a line; (2) line flows as a function of network characteristics and bus injections; (3) network losses; and (4) spot price differences at the ends of a line. We conclude with a note on the approximations and assumptions we have made.

□ **Losses on a line.** Consider a line k with resistance R_k, inductance X_k, impedance magnitude $z_k = (X_k^2 + R_k^2)^{1/2}$, and admittance magnitude $\Omega_k = z_k^{-1}$. Define

Z_{12}, Z_{21}:	the real power flows at each end of the line (Z_{12} is the power flow out of bus 1 towards bus 2);
L_k:	real power losses in line k: $L_k = Z_{12} + Z_{21}$;
$\Delta = \delta_1 - \delta_2$:	the difference of voltage angles at each end of the line;
$b = \cos^{-1}(R_k/z_k) = \sin^{-1}(X_k/z_k)$:	impedance-resistance angle compatible with the definition of z_k;
$\lvert V_1\rvert, \lvert V_2\rvert$:	voltage magnitude level at each end of the line.

Assume[31] $|V_1| = |V_2| = 1$. It follows that

$$Z_{12} = \Omega_k[\cos b - \cos(b - \Delta)] \tag{A1}$$

and

$$Z_{21} = \Omega_k[\cos b - \cos(b + \Delta)]. \tag{A2}$$

With the use of trigonometric identities equation (A1) implies

$$Z_{12} = -2\Omega_k\left[\sin\left(\frac{2b - \Delta}{2}\right)\sin\left(\frac{\Delta}{2}\right)\right].$$

Assume $b \gg \Delta$. We then have the approximation,

$$Z_{12} \simeq -2\Omega_k \sin b \sin\left(\frac{\Delta}{2}\right). \tag{A3}$$

Assuming $\Delta/2$ is small, we have the approximation,

$$Z_{12} \simeq -2\Omega_k \sin b \frac{\Delta}{2} = -\Omega_k \frac{X_k}{z_k} \Delta = -\Omega_k^2 X_k \Delta, \tag{A4}$$

and similarly for Z_{21}.

Assuming $R_k \ll X_k$, we approximate $z_k = (X_k^2 + R_k^2)^{1/2} \simeq X_k \simeq \Omega_k^{-1}$. Substituting above, we obtain

$$Z_{12} \simeq -Z_{21} \simeq -\Omega_k \Delta. \tag{A5}$$

With the use of (A1) and (A2), the line losses are given by

$$L_k = Z_{12} - Z_{21} = 2\Omega_k \cos b[1 - \cos \Delta]. \tag{A6}$$

Assuming Δ is small (as in (A4)) and using the definition for b yield

$$L_k = R_k \Omega_k^2 \Delta^2. \tag{A7}$$

Finally, substituting approximation (A5) yields

$$L_k = R_k(Z_{12})^2 = R_k(Z_{21})^2. \tag{A8}$$

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[30] When multiple data carriers serve a single customer, there will be opportunities for one carrier to improve its load factor and revenues at the expense of others, by offering prices closer to spatial spot prices. Smart customer PBX's already route calls over the cheapest available circuit; the main change needed will be to alter the "dispatch order" in response to real-time price updates from carriers.

[31] Note that voltage magnitude is a normalizing factor used throughout this Appendix. Readers should keep this in mind when trying to verify the units of subsequent results or considering the use of the relationships developed.

☐ **Line flows as a function of network characteristics and bus injections.** Define

$\underline{P}$: $[(N-1)\times 1]$ vector of bus injections (generation minus demand at bus i, $i = 2, 3, \ldots, N$);
P_1: injection at the swing bus ($i = 1$);
$\underline{\Omega}, \underline{R}$: $K \times K$ diagonal matrix of line admittances and resistances respectively, K lines;
$\underline{A}$: $[K \times (N-1)]$ network incidence matrix with 0, 1, -1 elements corresponding to network interconnections;
$\underline{Z}$: $[K \times 1]$ vector of line flows;
$\underline{\delta}$: $[(N-1)\times 1]$ vector of voltage angles at each bus; at swing bus $\delta_1 = 0$;
$\underline{\Delta} = \underline{A}\underline{\delta}$: $[(N-1)\times 1]$ vector of angle differences across all lines;
L: total line losses: $L = \sum_k L_k$.
$\underline{e}$: column vector of all ones;
$\underline{e}_k$: column vector of all zeros except for 1 in the kth line.

