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A Geometrical Approach to Two-Voice Transformations in the Music of Bela Bartok

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A Geometrical Approach to Two-Voice Transformations

in the Music of Béla Bartók

A Thesis Presented

By

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ABSTRACT

A GEOMETRICAL APPROACH TO TWO-VOICE TRANSFORMATIONS

IN THE MUSIC OF BÉLA BARTÓK

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A new analytical tool called “voice-leading class” is introduced that can quantify on an angular scale any transformation mapping one pitch dyad onto another. This method can be applied to two-voice, first-species counterpoint or to single-voice motivic transformations. The music of Béla Bartók is used to demonstrate the metric because of his frequent use of inversional symmetry, which is important if the full range of the metric’s values is to be tested. Voice-leading class (VLC) analysis applied to first-species counterpoint reveals highly structured VLC frequency histograms in certain works. It also reveals pairs of VLC values corresponding to motion in opposite directions along lines passing through the origin in pitch space. VLC analysis of motivic transformations, on the other hand, provides an efficient way of characterizing the phenomenon of chromatic compression and diatonic expansion. A hybrid methodology is demonstrated using Segall’s gravitational balance method that provides one way of analyzing textures with more than two voices. A second way is demonstrated using a passage from Bartók’s *Concerto for Orchestra*. Finally, the third movement of the String Quartet #5 is analyzed. Families of geometrically related VLC values are identified, and two are found to be particularly salient because of their relationship to major and minor thirds, intervals which play an important role in the movement. VLC values in this movement are linked to contour, form, motivic structure, pitch-class sets and pitch centricity, and are thus demonstrated to be useful for understanding Bartók’s music and the music of other composers as well.

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CHAPTER 1

INTRODUCTION

1.1 Context and definitions

Consider the passage from Béla Bartók's *Contrasts* shown in Figure 1 (mvt. II, mm. 1–18, violin and clarinet parts only). Note that most of the motion in this passage is contrary, one of the four categories of contrapuntal motion recognized by Fux. In fact, much of the motion—almost half, to be precise—is not only contrary, but inversionally symmetrical as well. (Inversionally symmetrical motion is indicated in the excerpt with asterisks.) Fux did not recognize inversionally symmetrical motion as a category distinct from contrary motion; nevertheless, there are compelling reasons to regard it as a category unto itself.[1]

Having discovered that inversionally symmetrical motion plays an important role in this passage, we might ask the following question: to what degree does the rest of the counterpoint in this passage approach or deviate from inversional symmetry?

Consider now the excerpt from Bartók's *Music for String Instruments, Percussion and Celesta* shown in Figure 2 (mvt. IV, mm. 28—43, outer voices only). The motion in this passage is predominantly similar; in fact it is largely parallel, not just in the diatonic sense but rather in the stronger, chromatic sense. Chromatic parallel motion accounts for more than a third of the contrapuntal motion in this excerpt, more than half if we discount repeated sonorities. (Chromatic parallel motion is indicated in the excerpt with asterisks.) By analogy with the example from *Contrasts* in Figure 1, and likewise given the knowledge that parallel motion is strongly represented in this passage, we may wonder, to what extent does the remaining

[1] Strictly speaking, Fux only recognizes three categories of motion: similar, contrary and oblique. Yet he recognizes a prohibition against parallel perfect intervals, a *de facto* recognition of parallel motion.

counterpoint in the passage approach or deviate from parallel motion? The purpose of this thesis is to introduce a precise but intuitive metric that can answer questions such as these by describing counterpoint not only qualitatively (i.e., parallel, contrary, oblique, etc…), but quantitatively as well. It does this by assigning different types of counterpoint places on an angular scale.

Figure 1. Bartók's *Contrasts,* mvt. II, mm. 1—18 (violin and clarinet parts only). Asterisks indicate inversionally symmetrical motion.

Figure 2. Mm. 28—43 of the fourth movement of Bartók's *Music for String Instruments, Percussion and Celesta* (Outer voices only). Asterisks indicate parallel motion.

The two examples discussed above embody two types of contrapuntal motion: inversionally symmetrical and parallel motion. These two types of motion are, geometrically speaking, diametrically opposed to one another. This can be seen by considering how the progression of one pitch dyad to the next is modeled by motion in an abstract two-dimensional pitch space (see Figure 3(a)). We map pitch dyads onto numerical ordered pairs by choosing zero to represent middle C and measuring the directed distance of each voice from middle C in half steps. Each pitch dyad corresponds to exactly one ordered pair in pitch space, notated

($v_1[n], v_2[n]$), where n represents ordinal position in a sequence of dyads.[2] Counterpoint can be classified by describing how each dyad progresses to the next.

As demonstrated by Dmitri Tymoczko in *A Geometry of Music* (Tymoczko 2011, 68), parallel motion is represented by motion parallel to the line $v_2 = v_1$ (a change of d units in one voice is matched by a change of d units in the other voice), and inversionally symmetrical motion is represented by motion parallel to the perpendicular line $v_2 = - v_1$ (a change of d units in one voice is matched by a change of -d units in the other voice). See Figure 3(b). Parallel and inversionally symmetrical motions are therefore geometrical opposites of one another.

Having divided the Cartesian plane into symmetrical octants (with two sets of perpendicular axes rotated 1/8 of a complete turn–45°–with respect to one another), we can now see how all types of contrapuntal motion, not just parallel and inversionally symmetrical motion, can be represented by directions in the plane (see Figure 4). Starting in the upper right quadrant and moving counter-clockwise through the other three, the types of motion represented are similar with both voices moving up, contrary with voice 1 moving down and voice 2 moving up, similar with both voices moving down and contrary with voice 1 moving up and voice 2 moving down. Within each quadrant, similar or contrary motion can be further subdivided according to whether the magnitude of the change in voice 1 is greater than or less than the magnitude of the change in voice 2 (note the labels in each octant indicating "voice 1 more" or "voice 2 more"). Finally, oblique motion is represented by motion along the voice 1 or voice 2 axis, and parallel and inversionally symmetrical motion are represented by the perpendicular axes rotated by 45° with respect to the voice 1/voice 2 axes.

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2 Since we are working in pitch space rather than pitch class space, the converse is true as well: each ordered pair in two-dimensional pitch space corresponds to exactly one pitch dyad.

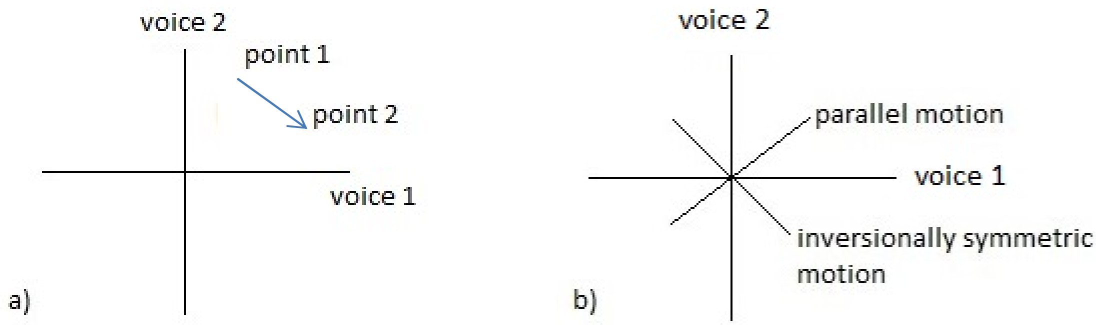

Figure 3. (a) Motion from one pitch dyad to another as represented in abstract two-dimensional pitch space. (b) Parallel and inversionally symmetrical motion represented by orthogonal axes in two-dimensional pitch space.

The proposed metric works by assigning to any dyad-to-dyad progression (which will hereafter be referred to as a "two-voice transformation") a direction in the plane, which not only describes in qualitative terms the type of contrapuntal motion involved, but actually quantifies the counterpoint on the angular scale shown in Figure 4.

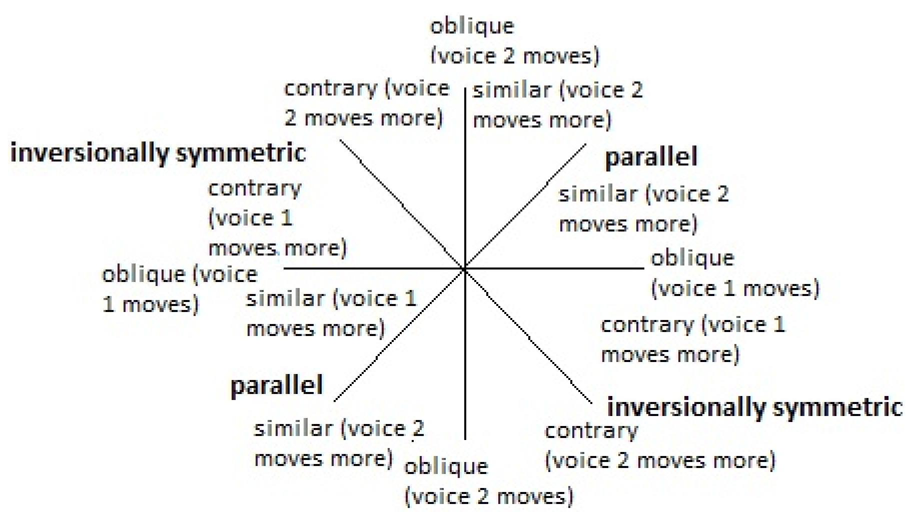

Figure 4. The various categories of motion represented by directions in the Cartesian plane.

To see how this can be done, refer to Figure 5. Pitch dyads are converted to numerical ordered pairs as above. The vector connecting the point represented by the first ordered pair (in which voices 1 and 2 have the argument "n") to the point represented by the second ordered pair (in which voices 1 and 2 have the argument "n+1") is calculated by subtracting the first ordered pair from the second. The tail of the resulting vector is placed at the origin, and the angle θ_n represents the angle that vector makes with the v_1 (horizontal) axis, with positive values measured in a counter-clockwise direction. Thus, θ_n specifies a direction in the plane for the transformation from the n^{th} dyad in a series to the subsequent one, and is given by the formula:

$$\theta_n = \arctan[(v_2[n+1]-v_2[n])/ (v_1[n+1]-v_1[n])].$$

Changing the direction of a difference vector by 180° does not change the value of the arctan function, because it changes the signs of both the numerator and the denominator of the arctan function's argument. For angles corresponding to difference vectors in the second (upper left) quadrant, therefore, we must add 180° to the angles calculated using the formula, and for angles corresponding to difference vectors in the third (lower left) quadrant we must subtract 180° from the angles calculated using the formula. The result is a function that maps pairs of pitch dyads onto angles in the range $-180° < \theta_n \leq 180°$. The choice of which of this angular segment's endpoints to include and which to exclude is arbitrary.

Since the same angle value corresponds to many vector differences between ordered pairs (positive integer multiples of a given difference vector will all have the same angle value), we adopt the term "voice-leading class" (or "VLC") to refer to the set of two-voice transformations described by a particular angle value. The value itself will be referred to as "voice-leading class value" or "VLC value."

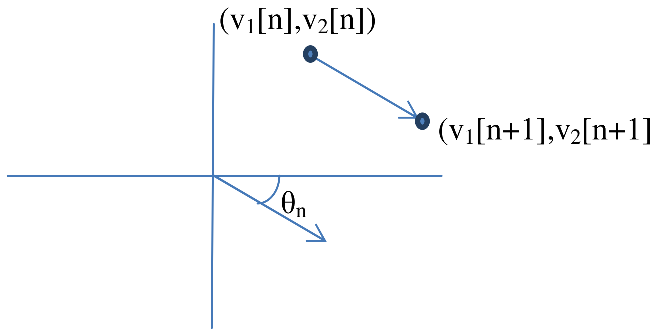

Figure 5. Set-up for calculating the angular metric;
this shows a negative value of θ_n.

A few examples will help to clarify how the metric works.

Example 1.

Suppose that we are given two pitch dyads, (C4,G4) and (F#4,C#5). Converting to numbers we obtain $(v_1[n],v_2[n]) = (0,7)$ and $(v_1\,[n+1],\,v_2\,[n+1]) = (6,13)$. Using the formula given above for θ_n, we obtain $\theta_n = \arctan(6/6) = \arctan(1) = 45°$, which corresponds to parallel motion.

Example 2.

Suppose we start with the pitch dyads (D3,A3) and (F3,G#3). Converting to numbers we obtain $(v_1\,[n],\,v_2\,[n]) = (-10,-3)$ and $(v_1\,[n+1],v_2\,[n+1]) = (-7,-4)$. Then $\theta_n = \arctan(-1/3) = -18.45°$, which corresponds to contrary, but not inversionally symmetrical, motion. Note that in this region of the v_1-v_2 plane, there is more motion in the v_1 direction than in the v_2 direction, whereas an angle value between -45° and -90° would indicate more motion in the v_2 direction than in the v_1 direction.

Example 3.

Suppose we are given the two pitch dyads (F3,G#3) and (D3,A3). Converting to numbers we obtain $(v_1\,[n],\,v_2\,[n]) = (-7,-4)$ and $(v_1\,[n+1],\,v_2\,[n+1]) = (-10,-3)$. Then $\theta_n = \arctan(1/-3) = -18.45°$, the same as in the last example. In this example, though, the vector difference between

(v_1 [n], v_2 [n]) and (v_1 [n+1], v_2 [n+1]) lies in the second quadrant; thus to calculate VLC value, we add 180° to the value calculated above, obtaining $\theta_n = 161.55°$.

Example 4.

Suppose we start with the pitch dyads (E2,G5) and (E2,G#5). Converting to numbers we obtain (-20,19) and (-20,20). Then $\theta = 90°$ (this must be manually calculated as a limiting value of the arctan function, since following the formula exactly entails dividing by zero). This corresponds to oblique motion in the v_2 direction.

Further basic properties of voice-leading class are discussed in Appendix A.

Having motivated and described the metric to be demonstrated in the rest of this thesis, we now provide brief surveys of geometrical methods in music theory and symmetry in the music of Bartók in order to situate the metric within a historical context.

1.2 Geometrical Methods in Music Theory

Since the time of Pythagoras, geometrical models and concepts have played a role in shaping music theory. Followers of Pythagoras ascribed great importance to the *tetractys,* a geometrical means of representing the number 10 (see Figure 6). The ratios of string lengths representing consonant intervals (the fourth, fifth, octave, twelfth and fifteenth) could be obtained as the ratios between the numbers represented by the four rows of the *tetractys* : 1, 2, 3 and 4 (Mathiesen 2010, 115).

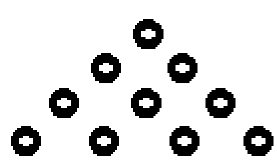

Figure 6. The Pythagorean *tetractys.*

Although the fact is often taken for granted, even the basic form of standard musical staff notation is implicitly geometrical, with the temporal dimension represented by a horizontal axis and the frequency or pitch dimension represented by a vertical axis. An important step in the evolution of modern staff notation was found in the two well-known treatises from the ninth or tenth centuries, the *Musica enchiriadis* and the *Scholica enchiriadis*, in which horizontal lines represented various pitch levels (Cohen 2010, 329-30).

An important conceptual leap in the progress of geometrical reasoning about music was made in 1711 when Johann David Heinichen first published the circle of fifths (Barnett 2010, 444). What this represented was a concept expressed in its own natural language: a linear progression of major and minor keys was transformed by the advent of equal-tempered tuning to become cyclical, and connecting the end points of a line segment to form a circle demonstrated this transformation perfectly.

More recent geometrical devices for displaying pitch relationships include Roger Shepard's helical model representing octave equivalence without sacrificing registral distinctions completely, and his double-helical model representing the information contained in the circle of fifths without sacrificing registral distinctions completely (Hook 2002, 125-6).

Joseph N. Straus, writing in 2011, stated that "in recent years, music theory has taken a geometrical turn, entering what might be called a new space age" (Straus 2011, 46). He was referring in large part to the sub-discipline that falls under the heading of "neo-Riemannian Theory", an updated version of work done in the nineteenth century by Hugo Riemann. Richard Cohn, in his introduction to an issue of the *Journal of Music Theory* devoted to the subject, describes the foundations of neo-Riemannian theory thus:

> The neo-Riemannian response recuperates a number of concepts… [including]...triadic transformations, common-tone maximization, voice-leading

> parsimony, "mirror" or "dual" inversion, enharmonic equivalence, and the "Table of Tonal Relations"….Neo-Riemannian theory strips these concepts of their tonally centric and dualist residues, integrates them, and binds them within a framework already erected for the study of the atonal repertories of our own century (Cohn 1998, 169).

A crucial step in the emergence of neo-Riemannian Theory was a dissertation written in 1989 by Bryan Hyer that revived the "Table of Tonal Relations" or *Tonnetz.*

Although Hugo Riemann made the *Tonnetz* famous, the first one was introduced by the mathematician Leonhard Euler (Tymoczko 2011, 412); another early example was introduced by Gottfried Weber (Bernstein 2010, 784–6). Many *Tonnetze* followed, including one introduced by Schoenberg that is very similar to Weber's version (Bernstein 2010, 804). These tables showed relationships between keys such as dominant/sub-dominant, relative major/minor and parallel major/minor. Hyer's *Tonnetz* is in the form of a four-dimensional hyper-torus, wherein the four dimensions correspond to dominant relationships, Leittonwechsel relationships, parallel relationships and relative relationships (Hyer 1989, 210).

Finally, any discussion of geometrical methods in music theory would be remiss if it did not outline some of the contributions made by Dmitri Tymoczko in his book, *A Geometry of Music* (Tymoczko 2011). These contributions include the description and quantitative (geometrical) measurement of voice-leading in the context of two or more voices. In that respect, the present work is in some sense an alternative to the methods of analysis that Tymoczko employs in the analysis of two-voice counterpoint.

For example, since Tymoczko works predominantly with pitch classes, the musical spaces he employs are often more abstract than the two-dimensional non-periodic pitch space used here. Tymoczko shows that two-dimensional pitch-class space is topologically equivalent to the Möbius strip (Tymoczko 2011, 69), and higher-dimensional pitch-class spaces are shown to be equivalent to higher-dimensional analogs of the Möbius strip (Tymoczko 2011, 85-115). As discussed in the next section, however, for our purposes there are compelling reasons to work in pitch space rather than pc-space.

1.3 Symmetry and the music of Bartók

In order to test the validity of voice-leading class as an analytical metric, musical examples that utilize the full 360 degrees of VLC's range are needed. It is for this reason that Bartók's music has been selected for the present study: because of his frequent and variegated use of contrapuntal symmetry, it is hypothesized that his music will better meet the above requirement than the work of a composer who does not frequently employ symmetry. That is because music which makes frequent use of symmetry puts inversionally symmetrical motion on an equal footing with parallel motion, thus utilizing all of VLC's possible values.

Uses of contrapuntal symmetry in Western music trace their origins far back in history. Certainly, by the time of Bach, examples were plentiful. For example, the fugue in B flat minor from Book II of the *Well-Tempered Clavier* contains inverted versions of the subject—a subject which is well-suited to inversion because of its use of tritones, which remain the same under inversion (see Figure 7). As a matter of fact, this fugue features a recasting of the entire exposition in inverted form, and a dual mirror presentation of the fugue subject near the end. See Gauldin (1988, 194—197) for a discussion of mirror inversion in fugues.

Bartók's contemporaries Schoenberg, Berg, and Webern came upon the notion of symmetry as part of a new way of organizing music in the post-tonal era. Bartók's use of symmetry was different, as it was informed by two other traditions that had less of an influence on his Germanic contemporaries: the folk idioms of his native Hungary and the impressionism of French composers including Debussy (Antokoletz 1984, 2). The music of Debussy, influenced by the music of the Russian nationalist composers Glinka, Borodin, Rimsky-Korsakov and

Mussorgsky, incorporated symmetrical intervallic spaces in novel ways (Perle 1955, 190–1). Debussy's well-known use of whole-tone scales is an example, but also noteworthy is his use of octatonic scales (Forte, 1991). The synthesis of Hungarian folk idioms, French impressionism, and the late tonal languages of Wagner and Strauss resulted in a unique style which, though it paralleled in certain respects the style of the Second Viennese School, was entirely Bartók's own (Antokoletz 1984, 20-1).

Figure 7. The subject of Bach's Fugue No. 22 from Book II of the *Well Tempered Clavier* in its original form, mm. 1—4 (top) and inverted form, mm. 42—45, (bottom).

Bartók's early use of symmetry began by highlighting the symmetrical aspects already present in Hungarian folk songs: minor-seventh chords and pentatonic scales (Antokoletz 1984, 29). Furthermore, he frequently transformed diatonic melodies into symmetrical orderings of pitches by fourth or fifth (Antokoletz 1981, 9-10). These techniques can be found in the Fourteen Bagatelles for piano and the Eight Improvisations for piano (Antokoletz 1984, 28–32 and 55–62). In the third Improvisation, Bartók utilizes a symmetrical scale (the second mode of the acoustic scale), and foregrounds the enharmonically spelled augmented triad based on the first scale degree, a symmetrical vertical sub-structure assembled from the symmetrical horizontal structure of the scale (Antokoletz 1984 60–62).

His later (and arguably more developed) uses of symmetry consist of inversionally related tones which may or may not be subsets of symmetrically structured scales other than the twelve-note chromatic scale. They include the symmetrical distribution of notes on which the fugue subject enters in the first movement of the *Music for String Instruments, Percussion and Celesta* (Bernard 1986, 188). This is an example of "registrally represented symmetry," which is taken to be "more significant structurally" than symmetry that is *not* "registrally represented" (Bernard 1986, 186; see also Bernard 2003). Bartók himself stated that registral placement of symmetrical notes is "crucial to their effect" (Bernard 1986, 188, from Bartók 1920). For this reason, among others, pitch space rather than pitch-class space is used in this study.

Bernard points out that "it should be possible…to discover a hierarchy of relationships in which smaller symmetries contribute to larger ones" (Bernard 1986, 192). He goes on to identify one such case, *Mikrokosmos* No. 141, in which "the individual [local] axes of symmetry, if taken as a series, form a symmetrical pattern of their own" (Bernard 1986, 188). Finally, Robert Katz points out a similar situation in *Mikrokosmos* No. 143, where "the primary axis or tonal center of the work is established by the means of the complementary symmetrical balancing of subsidiary axes" (Katz 1993, 333–4).

Thus, drawing on a Germanic tradition more or less shared with his coevals Schoenberg, Webern, and Berg, Bartók incorporated Hungarian folk music and French Impressionism as well, creating a sophisticated language that made frequent use of symmetrical melodic and harmonic inversion. It is for this reason that Bartók's music has been chosen for this study.

The next chapter demonstrates how voice-leading class can be used for analysis, with source materials supplied by *Constrasts* and *Music for String Instruments, Percussion and Celesta.* Chapter 3 shows how VLC analysis can be applied to the phenomenon of chromatic compression and diatonic expansion. Chapter 4 situates voice-leading class analysis in the context of other transformational approaches, identifying a promising hybrid methodology. Chapter 5 presents an alternative to voice-leading class for use with more than two voices. Finally, Chapter 6 presents a case study of VLC analysis applied to the *Scherzo* of the String Quartet #5.

CHAPTER 2

ANALYSIS USING VOICE-LEADING CLASS

2.1 Overview

In this chapter, the two examples from Chapter 1 (*Contrasts* and *Music for String Instruments, Percussion and Celesta)* will be analyzed using the tools of voice-leading class. These tools include: 1) the identification of geometrically related pairs of VLC values in a given set of data, 2) VLC multiplicity analysis, in which multiple difference vectors corresponding to a single VLC value are taken as evidence of a special importance for that value in structuring the passage in question–recall that one VLC value can correspond to more than one difference vector between dyads–and 3) VLC frequency analysis, which counts the number of occurrences of each VLC value that appears in a given set of data. Global properties of voice-leading class—in particular, the forms of highly structured VLC frequency histograms—are taken as evidence that voice-leading class is a meaningful musical metric.

2.2 Contrasts

The first example is the passage from *Contrasts* discussed above (mvt. II, mm. 1–18, violin and clarinet only). We begin by tabulating the data (see Table 1), with successive columns for dyad number, n^{th} violin pitch, n^{th} clarinet pitch, and VLC value corresponding to ordered pairs n and n+1.

There are several observations we can make that might have a bearing on how we understand this passage and how a performer should approach it. In particular, we hypothesize that motion in opposite directions along a line in pitch space should be heard and played in such a way as to emphasize this relationship; we observe several such directional pairs in the data. For example, the first and sixth VLC values, the third and seventh VLC values, the fifth and eighth VLC values, the ninth and twelfth VLC values, and the fourteenth and seventeenth VLC values are all supplementary angle pairs. Furthermore, in the case of the third and seventh VLC values, this property can only be deduced using voice-leading class or another metric with the same information content, because the difference vector corresponding to one direction along the line in pitch space, (2,-4), is negative two times the difference vector corresponding to the other direction along the line in pitch space, (-1,2). In other words, the ratio of values is what determines the correspondence here, not the actual values themselves.

Table 1. Ordered pitch pairs and corresponding VLC values from *Contrasts*, mvt. II, mm. 1—18.

Dyad #	Violin	Clarinet	VLC Value	Dyad #	Violin	Clarinet	VLC Value
1	-3	6	-33.69	26	0	-8	0.00
2	0	4	-36.87	27	30	-8	135.00
3	4	1	-63.43	28	7	15	-45.00
4	6	-3	123.69	29	9	13	-45.00
5	4	0	126.87	30	12	10	135.00
6	1	4	146.31	31	9	13	135.00
7	-2	6	116.57	32	6	16	-45.00
8	-3	8	-53.13	33	7	15	-45.00
9	0	4	116.57	34	10	12	168.69
10	-1	6	-12.72	35	5	13	-45.00
11	30	-1	154.36	36	7	11	-45.00
12	5	11	-63.43	37	10	8	135.00
13	6	9	-90.00	38	7	11	135.00
14	6	7	-26.57	39	4	14	135.00
15	8	6	0.00	40	3	15	-45.00
16	10	6	-45.00	41	9	9	-84.29
17	11	5	153.43	42	10	-1	111.80
18	9	6	116.57	43	8	4	135.00
19	7	10	153.43	44	5	7	-36.87
20	5	11	135.00	45	9	4	-56.31
21	3	13	-45.00	46	11	1	135.00
22	4	12	-108.43	47	10	2	135.00
23	-2	-6	-45.00	48	8	4	-45.00
24	-1	-7	-45.00	49	10	2	135.00
25	1	-9	135.00	50	7	5	

Next we turn to a VLC multiplicity analysis, which identifies any VLC values that correspond to more than one distinct difference vector. See Table 2. There are two pairs of VLC values that have more than one difference vector, the pairs corresponding to motion in opposite directions along two axes in pitch space. One of these axes represents inversionally symmetrical motion and the other was identified above as corresponding to one of the geometrically related pairs of VLC values. Thus, the multiplicity tool supports both the significance of inversional symmetry in this passage and the significance of identifying geometrically related pairs of VLC values.