The energy balance constraint, $\underline{e}'\underline{P} + \underline{P}_1 - L = 0$, reduces degrees of freedom by 1. It is thus necessary to select arbitrarily and to specify one of the nodes; i.e., $\delta_1 = 0$.

Since the sum of all powers entering a bus is zero,

$$\underline{P} = \underline{A}'\underline{Z}. \tag{A9}$$

Combining equation (A9) and the matrix form of (A5) yields

$$\underline{P} = \underline{A}'\underline{\Omega}\underline{A}\underline{\delta}. \tag{A10}$$

Solving for $\underline{\delta}$, we obtain

$$\underline{\delta} = (\underline{A}'\underline{\Omega}\underline{A})^{-1}\underline{P}. \tag{A11}$$

Substituting into the matrix form of (A5) yields

$$\underline{Z} = \underline{H}\underline{P}, \tag{A12}$$

where $\underline{H} = \underline{\Omega}\underline{A}(\underline{A}'\underline{\Omega}\underline{A})^{-1}$. $\underline{H}$ is called the *transfer admittance matrix.*

☐ **Network losses.** By using (A8), the total line losses L become $L = \underline{Z}'\underline{R}\underline{Z}$. Substituting $\underline{Z}$ from (A12) yields

$$L = \underline{P}'\underline{B}\underline{P}, \tag{A13}$$

where $\underline{B} = (\underline{A}'\underline{\Omega}\underline{A})^{-1}\underline{A}'\underline{\Omega}\underline{R}\underline{\Omega}\underline{A}(\underline{A}'\underline{\Omega}\underline{A})^{-1}$. Assuming $\underline{R} = \alpha\underline{\Omega}^{-1}$, i.e., a constant resistance-inductance ratio across all transmission lines, we obtain

$$\underline{B} = \alpha(\underline{A}'\underline{\Omega}\underline{A})^{-1} = (\underline{A}'\underline{R}^{-1}\underline{A})^{-1}. \tag{A14}$$

☐ **Spot price differences at the ends of a line.** Define

$\underline{p}^*$: $(N-1)$ vector of spot prices;
p_1^*: spot price at swing bus.

From equation (11) we have

$$\underline{p}^* = \theta\left(\underline{e} - \frac{\partial L}{\partial \underline{P}}\right) - \frac{\partial \underline{Z}}{\partial \underline{P}}\underline{\eta}$$

$$p_1^* = \theta, \tag{A15}$$

where $\underline{\eta}$ is the column vector of Lagrangian multipliers.

Using (A12) and (A13), we obtain

$$p^* = \theta(\underline{e} - 2\underline{B}\underline{P}) - \underline{H}'\underline{\eta}. \tag{A16}$$

With the additional approximation of a constant resistance-inductance ratio that gives (A14) and with the substituting of $\underline{P}$ from (A10), we get

$$p^* = \theta(\underline{e} - 2\alpha\underline{\delta}) - \underline{H}'\underline{\eta}. \tag{A17}$$

Define further $\underline{d}$ as the vector of price differences across the ends of lines, i.e., $\underline{d} = \underline{A}\underline{p}^*$. Equation (A17) then yields

$$\underline{d} = -2\alpha\theta\underline{A}\underline{\delta} - \underline{A}'\underline{H}'\underline{\eta} = -2\alpha\theta\underline{\Omega}^{-1}\underline{Z} - \underline{A}'\underline{H}'\underline{\eta}. \tag{A18}$$

Equation (A18) shows that the price difference across each line, disregarding $\underline{\eta}$ terms, is proportional to the voltage angle difference across each line or the real power flow through the line.

□ **Note on the approximations and assumptions made.** The assumptions made in equations (A3), (A4), and (A6) are $b \gg \Delta$, Δ small, $X \gg R$, and constant voltage magnitude levels. These assumptions are reasonable and are often used in actual utility operations and planning for transmission lines. The additional assumption made in (A14) postulating a constant resistance-inductance ratio for all transmission lines may lead to larger approximation errors than are tolerable for operational decisions. But the results thus obtained are useful in increasing one's intuitive understanding of spatial network pricing and can be used in long-term planning studies where general behavior modelling is important, but high accuracy is not required.

The assumptions made in equations (A3)–(A5) can become more inaccurate for lower voltage, subtransmission and distribution lines. Thus, when dealing with certain actual operational conditions, it may be necessary to use more accurate approximations or even to use the full "AC load flow" equations. It should be noted that standard computer programs are available for performing such calculations and are used when necessary by power system engineers.

A short review of the relevant engineering literature is available from the authors upon request.

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