Table 2. VLC multiplicity analysis for *Contrasts*, mvt. II, mm. 1—18.

VLC Value	Multiplicity
-45.00°	3
-63.43°	2
135.00°	4
116.57°	2

We turn now to a VLC frequency analysis of the passage; this simply entails counting how frequently each voice-leading class value appears in the passage. The results are shown as a histogram in Figure 8. The form of this histogram bears further investigation. First of all, we see that the data are bimodal, centered on -45° and 135°. This is consistent with our observation that inversional symmetry is important in this passage.

That is not the only observation we can make about the data, however; it is tempting to superimpose two normal distributions on this histogram as shown in Figure 9. This data is highly sensitive with regards to the initial conditions used to fit Gaussian curves to it, though.[3] Therefore the simple expedient of substituting the means and standard deviations of the two

[3] This was confirmed by a statistician working for Stata Corp., maker of the statistical software used to generate the graphs in this thesis.

clearly visible clusters into the expressions for two normal distributions was used to generate Figure 9. The structure of this histogram, approximated here by two normal distributions, suggests that voice leading class is a meaningful metric.

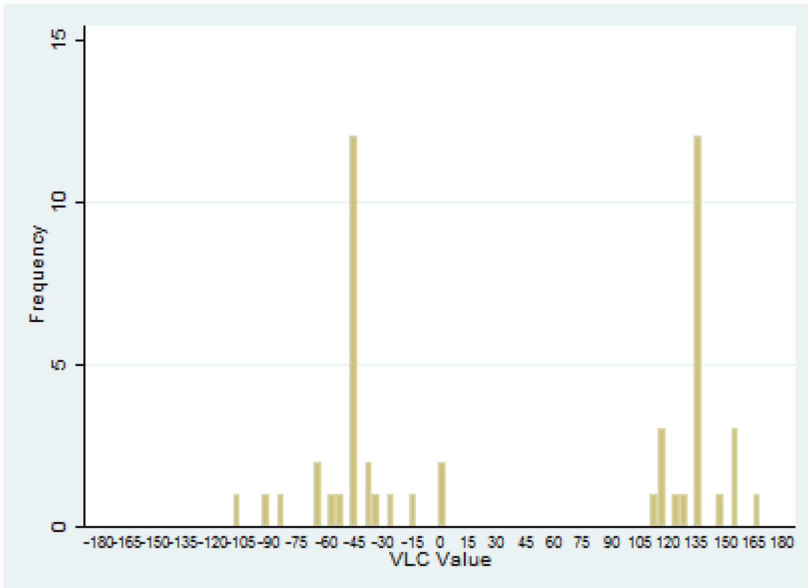

Figure 8. VLC frequency analysis for the violin and clarinet parts, mm. 1—18 of the second movement of *Contrasts*.

2.2.1 Validity of Voice-Leading Class as a Metric

In this section, we discuss the proposition that the structure inherent in the VLC frequency histogram for *Contrasts* proves that voice-leading class is a valid numerical metric that has the potential to reveal meaningful information about musical passages. We begin with the assumption that the composer–intentionally or intuitively–was listening for two-voice transformations that are, in some musical sense, "close" to inversionally symmetrical ones. This is a reasonable assumption to make given the clearly visible clusters of VLC values around θ = -45° and θ = 135°; it seems unlikely that this type of structure would manifest itself without the action of a compositional will.

Given that there are clusters of VLC values around θ = -45° and θ = 135°, which we take as evidence of compositional will, intentional or intuitive, what else does the histogram reveal? I propose that it reveals as fact that proximity of voice-leading class value indicates similarity of an innate musical quality of two-voice transformations—which we have called "voice-leading class". To see why, imagine for a moment that listeners do not perceive voice-leading class at all, but only perceive *categories* of voice-leading such as parallel, similar, inversionally symmetrical and contrary. In this case, although it might be reasonable to expect peaks in the VLC histogram corresponding to inversionally symmetrical motion, it is unlikely that there would be clusters of VLC values on either side of those peaks, because presumably, listeners would not hear transformations with VLC values *close* to -45.00° or 135.00° as being, in some musical sense, *close* to transformations manifesting inversional symmetry. The fact that the VLC frequency histogram for this example exhibits this type of quasi-symmetrical structure about its peaks therefore seems to indicate that VLC value measures a musically significant property and is thus a useful metric.

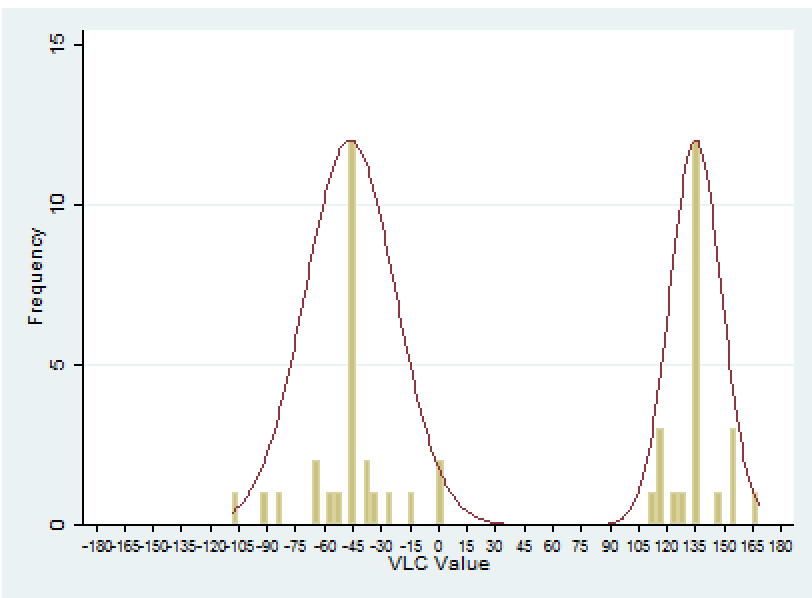

Figure 9. Estimated fit of two normal distributions to data from the second movement of *Contrasts*.

2.3 *Music for String Instruments, Percussion and Celesta*

The second example is the passage from the *Music for String Instruments, Percussion, and Celesta* discussed in Chapter 1 (fourth movement, mm. 28–43), during which a double string quartet plays in rhythmic unison. For the sake of simplicity, only the soprano and bass voices (i.e. violin I and ‘cello II) are analyzed here. The VLC frequency histogram for this passage is shown in Figure 10. (The raw data for this example, as well as for the examples that follow, may be found in Appendix B).

Note that, in Figure 10, the data have been fit with two Gaussian curves. As was the case in the previous section, the data for this passage are extremely sensitive with respect to initial conditions, so the parameters for the Gaussian distributions were supplied by simply calculating the means and standard deviations of the two clearly visible clusters in the data. Note that the curve on the right is shifted slightly to the right of the peak at 45°; this is due to the adjacent columns in the histogram near 115°.

Note that the distribution centered on 45° has a much wider spread (larger standard deviation) than the one centered on -135°. Musically, this might mean that two-voice transformations with VLC values close to -135° are much more perceptible than transformations with VLC values close to 45°. In fact, the distribution centered on 45° might not even be perceptible at all because it is so wide. This is perhaps a situation in which perception and cognition experiments might be useful in revealing the nature of voice-leading class and its usefulness as a metric.

The VLC multiplicity tool supports the primacy of parallel motion in structuring this passage; the results are shown in Table 3. The VLC frequency and multiplicity analyses go hand-in-hand: the only VLC values for which non-trivial multiplicities occur are also the VLC values for which the frequency histogram attains its highest values.

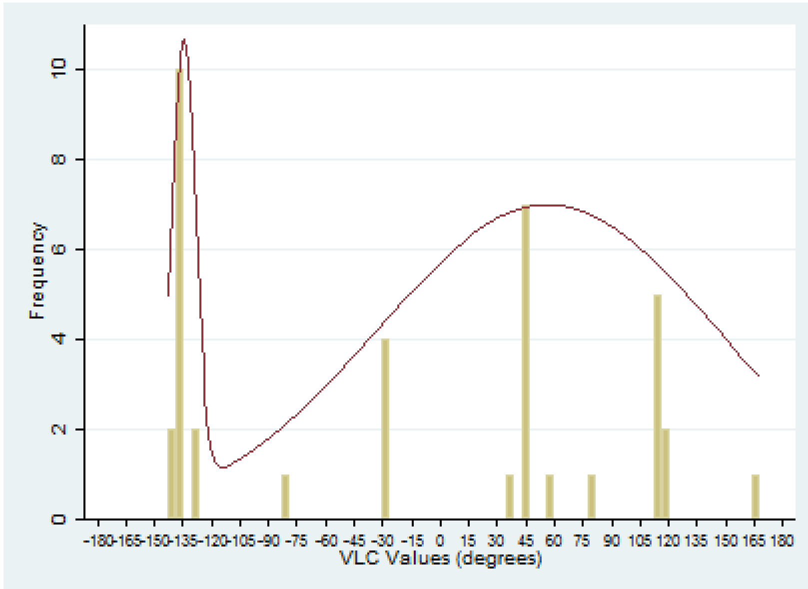

Figure 10. VLC frequency histogram with estimated Gaussian curvefit for *Music for String Instruments, Percussion, and Celesta,* mvt. 4, mm. 28—43, outer voices.

Table 3. VLC multiplicity analysis for *Music for String Instruments, Percussion and Celesta* mvt. 4, mm. 28—43

VLC Value	Multiplicity
-135.00°	5
45.00°	2

CHAPTER 3

CHROMATIC COMPRESSION AND DIATONIC EXPANSION

3.1 Motivic transformation and VLC analysis

In its original formulation, voice-leading class is rooted in the quantitative description of two-voice, first-species counterpoint. However, the tools of voice-leading class, once defined, can be removed from their original context and used abstractly to analyze other musical situations.[4] For example, the tools of voice-leading class can be used to analyze motivic transformation. A special case of motivic transformation, frequently encountered in Bartók and discussed later in this chapter, is the phenomenon of chromatic compression and diatonic expansion.

The study of motivic transformation has a long history in music theory. In the last hundred years, contributors to the field have included Arnold Schoenberg (for a synopsis see Carpenter and Neff 1995, 15-44), Robert Morris (1987) and Ian Quinn (2001). Morris provides a summary of several measures of similarity (called similarity relations) between motives expressed as pc-sets, and Quinn makes the case for a greater degree of affinity between those measures than is ordinarily assumed.

Here it is shown that voice-leading class provides yet another way to quantitatively describe the relationships between two forms of a motive. In order use voice-leading class this way, the same formula for θ is used as in Chapter 2, but the dyads used in the formula are constructed differently. Instead of constructing them from pairs of vertically aligned pitches, they

[4] Even within the context of first-species counterpoint, a variety of musical situations can arise. For example, in the excerpt from *Contrasts* discussed in Chapters 1 and 2, the piano part does not interfere with the first-species counterpoint between the violin and clarinet. Alternatively, there can be more than one pair of voices in note-against-note counterpoint within a single texture, as is the case in the third movement of the String Quartet #4.

are constructed by taking successive pairs of corresponding pitches from the pair of motive forms under consideration. This is illustrated in Figure 11. (At times it will be useful to treat two successive motive forms as though they were vertically aligned, as a graphical and conceptual expedient.)

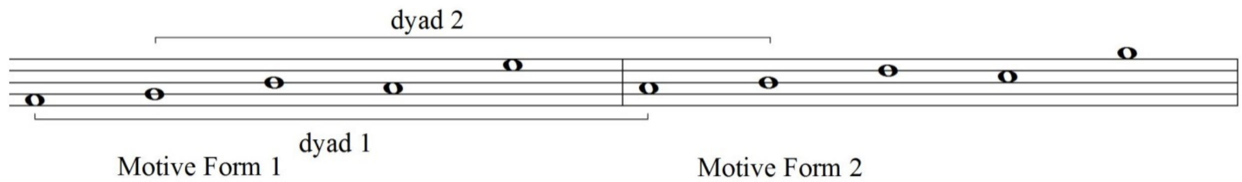

Figure 11. Construction of dyads for the analysis of motivic transformation.

It is a bit of a conceptual leap to go from note-against-note counterpoint to motivic transformation wherein the two forms of the motive to be analyzed are not vertically aligned. In fact, one may even question the audibility of any structure ascribed to such a transformation by VLC analysis. In the case of vertically aligned dyads, transformations are clearly audible as two-voice counterpoint, but in the case of horizontally displaced dyads, the aural identity of the transformations may not be so clear. This does not mean, however, that motivic transformations as described by voice-leading class are not meaningful.

Jonathan Kramer makes this point more generally, arguing that the fact that a listener cannot *hear* a given structure in a piece of music does not imply that the structure is musically unimportant: "[The analyst] realizes that some things which cannot literally by 'heard'—that is, cannot be accurately identified, named, or notated—may still have discernible musical reasons for being in a piece" (Kramer 1988, 328). As an example he cites the 12-tone rows used by the serialist composers, which are often times inaudible to all but the most expert listeners.

It would seem that perception and cognition experiments might be in order to test whether or not listeners can detect structures involving voice-leading class. In this regard, it is worth noting that little is currently known about the perception of the type of melodic structures considered here—structures that Patel refers to below as “parallel”. Patel makes this point as follows:

> The study of parallelism is based on the measurement of structural and perceptual similarity. Since similarity is a matter of degree, and is influenced by many factors, the study of parallelism has lagged behind other aspects of music which have more discrete and easily measurable characteristics....Thus despite its fundamental role in melodic perception, parallelism and its perception is only beginning to be investigated in a quantitative framework (Patel 2003, 328-9).

Perhaps our understanding of voice-leading class analysis as applied to motivic transformations will improve as our understanding of the perception of parallelism improves.

3.2 Chromatic compression and diatonic expansion

The music in Figure 12 is taken from *Mikrokosmos* No. 64. Note that Figure 12(b) is an intervallically compressed version of 12(a). This is an example of what is called chromatic compression—its opposite is called diatonic expansion.

Figure 12. (a) *Mikrokosmos* No. 64(a). (b) *Mikrokosmos* No. 64(b).

The concept of chromatic compression and diatonic expansion was introduced by Bartók in a lecture he gave at Harvard in 1943:

> The working with these chromatic degrees gave me another idea which led to the use of a new device. This consists of the change of the chromatic degrees into diatonic degrees...You know very well the extension of themes in their value called augmentation, and their compression in value called diminution...Now, this new device could be called 'extension in range' of a theme. For the extension we have the liberty to choose any diatonic scale or mode (Bartók 1976, 381).

This chapter explores the phenomenon of diatonic expansion and chromatic compression from three points of view: the modular transformation method of Santa, the pitch-cell method of Antokoletz, and the present method of VLC analysis. In comparison to the other two methods, VLC analysis is shown to offer a precise and intuitive, though non-explanatory, description of the phenomenon.

3.3 Santa's MODTRANS Function

To begin, consider how Matthew Santa models chromatic compression and diatonic expansion. He introduces a function, MODTRANS, with which any scalar module may be mapped onto any other by specifying a point of synchronization and counting up or down through what Santa calls "step classes" in each module (Santa 1999, 202). In the example given above, *Mikrokosmos* No. 64, the minor (diatonic) scale of 64(a) maps onto the chromatic scale of 64(b) with E as the point of synchronization. See Figures 13(a) and 13(b), wherein the numbers beneath each staff indicate step class.

VLC analysis provides an alternative to MODTRANS: we simply calculate VLC values using the series of steps described in Section 3.1 for describing motivic transformation. The results of this calculation are shown in Figure 13(c), where the two forms of the melody have been vertically aligned with one another. Note that where half-steps map to half-steps, the relevant VLC value is 45°, corresponding to "virtual parallel motion", but where two intervals are comprised of different numbers of half-steps, the relevant VLC value *differs* from 45°.[5] Thus VLC analysis makes explicit both where compression occurs and how much compression occurs.

Figure 13. A simple example of chromatic compression, *Mikrokosmos #64*, analyzed using Matthew Santa's MODTRANS function (a and b) and voice-leading class (c).

Before continuing, let us deduce what we might expect to obtain from a VLC analysis of this phenomenon in general. Suppose we allow voice 2 to represent an intervallically expanded form of a given motive in voice 1; that is, suppose that for every interval between notes in the first voice, the corresponding interval in the second voice not only has the same direction as the interval in the first voice, but is greater than or equal to it in magnitude. Then, by referring to Figure 4 (page 5), we see that VLC values for this pair of voices will all lie in one or both of the two ranges [45°,90°] or [-135°,-90°] where the end-points are included to allow parallel and oblique motion as forms of expansion. If, instead of being an expanded version of the first voice,

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5 If the half steps were descending, the relevant VLC value would be -135°.

the second voice is a compressed version of the first voice, all VLC values will lie in the range [0°,45°] or [-180°,-135°].

An alternative situation occurs when a motive is not only expanded, but also inverted. In this case, we are interested in the regions in the plane corresponding to contrary motion with the label "voice 2 more": [90°,135°] and [-45°,-90°]. And if voice 2 is inverted and *compressed* with respect to voice 1 we are interested in regions of the plane corresponding to contrary motion with the label "voice 1 more": [-45°,0°] and [135°,180°]. We will see an example of inversion and expansion below.

3.4 *Music for String Instruments, Percussion and Celesta*

The next, more complex example comes from Bartók's *Music for String Instruments, Percussion and Celesta*, in which the chromatic fugue subject from the first movement reappears at the end of the fourth movement expanded into a form based on the C acoustic scale. These two related melodies are shown in Figures 14(a) and 14(b). A number of authors have analyzed the relationship between them, including Elliot Antokoletz and Matthew Santa.

Antokoletz posits that three symmetrical pitch-class cells, customarily labeled X, Y, and Z, serve as a bridge between the chromatically compressed fugue subject and its diatonically expanded counterpart (2006).[6] He also demonstrates how the two tritones in the C acoustic scale (Bb–E and C–F#) are compressed back into the two tritones separated by a half-step that together comprise cell Z, preparing a return to the chromaticism of the opening movement to end the piece (Antokoletz 1984, 134—7).

[6] The first two pitch-class cells, X and Y, were introduced by Perle (1955), and the third cell, Z, was introduced by Treitler (1959).

Figure 14. Fugue subject from *Music for String Instruments, Percussion, and Celesta.* Shown in original (compressed) form (a); in diatonic (expanded) form (b); and in both forms compared to one another (c).

Santa, on the other hand, uses his MODTRANS function to describe how the chromatic scale of the first version of the melody maps onto the acoustic scale of the second version of the melody–see the numbers corresponding to “step classes” beneath Figures 14(a) and 14(b) (Santa 1999, 207). The one-to-one correspondence between step classes is impressive.

Figure 14(c) displays the two forms of the fugue subject vertically aligned with one another. The voice-leading class values calculated according to the steps outlined above are displayed beneath the staff. They provide a direct description of the transformation as it appears on the musical surface, whereas the pitch-cell method and the MODTRANS method operate on a higher structural level. Thus the other methods provide an explanatory function that VLC analysis lacks, but also preclude efficient, direct descriptions of the transformation as it actually occurs.

Note that, since we have chosen to calculate VLC values corresponding to a transformation from the form of the subject on the lower staff to the form of the subject on the upper staff, all values of θ are between 45° and 90° or -90° and -135°. This is in accordance with the expectations discussed earlier.

3.5 Inversion and expansion

We turn, finally, to an inverted and expanded version of the fugue theme from the *Music for String Instruments, Percussion and Celesta* (mvt. IV, mm. 209—212); this is illustrated on the top staff in Figure 15. Note that, because of the inconsistent use of half steps in this excerpt (the same pitch, A3, in the original form maps to two different pitches in the modified form), it would be impossible to assign the modified form to a particular scale or scalar modulus, and thus it would be impossible to use Santa's MODTRANS approach. While Antokoletz does analyze this melody in terms of his three prominent symmetrical pitch class sets, voice-leading class offers a simple and highly descriptive way of characterizing the transformation of the original fugue theme (given on the bottom staff of Figure 15) into this inverted and expanded version. VLC values for this transformation are shown beneath the staff in Figure 15. Note that the VLC values for this transformation are all in the ranges from 90° to 135° or -45° to -90°, consistent with the previous discussion.

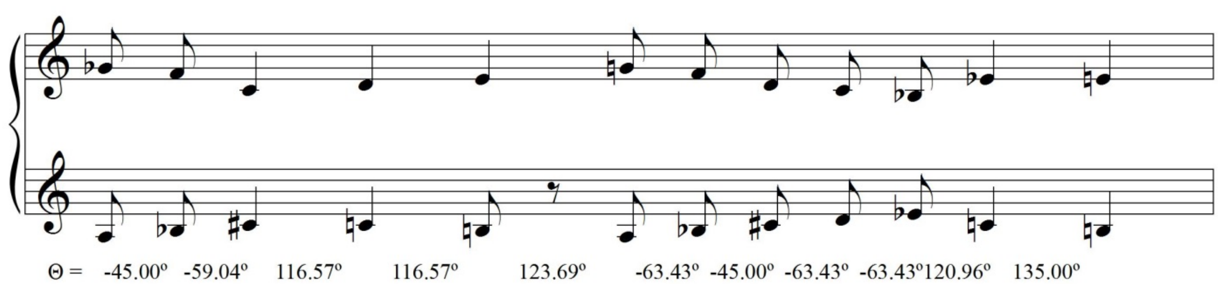


Figure 15. Inverted and expanded version of the fugue subject from the *Music for String Instruments, Percussion and Celesta* (top staff), original form (bottom staff) and VLC values (below bottom staff)

Thus, VLC analysis provides a clear, intuitive and precise way of characterizing chromatic compression and diatonic expansion, clearly indicating where compression or expansion occurs and to what extent, without being reliant on any particular scales or pitch-class sets.

CHAPTER 4

OTHER TRANSFORMATIONAL APPROACHES

4.1 Overview

In this chapter we compare the method of voice-leading class to several other transformational approaches, including Klumpenhouwer networks, O'Donell's dual transformation approach, and Segall's gravitational balance approach.[7] The gravitational balance method is found to be compatible with voice-leading class and a hybrid methodology is proposed.

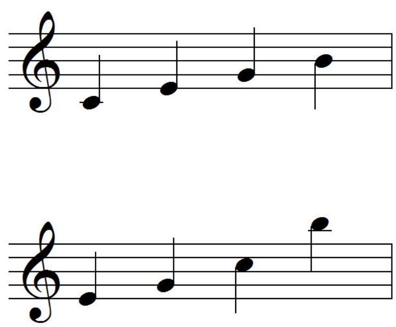
4.2 Klumpenhouwer Networks and Dual Transformations

The first approach considered here models intervallic relationships using tetrachordal Klumpenhouwer networks, or k-nets (Lewin 1992). K-nets provide a means of illustrating transpositional and inversional relationships among groups of pitch classes in such a way that musically interesting features may be easily discerned. A given set of pitch classes may be described by multiple k-nets, depending on the intra-set relationships the analyst wishes to highlight. Conversely, due to the fact that k-nets deal only with pitch classes and not actual pitches, a single k-net necessarily corresponds to many pitch sets. Figure 16 illustrates this property.

Although it may at first seem that a tetrachordal k-net possessing a pair of T operations on opposite edges of a graph might readily yield a VLC value equal to arctan(n/m) for T operations with subscripts m and n, bear in mind that k-nets utilize relationships in pc-space

[7] References for Klumpenhouwer networks include Lewin (1992) and Lewin (2002). Dual transformations are discussed in O'Donnell (1997). The gravitational balance approach may be found in Segall (2010).

rather than pitch space. Thus, while k-nets embody more information than voice-leading class in some ways, such as the analyst's interpretations of certain pc-relationships as being either transpositional or inversional, in other ways they contain less, since information on register is lost in constructing a k-net. Therefore, k-nets are incompatible with voice-leading class analysis.[8]



$$\begin{tikzcd} C \arrow[r, "T_4"] \arrow[d, "T_7"'] & E \arrow[d, "I_3"] \\ G \arrow[r, "T_4"'] & B \end{tikzcd}$$

Figure 16. Two pitch sets distinguished from each other
by register are modeled by the same Klumpenhouwer network
(one of many that could be used to model these pitch sets).

The second approach considered here is the dual-transformation approach introduced by O'Donnell (O'Donnell 1997). In a dual transformation, horizontally adjacent sets are each partitioned into two vertically adjacent subsets, and two T/I operations are grouped into a single "dual transformation" that operates on the first set by assigning one operation to each of the vertically adjacent subsets. Although it may seem that a dual transposition, notated $<T_m / T_n>$,

[8] If we were to modify K-nets, by including information on register–thus constructing them with pitches rather than pitch classes–they would become compatible with VLC analysis, but only superficially. That is to say they would at best merely become intermediate stages between the musical surface and the calculation of VLC values.

might readily yield an angle value - arctan(n/m) - that has an analogous interpretation to the two-voice case, again we are confounded by the use of pitch-class space rather than pitch space. [9]

4.3 Gravitational Balance and a Hybrid Methodology

An approach that does not compromise information on register, but includes it in a context based on k-nets, is the "center of gravity" method proposed by Christopher Segall (Segall 2010). This method is based on the following observation about K-nets, called "Whincop's Observation":

> Whincop's Observation says that any complete (well-formed) K-net N can be articulated into two subnetworks N1 and N2, such that all arrows within N1 are T-arrows, all arrows within N2 are T-arrows, and all arrows between locations of N1 and locations of N2 are I-arrows (Lewin 2002, 221).

"Well-formed" in this context essentially means that the composite operations formed by traversing any two paths from a given first node to a given second node of a network will be the same (Lewin 2002, 221). The more precise and technical requirement can be found in part D of Definition 9.2.1, on page 195 of Lewin 1987. Whincop's conjecture is proven true by Lewin (2002, 221–2).

Based on this observation, Segall creates pairs of pitch-based (rather than pc-based) T-nets from a number of different K-nets. He then proceeds to calculate the numerical averages (arithmetic means) of each T-net, and from there he averages each pair of averages. He calls the results the "gravitational centers" of each pair of T-nets. (Note that these results are dependent upon how each k-net is broken down into constituent T-nets).

[9] This offers a promising avenue for future exploration, since it might make possible a hybrid methodology allowing set relationships, rather than merely relationships between ordered pairs of pitches, to be quantified according to voice-leading class.

As an example of a gravitational center that shifts and then returns again, Segall presents the first four measures of *Mikrokosmos* no. 131, "Fourths" (Segall 2010, 131), reproduced here as Figure 17(a). As stated by Segall, the fact that both of the pianist's hands play perfect fourths throughout the passage "invites either a k-net or a dual transformational approach" (Segall 2010, 130). The additional approach that Segall offers, however, is the gravitational balance method, using the segmentation of the music for the pianist's right and left hands as the constituent T-nets of each tetrachord (Segall 2010, 130). This is shown in Figure 17(b), where the noteheads represent the center of gravity of each tetrachord and quarter-tone notation is employed.

Voice-leading class provides yet another way of understanding the chords in this passage. To see how, consider carrying out all but the last step of the preceding analysis. That is, calculate the centers of gravity for each tetrachord's constituent perfect fourths, but do not average them together. This yields, in effect, a passage in two-voice counterpoint that can be analyzed using voice-leading class. Figure 17(c) shows the values of θ for each transformation mapping one pair of centers-of-gravity to the next. It is the deviations from θ=-45° (or 135°) that are of interest, showing the places where inversionally symmetrical motion is abandoned in favor of non-symmetrical contrary motion. Note that applying voice-leading class analysis to the results of the intermediate stage of the gravitational balance calculation yields more specific information about the interactions of the individual centers of gravity than does the final stage of the gravitational balance method. This is therefore a useful combination of methodologies.

Figure 17. (a) The first four bars of Bartók's *Mikrokosmos* No. 131. (b) Segall's centers-of-gravity calculated for each chord in part a. (from Segall 2010, Example 8b). (c) Centers-of-gravity for the notes in treble and bass clef, with voice-leading class values underneath the bass clef staff.

A more extended example of this hybrid methodology is shown in Figure 18(a), taken from the *Sonata for Two Pianos and Percussion*, 1st mvt., mm. 57—60. Following the example given above, the tetrachords played by the right and left hands form natural T-nets to be condensed to one note each. This is illustrated in Figure 18(b), again using quarter-tone notation. These ordered pairs of pitches are then analyzed using voice-leading class.

There are no non-trivial VLC multiplicity values (i.e. each VLC value corresponds to only one difference vector), and the VLC frequency analysis does not reveal any clear patterns—other than a lack of variety of VLC values in this passage. In musical terms, the reason for this lack of variety is that the entire passage can be partitioned into descending major thirds in the left-hand and ascending augmented seconds in the right-hand (enharmonically speaking). The VLC value for this particular contrapuntal gesture is -143°. All of the other VLC values (there are only four) correspond to transitions between successive transpositional levels of the LH and RH parts. For example, the VLC value corresponding to a transition between transpositions of the gesture separated by positive six units in the LH part and negative one unit in the RH part is -9°.

The VLC values calculated using the partially executed center-of-gravity method thus provide a concise description of the counterpoint in this passage.

Figure 18. (a) *Sonata for Two Pianos and Percussion*, 1st mvt., mm. 57–60, piano 2
(b) Same passage reduced to two lines using center-of-gravity method
VLC values are given beneath the staff

This hybrid methodology offers great promise, by providing a means of translating textures consisting of many voices into a two-voice form that can be analyzed using the tools of voice-leading class. Intuitively speaking, this hybrid method describes what we might call "effective counterpoint"; it describes the type of counterpoint we might expect to hear if our ears somehow averaged together registrally similar pitches. It greatly expands the number of passages that can be analyzed using voice-leading class.

CHAPTER 5

VLC-TYPE ANALYSIS APPLIED TO MORE THAN TWO VOICES

Section 5.1 Overview

In this chapter we explore an alternative definition of voice-leading class for use with more than two voices.[10] The musical example chosen here comes from the 2nd movement of the *Concerto for Orchestra*, mm. 123—146, in which a brass choir consisting of two trumpets, two trombones and a tuba play in rhythmic unison with few exceptions throughout the entire passage. See Figure 19.

Figure 19. Mm. 123—134 of the Second Movement of the *Concerto for Orchestra* (percussion excluded).

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10 An analogy can be made to the metamorphosis from theory encompassing two voices to theory encompassing triads, as described in Crocker 1962.

Section 5.2 Definition and Properties

The alternative method entails calculating the angle between a given difference vector in multi-dimensional pitch space and each individual basis vector in the space–that is, each vector of the form (0,0,..1..,0), that has only one non-zero element. The angle can be calculated using a simple formula relating the cosine of the angle to the scalar product of the vectors:

$$\cos Ø = v_B \bullet v_D / |v_B||v_D|$$

where v_B and v_D are the basis vector and the difference vector, respectively, and the absolute value signs indicate vector magnitudes. Taking the arc-cosine of both sides of the equation yields

$$Ø = \arccos\,(v_B \bullet v_D / |v_B||v_D|).$$

The symbol Ø (phi) is used here to avoid confusion with θ; additionally, the term "alternative voice-leading class value" will be used in place of "voice-leading class value" to refer to Ø.

First we note that alternative voice-leading class value reduces to the definition given previously for voice-leading class value, the only difference being that the formula for alternative voice-leading class value maps to a range of [0°,180°] instead of a range of (-180°,180°], as does voice-leading class value.[11] It is still an informative metric.

Section 5.3 A Sample Analysis Using Alternative Voice-Leading Class

We turn now to the alternative voice-leading class analysis of the passage from the *Concerto for Orchestra* discussed above. Alternative VLC frequency histograms have been calculated for projections on the basis vectors corresponding to each of the five voices and are given below in Figures 20-24. The most noteworthy feature of these graphs is that the graphs for

[11] To see this, note that in the two-dimensional case, Ø is by definition the angle between the difference vector and one of the two basis vectors, which can be chosen to correspond to the v1 axis. This equates to the definition of θ given in Section 1.1, except for the fact that reflecting a given difference vector about the v1 axis does not change the projection of the difference vector onto that axis, so the range of Ø is therefore limited to [0°, 180°].

voices 2,3, and 4 have clearly visible peaks at Ø = 90°. As is the case for θ, Ø = 90° corresponds to oblique motion. This can be readily understood by noting that Ø = 90° indicates that the difference vector and the basis vector are orthogonal, that is, the difference vector has no projection on the basis vector. Thus, if Ø = 90° for a particular voice, that voice does not change; if it did, the difference vector would necessarily have a non-zero projection on the corresponding basis vector.

The histograms for voices 1 and 5, however—the outer voices—show a different frequency pattern. In particular, they both have visible peaks around Ø = 60° or so; perhaps that is motivic for this passage. Further analysis would be necessary in order to understand this structure.

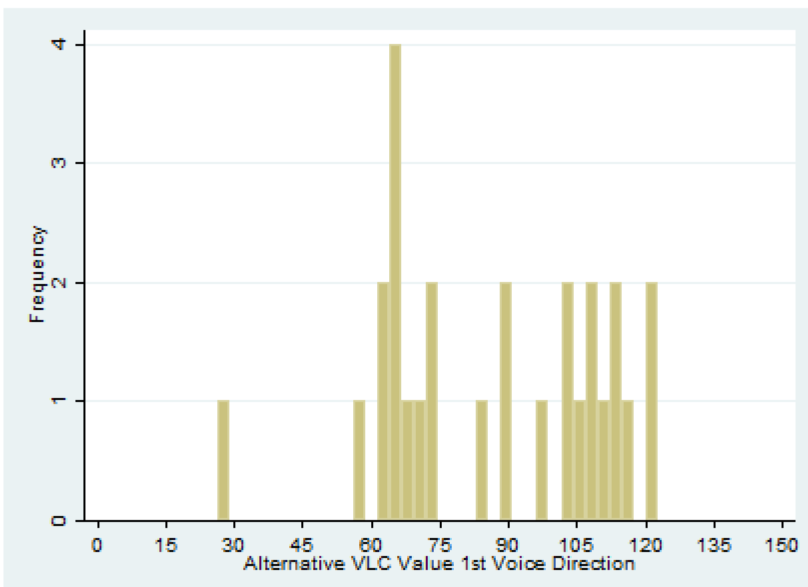

Figure 20. Alternative VLC Values Calculated Using Basis Vector Corresponding to 1st of 5 Voices

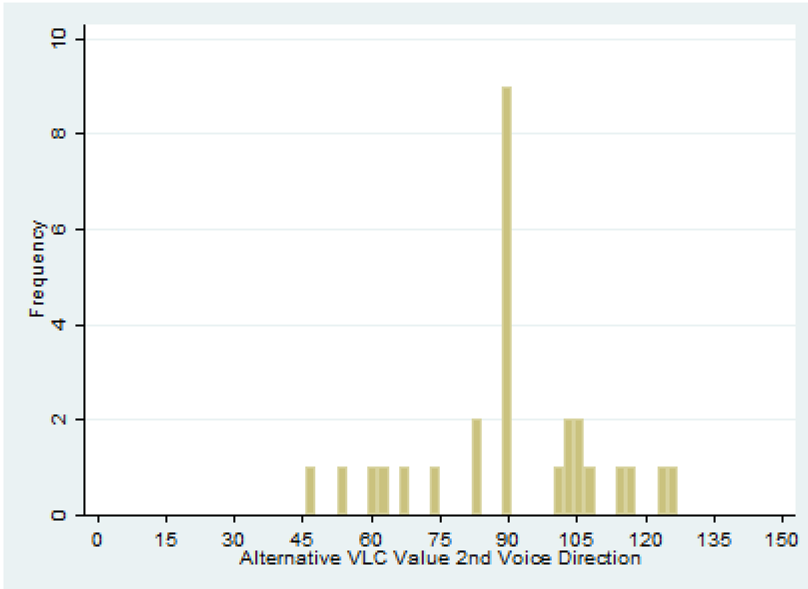

Figure 21. Alternative VLC Values Calculated Using Basis Vector Corresponding to 2nd of 5 Voices

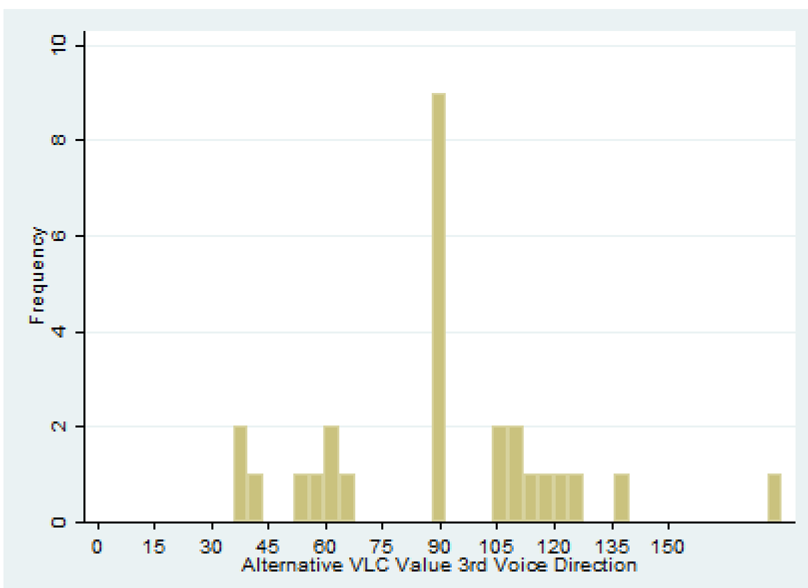

Figure 22. Alternative VLC Values Calculated Using Basis Vector Corresponding to 3rd of 5 Voices

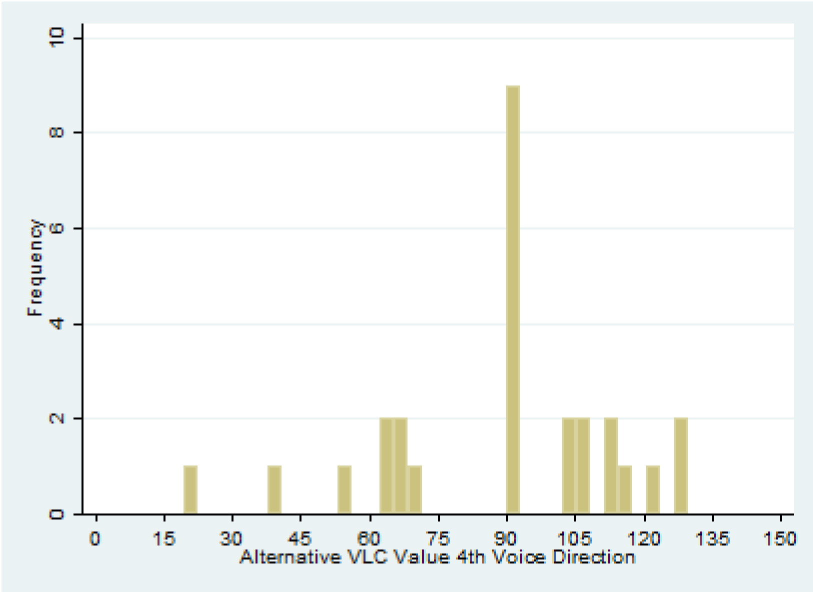

Figure 23. Alternative VLC Values Calculated Using Basis Vector Corresponding to 4th of 5 Voices

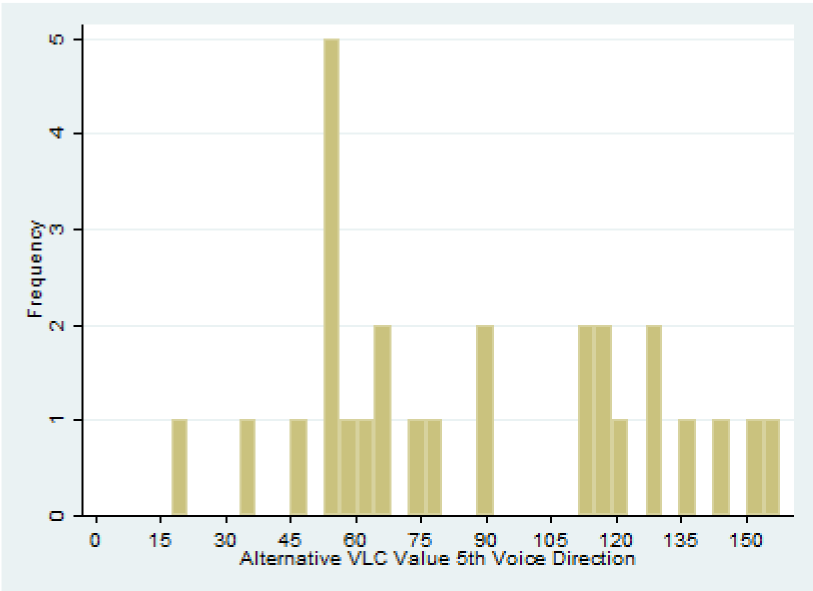

Figure 24. Alternative VLC Values Calculated Using Basis Vector Corresponding to 5th of 5 Voices

CHAPTER 6

CASE STUDY: THE SCHERZO OF THE STRING QUARTET #5

6.1 Overview

In this chapter, the third movement of the String Quartet #5—the *Scherzo*—is analyzed using voice-leading class. It is shown that the tools of voice-leading class (in particular as they apply to motivic transformation) are useful in understanding contour, form, motivic structure, pitch-class set structure and pitch centricity in this movement.

The overall form of the movement is sketched in Figure 25. The basic materials out of which the *Scherzo* and the *Scherzo Da capo* are constructed are similar: 1) a texture I have called "*perpetuum mobile*", consisting of continuously changing forms of an eight- or nine-note motive initially presented at the beginning of the movement—see Figure 26 (note the great variety of motivic form contained therein); 2) a texture identified by its rehearsal letter, "A", which incorporates sixteenth notes and sixteenth note triplets, producing a more boisterous affect than "*perpetuum mobile*"–see Figure 27; and 3) syncopated quarter notes, which intensify the rhythmic vigor present throughout the movement–see Figure 28.

To apply the methods of voice-leading class to this movement, successive pairs of eighth-note motive forms are used to construct dyads as in Chapter 3. A histogram showing the aggregate of all two-voice transformations is given in Figure 29. The fact that it consists of four more-or-less normal distributions, centered on the two VLC values for parallel motion and the two VLC values for inversionally symmetrical motion, is intriguing. More work is indicated to determine whether this is a unique property of this movement or is more or less universal. Or perhaps it is correlated with the extremely variegated, at times almost random-sounding, shapes of the motive-forms in this movement, since normal distributions are the most commonly occurring distributions among many types of random variables.

SCHERZO

1. mm. 1–13: *Perpetuum Mobile* leads to fermata prepared with repeated motive forms
   m. 14: preview of final motive form
2. mm. 14–23: *Perpetuum Mobile*
3. mm. 24–35: "A" prepares syncopated quarters with repeats of two motive forms
4. mm. 36–49: syncopated quarter notes
5. mm. 50–53: *Perpetuum Mobile*
   initial motive form returns once
6. mm. 54–57: prepares transition to trio with repeats of two motive forms
   m. 64: Scherzo ends with C# in bass
   (also final note of the movement)

TRIO

SCHERZO *da capo*

7. mm. 1-8: *Perpetuum Mobile*
8. mm. 9-12: alternating motive forms
   D-F-A-C-A-F-D-B, B-D-F-A-F-D-B-G#
   perhaps mirror fermata placement
   in first Scherzo
9. mm. 13-18: *Perpetuum Mobile*
10. mm. 19-28: prepares "A" with alternating motive forms C#-E#-G#-B-G-E-C#-A#,
    E-G-B-D-B-G#-E#-C#
11. mm. 29-40: "A" prepares syncopated quarters
12. mm. 41-57: Syncopated quarters prepare *Agitato* coda

Figure 25. Sketch of form in the *Scherzo* of the String Quartet #5.

Figure 26. *Perpetuum Mobile* texture in the *Scherzo* of String Quartet #5 (mm. 14—23). Note the variety of contour present in the various forms of the motive.

Figure 27. 'Letter "A"' texture in the *Scherzo* of String Quartet #5 (mm. 24—26).

Figure 28. ‘Syncopated quarters’ texture in the *Scherzo* of String Quartet #5 (mm. 36—38).

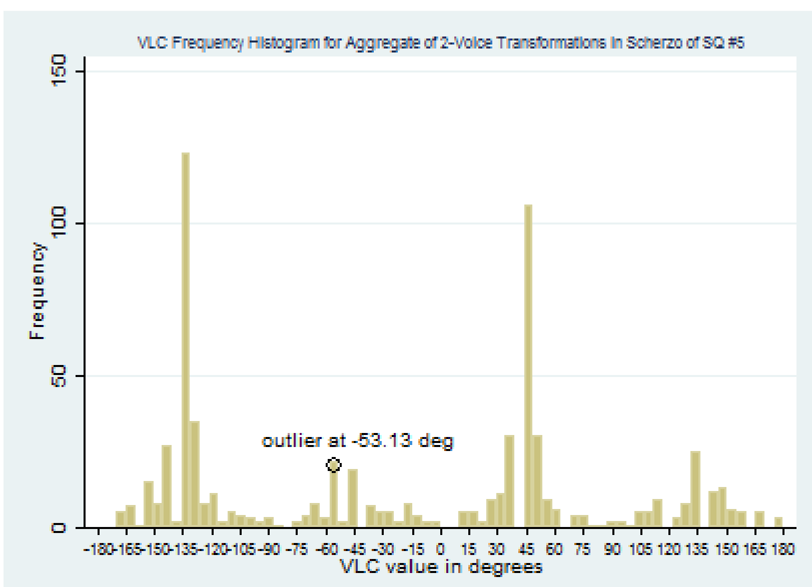

Figure 29. VLC frequency histogram for aggregate of all two-voice transformations in the *Scherzo* of the String Quartet #5.

Note, too, that there is an outlier around -50° (-53.13° to be exact). We will return to this shortly. For now, however, we turn to an analysis of contour in this movement.

Section 6.2 Contour and Voice-leading Class

Figures 30 through 37 show the frequency histograms for the two-voice transformations in this movement, grouped according to placement within respective motive form. In other words, Figure 30 shows transformations from the first to the second note in each motive form, Figure 31 shows transformations from the second to the third, and so on. Although not explicit, these graphs contain complete information on the melodic contour of each motive form. To see this, note that positive VLC values correspond to positive changes, and negative VLC values to negative changes, in the "voice 2" dimension–here corresponding to the second of each pair of motive forms under consideration. VLC values, however, contain substantially more information than simply an indication of a positive or negative contour direction.

In particular, VLC values show how an up or down motion of voice 2 relates to the up or down motion in the corresponding place in the previous motive form. Note by inspection of Figures 30 through 37 that the relationships between corresponding places in successive motive forms are primarily similar–in fact, primarily parallel.[12] In other words, an ascent (or descent) by n half-steps in the first motive form usually corresponds to an ascent (or descent) by n half-steps in the second motive form. This means we expect long stretches of "parallel" motion between each pair of notes in successive motive forms, interrupted only occasionally by contrary motion. Note, however, that by virtue of the fact that we have isolated pairs of notes within each motive form, those interruptions need not take place for every pair of notes simultaneously.

12 Some, but not all, of the preponderance of parallel motion is due to repetitions of a given motive form in different octaves.

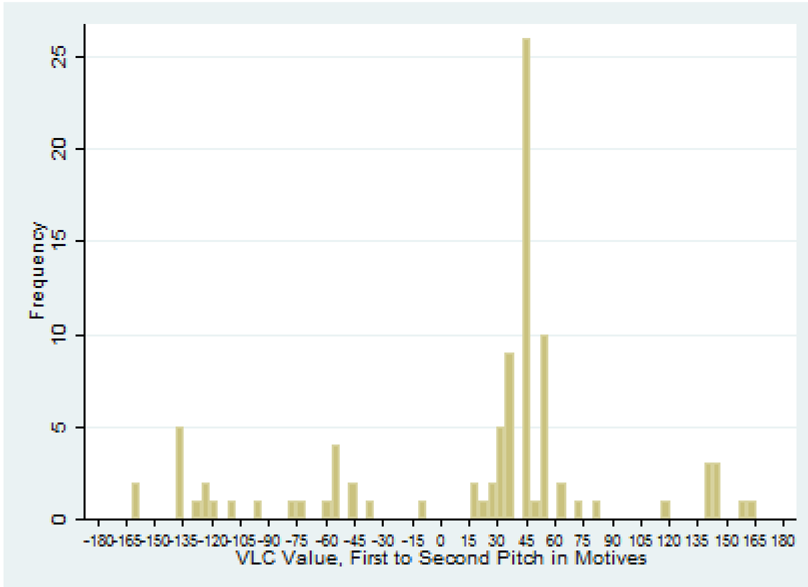

Figure 30. VLC frequency histogram for two-voice transformations mapping the first to the second pitches of successive motive forms.

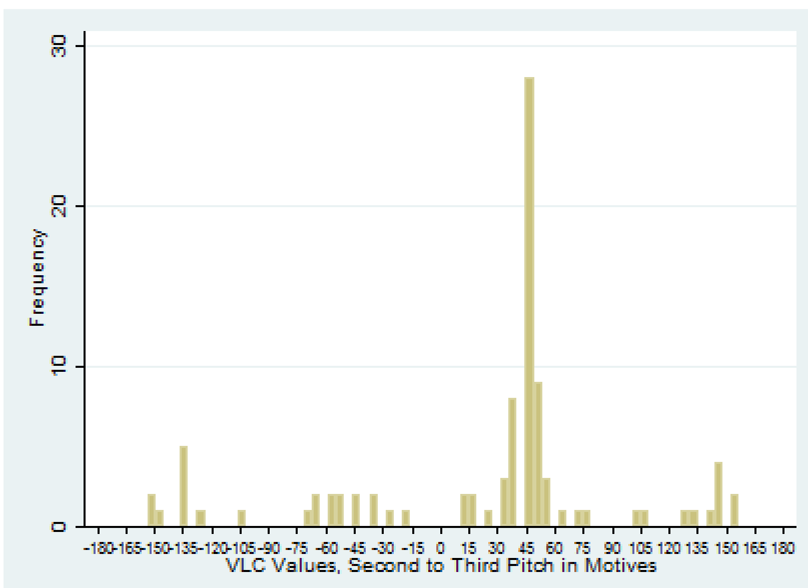

Figure 31. VLC frequency histogram for two-voice transformations mapping the second to the third pitches of successive motive forms.

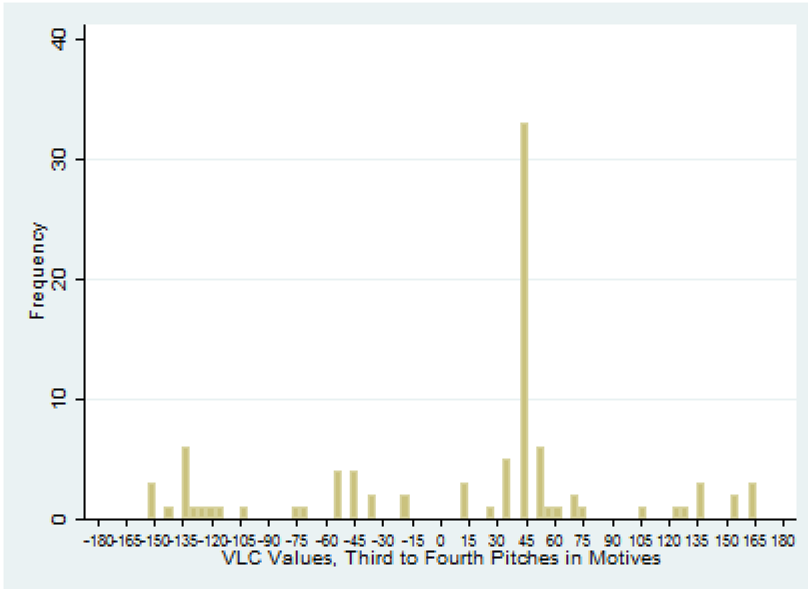

Figure 32. VLC frequency histogram for two-voice transformations mapping the third to the fourth pitches of successive motive forms.

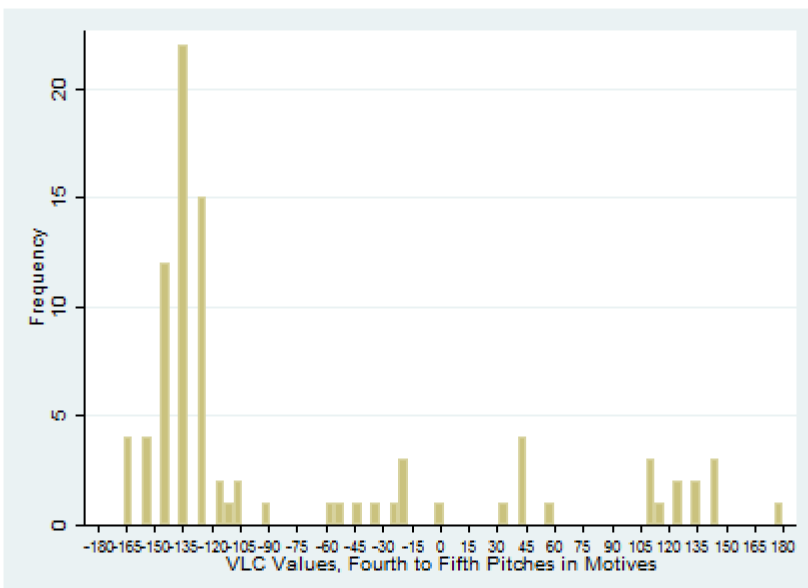

Figure 33. VLC frequency histogram for two-voice transformations mapping the fourth to the fifth pitches of successive motive forms.

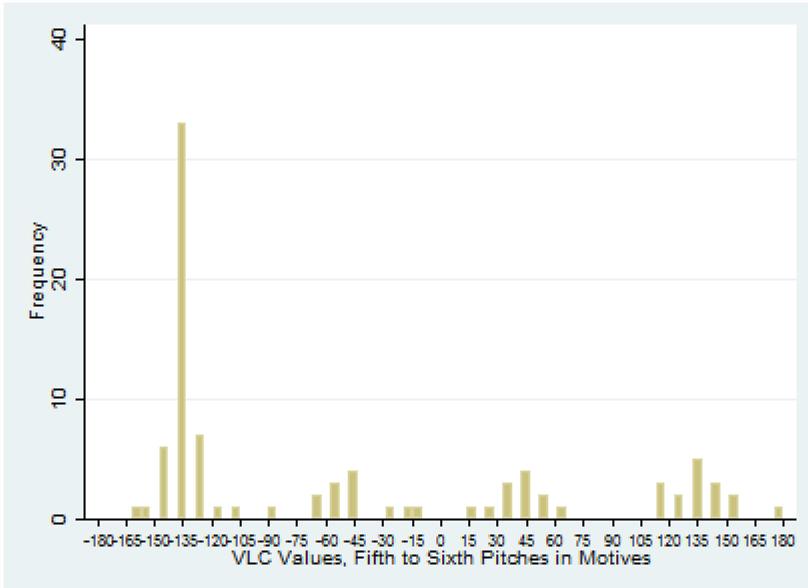

Figure 34. VLC frequency histogram for two-voice transformations mapping the fifth to the sixth pitches of successive motive forms.

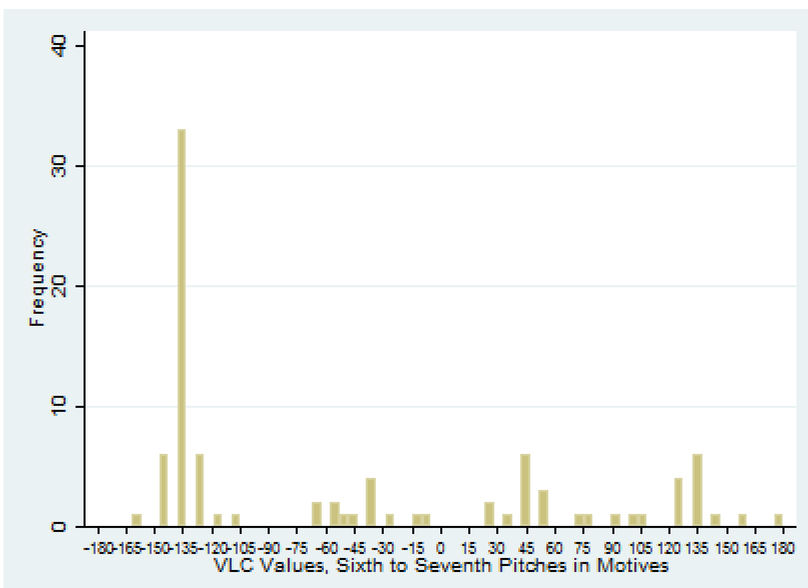

Figure 35. VLC frequency histogram for two-voice transformations mapping the sixth to the seventh pitches of successive motive forms.

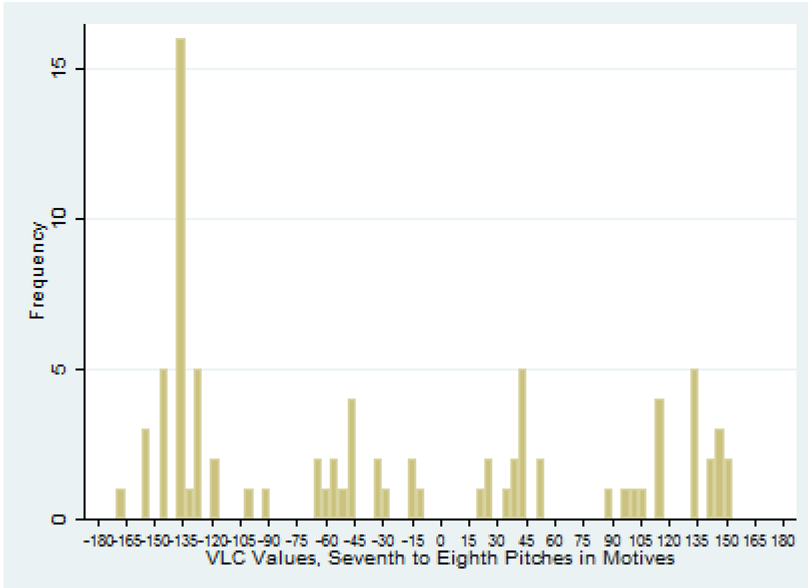

Figure 36. VLC frequency histogram for two-voice transformations mapping the seventh to the eighth pitches of successive motive forms.

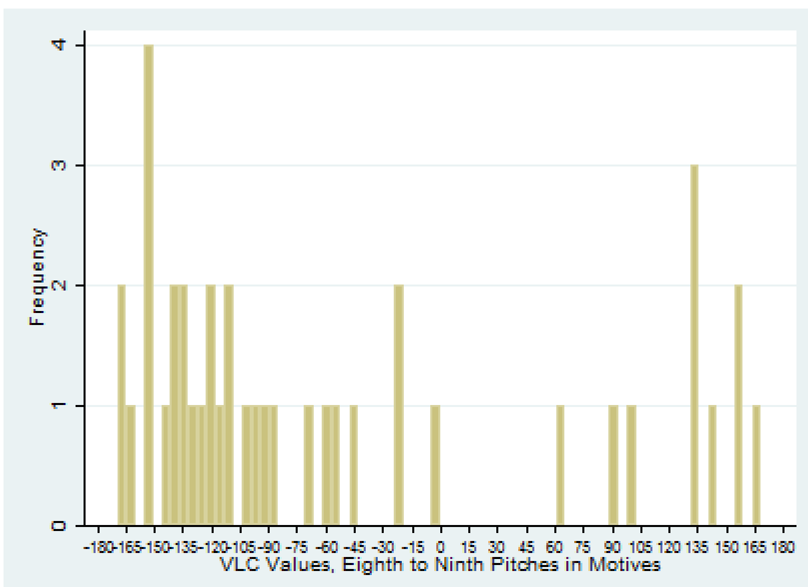

Figure 37. VLC frequency histogram for two-voice transformations mapping the eighth to the ninth pitches of successive motive forms.

Having thus demonstrated that VLC values carry more information than simply a qualitative description of contour, we may still use the information on contour contained within the VLC values described here to make relevant observations about the motive forms employed in this movement. To be precise, we infer that the most common direction, up or down, traversed by a given pair of notes within a given form of the motive, corresponds to the up or down melodic contour of the first form of the motive presented in the movement. (See Figure 38(a)). In other words, the first three graphs indicate positive contour direction, corresponding to the first four notes in the initial motive form, while the remaining graphs indicate negative contour direction, corresponding to the last five notes in the initial motive form. This is only a pair-wise description, so there are some motive-forms that do not match the contour of the initial form in every pair, but contribute to this correspondence nonetheless.

A final observation about contour comes from comparing the initial motive form in this movement to the final: the contour is completely inverted (see Figure 38(b)). Using VLC values, in fact, we can say more: all of the VLC values that result from the hypothetical transformation of the initial into the final form of the motive are geometrically related. (See Table 4.) In other words, they may be expressed in terms of geometrical operations on a single VLC value, which turns out to be the outlier in Figure 29, -53.13°. To see this, let $\theta_0 = -53.13°$. Then the angles in the transformation in question may be expressed, with the exception of -45° and 135°, as θ_0 (-53.13°), $-90° - \theta_0$ (-36.87°), $90° - \theta_0$ (143.13°) and $180° + \theta_0$ (126.87°). These arithmetic operations correspond to reflections about the $\theta = -45°$ or $\theta = 135°$ axes.

More will be said about geometrically related angles in the next section.

Figure 38. (a) Initial form of the motive in the *Scherzo* of String Quartet #5.
(b) Final form of the motive in the *Scherzo* of String Quartet #5.

Table 4. The hypothetical motivic transformation mapping the initial form of the motive to the final form of the motive in the *Scherzo* of the String Quartet #5.

-45.00°
-36.87°
-53.13°
143.13°
135.00°
135.00°
126.87°
-45.00°

Section 6.3 VLC Analysis and Form

Transitions between the sections of the form outlined in Figure 25 are frequently announced by repetitions or alternations of eight- or nine-note motive forms (though not all repetitions or alternations correspond to formal divisions and not all formal divisions are announced with repetitions or alternations). For example, the fermata in the *Scherzo* is prepared with a repeated motive form. Repeated motive forms during letter "A" of the *Scherzo* prepare the syncopated quarters that follow. The transition from the *Scherzo* to the Trio is prepared with two repeated motive forms.

This pattern resumes in the *Scherzo da capo*, and it is here that the tools of voice-leading class may be brought to bear. That is because of a special property of certain pairs of motive-forms that are repeatedly alternated in the *da capo*.

This property follows from the fact that in some transformations, pairs of successive VLC values sum to 90° (modulo 360°). See Table 5 for an example. What this means from a geometrical point of view is that the difference vectors corresponding to the two VLC values that sum to 90° are symmetrically positioned with respect to the $\theta = 45°$ axis. See Figure 39.

By comparison, consider what happens when we invert the order of the two motive forms being compared. When we do this, voice 1 is effectively exchanged with voice 2, and consequently, each difference vector is reflected about the $\theta = 45°$ axis. (See Appendix A for more on the idea of exchanging voices).

As a result, a particularly rich network of musical relationships results when a motivic transformation possessing one or more pairs of successive VLC values that sum to 90° alternates with its voice-exchanged counterpart, which happens when two forms of the motive (or transpositions thereof) alternate with one another on the musical surface. When this happens, the pairs of complementary VLC values exchange position in each successive transformation, which can be considered a kind of special effect utilizing motivic transformation.

This is exactly what happens in mm. 9–12 of the *da capo*, where it serves as a demarcation point between two *perpetuum mobile* sections, perhaps reminiscent of where the fermata occurs in m. 13 of the *Scherzo* before the Trio. Refer to Table 6 for a listing of the successive motivic transformations in terms of voice-leading class and to Figure 40 for mm. 9—12 of the score.

Table 5. An example of a motivic transformation from the *Da capo* of the *Scherzo* of SQ #5 exhibiting pairs of successive VLC values that sum to 90° (mod 360°).

45.00°
53.13°
36.87°
-143.13°
-126.87°
-135.00°
-135.00°

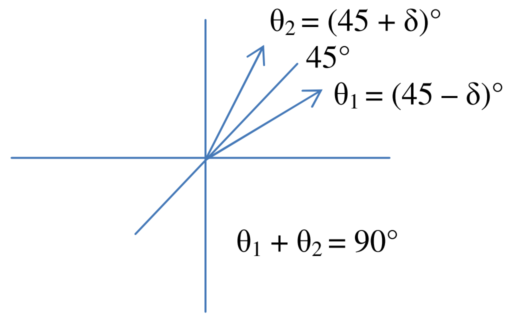

Figure 39. Illustration of the fact that angles which are symmetrically distributed about the 45° axis in pitch space sum to 90°.

Table 6. Three successive motivic transformations from m. 9-12 of the *da capo.*

45.00°	45.00°	45.00°
36.87°	53.13°	36.87°
53.13°	36.87°	53.13°
-126.87°	-143.13°	-126.87°
-143.13°	-126.87°	-143.13°
-135.00°	-135.00°	-135.00°
-135.00°	-135.00°	-135.00°

Figure 40. Alternation of motive forms in m. 9-12 of the *da capo* of the *Scherzo* in the String Quartet #5.

It happens again in mm. 19–28 of the *da capo*, where it clearly prepares the transition from the second *perpetuum mobile* section to the "A" section (See Table 7 for VLC values and Figure 41 for the score for these measures).

Table 7. Three successive motivic transformations from m. 19-28 of the *da capo.*

36.87°	53.13°	36.87°
53.13°	36.87°	53.13°
45.00°	45.00°	45.00°
-143.13°	-126.87°	-143.13°
-135.00°	-135°	-135.00°
-143.13°	-126.87°	-143.13°
-116.57°	-153.44°	-116.57°

Figure 41. Mm. 19—22 of the *da capo*, representing alternation between two motive forms.

Finally, note that the effect of exchanging pairwise VLC values on successive motivic transformations depends on there being pairs of VLC values that sum to 90°. As a counterexample, see Table 8, which displays a fictitious motivic transformation without any pairs that sum to 90°, and its inverse, which lacks the effect of swapping pairwise VLC values.

Table 8. Fictitious motivic transformation with no pairs of VLC values summing to 90° (left) and its voice-exchanged counterpart (right).

45°	45°
52°	38°
15°	75°
22°	68°
-36°	126°
-5°	95°
17°	73°

Section 6.4 Pitch class sets and VLC analysis

Thus far, voice-leading class analysis of this movement has highlighted interesting properties related to contour and form. In this section, the structure of the *Scherzo* and the *Scherzo da capo* will be linked to a pair of important pitch class sets articulated just before and during the Trio.

In what follows it will be useful to define formally families of geometrically related angles. These can be divided into families representing similar motion and families representing contrary motion. For families representing similar motion, all constituent VLC values are related to one another by reflection around the $\theta = 45°$ or $-135°$ axis (they amount to the same operation mod 360°), or by inversion through the origin. For families representing contrary motion, constituent VLC values are related to one another by reflection around the $\theta = -45°$ axis or the $\theta = 135°$ axis, or by inversion through the origin.

Specifying a single VLC value and whether a family represents similar or contrary motion is sufficient to determine all members of a family, which can consist only of two or four members. For an example of a family of geometrically related values, refer to Table 4 and the accompanying discussion in Section 5.2: 53.13°, 36.87°, -143.13° and -126.87°.

All of the VLC values discussed in the previous section belonged either to families of similar motion or families of contrary motion containing $\theta = 53.13°$ or $\theta = -53.13°$. Furthermore, these families are by far the most common found in the data for this movement: of the 95 motivic transformations in the movement, 33 have at least two distinct VLC values belonging to one of these families, while only 10 have more than one member of any other family. This should not come as a surprise, since the value 53.13° and its cohorts represent transformations of minor into major or major into minor thirds, and the motive forms in this movement are predominantly constructed of thirds.

Nevertheless, the predominance of thirds in this movement's motive forms, as well as the fact that -53.13° appears as an outlier in the aggregate VLC frequency histogram (see Section 5.1), suggest that we look for other connections based on thirds as well. Such connections can be found in the measures leading up to the Trio and during the Trio itself.

To be specific, one of the most clearly audible features of the transition to the Trio is the dyad (A_4,C_5) in the viola against the C#$_2$ in the 'cello (m. 64). See Figure 42. The C# is prepared in the four measures leading up to m. 64 with occurances of C# in several registers and in all of the instrumental parts. Furthermore, the pitch-class dyad (A,C) recurrs in various registers in the first fourteen measures of the Trio, always voiced as a minor third. Departing for a moment fom our adopted convention of working with pitches-in-register, then, we may identify the pitch-class set (C#,A,C) as being of some significance for this piece. For our analytical purposes, that significance lies in the fact that it contains one ic-3 dyad and one ic-4 dyad, in keeping with the prominence of major and minor thirds in the movement's motivic forms.

This pc-set reappears in the 'cello part in mm 40-51 of the Trio, inverted to become (B-flat,C#,D). (See Figure 43). This set is highlighted not only by the insistent repetition of C#$_3$ and B-flat$_2$ in the 'cello part but also by the minor second dissonance with D_3 in the'cello part and the announcement of the start of this pattern with the octave D's in the viola part.

So the prominence of transformations involving major and minor thirds in this movement, revealed by voice-leading class, is echoed in two pitch-class sets just before and during the Trio. This provides further evidence that major and minor thirds permeate and inform multiple aspects of the movement's structure.

Figure 42. Manifestation of the pitch-class set (C#,A,C) from mm. 63—66 of the *Scherzo*.

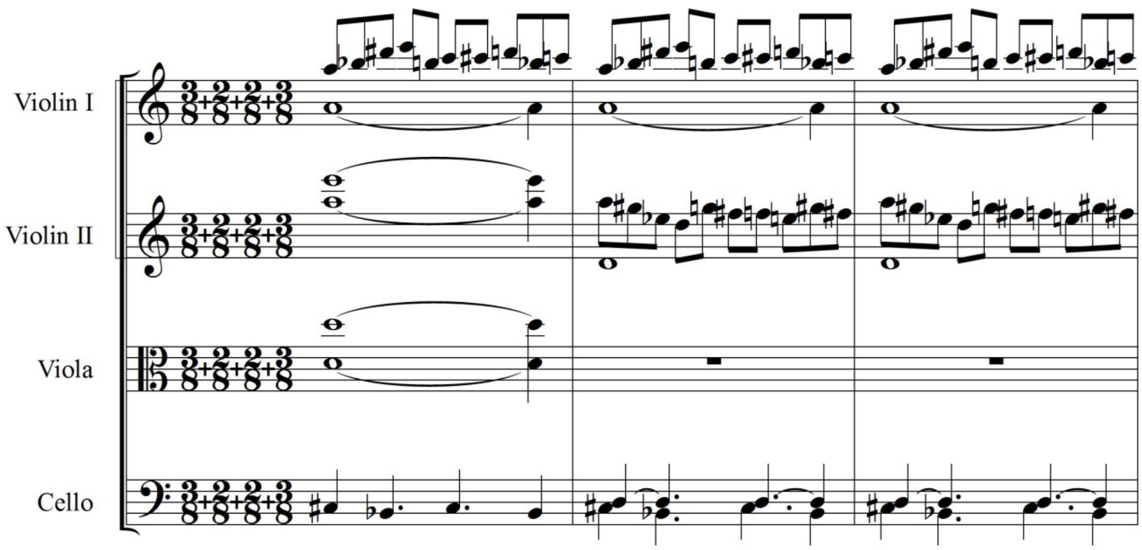

Figure 43. Manifestation of the pitch-class set (D,B-flat,C#) from mm. 40-42 of the Trio.

Section 6.5 Pitch centricity and form

This movement relies to a certain extent on the centricity of the pitch-class C#. For example, the movement ends with a clear V-i cadence in C#-minor, and C# is articulated as a kind of pitch center in many other places as well. For example, the first motive-form in the movement begins and ends on C#. A "5-line" preceded by the lowered sixth scale degree ends on C# at the first letter "A". The first motive-form is repeated several times leading into the syncopated quarter notes at measure 35, and again after the syncopated quarter notes at the first letter "B". Even more significantly, the initial *Scherzo* ends with C# in several octaves, all of them preceded with $G\#_2$ in the 'cello. (See the previous section.) In the *da capo* the initial motive form is repeated going into letter "A", and the aforementioned V-i cadence at the end of the movement is even preceded by a secondary dominant, $D\#^7$.

This $D\#^7$ may explain the presence of D# as a conflicting tonal center, beginning with the pizzicato 'cello notes at the very outset of the movement and continued in several other places throughout. Other conflicting tonal centers include C (after the first letter "A") and G (*da capo*, letter "A"); the latter is prepared by a repeated tritone in the 'cello from $C\#_3$ to G_2.

Here our primary interest lies in the centricity of C#, however, not only because of its predominance at the musical surface, but also because its presence in the *Scherzo* and the *Scherzo da capo* help frame the Trio. This is of interest because this movement serves as a point of temporal symmetry for the entire quartet (note the order of movements-Allegro, Adagio, Vivace, Andante and Allegro vivace-and their corresponding tempi). The Trio, then, becomes a focal point for the entire piece.

In the previous section it was demonstrated that the clear articulation of the centricity of C# just before the Trio was linked, via the pitch dyad (A_4,C_5), to the structurally significant pitch-class set, (C#,A,C). It was also observed that an inversion of that set, (B-flat,C#,D) was clearly articulated in the 'cello part during mm. 40-51 of the Trio. Noting that the first motive form of the *da capo* begins on D and ends with B-flat and C#, perhaps it makes sense to view the articulation of the transformed set (B-flat,C#,D) as a kind of "modulation" from the centricity of C# to the (weaker) centricity of D. This leaves us waiting for the return to the original pitch centricity of C#, which occurs at rehearsal letter "C" of the *da capo*, thus rounding out the symmetrical form of the movement and of the piece as a well.

CHAPTER 7

CONCLUSION

Based on Dmitri Tymoczko's observation that parallel and inversionally symmetrical motion in two voices can be represented by orthogonal axes in two-dimensional pitch space rotated by 45° with respect to the original axes, an angular metric has been proposed here that can characterize any transformation mapping dyads to dyads in pitch space. This metric works by assigning to any two-voice transformation an angle indicating a direction in the Cartesian plane. Angles from -180° to -90° or from 0° to 90° correspond to similar motion, while angles from -90° to 0° or from 90° to 180° correspond to contrary motion. Parallel and inversionally symmetrical motion correspond to 45°/-135° and -45°/135°, respectively, and multiples of 90° (including 0°) correspond to oblique motion.

The metric, called voice-leading class (or "VLC"), applies to numerous musical situations, including two-voice first-species counterpoint (which provides the conceptual basis for voice-leading class) and textures involving single-voice motivic transformation. In the former case, the meaning of voice-leading class is intuitive, representing as it does a generalization and quantification of Fux's categories of counterpoint. In the latter case, however, it must be understood as a descriptor of the musical surface which may or may not be easily perceptible.

In the case of first-species counterpoint, VLC analysis of passages from *Contrasts* and *Music for String Instruments, Percussion and Celesta* reveals a high degree of structure. For example, the data from *Contrasts* reveals pairs of geometrically related VLC values—in particular, VLC values that correspond to motion in opposite directions along a given axis in pitch space. VLC multiplicity analysis, which counts the number of distinct difference vectors corresponding to each VLC value in a data set, confirms the importance of these geometrically related pairs and also corroborates the importance of inversionally symmetrical and parallel motion in *Contrasts* and *Music for String Instruments, Percussion, and Celesta,* respectively. Finally, VLC frequency histograms exhibiting quasi-normal distributions around the values corresponding to parallel or inversionally symmetrical motion lend support to the conclusion that voice-leading class is a viable metric.

Voice-leading class provides a robust descriptor of chromatic compression and diatonic expansion. In comparison to the pitch-cell method of Antokoletz or the modular transformation method of Santa, it is more direct and more robust because it can describe any succession of intervallic transformations, regardless of any particular pitch-cells or scalar modules. (VLC analysis, however, is not *explanatory* in the same way that those methods are, but only descriptive.)

It is worth digressing for a moment to point out the potential usefulness of voice-leading class as an aid to the performer, in particular as an aid to determining how dynamics are to be used. This is particularly clear in the case of chromatic compression and diatonic expansion: one might assume that dynamics would follow the up or down contour of the line, but perhaps we can go one step further. When a compressed melody appears again in expanded form, we might turn to VLC values as an indicator of the extent to which each note represents an actual expansion of the original melody. For large amounts of expansion in an upward directed line, we might employ large increases in dynamics; for large amounts of expansion in a downward directed line, we might employ large decreases in dynamics. Note that this is not the same as making dynamic changes proportional to the size of the intervallic distance between notes in the expanded melody; it is a contextual mapping dependent not only on the intervals in the expanded version of the melody but also on the intervals in the original, compressed version of the melody.

In comparison to other models of atonal voice leading, this metric retains a different profile of two-voice transformations than k-nets, dual transformations or Segall's "center-of-gravity" method. To reiterate, it generalizes—and quantifies—the types of note-against-note counterpoint described by Fux. A crucial difference between VLC on the one hand and k-nets and dual transformations on the other is that voice-leading class measures relationships in pitch space rather than pitch-class space. Segall's "center-of-gravity" method, however, operates in pitch space and can fruitfully be combined with the tools of voice-leading class to yield a hybrid methodology that provides one of several ways of analyzing textures with more than two voices.

The simplest way to analyze textures with more than two voices is simply to measure θ for every possible pair of voices. For three voices, there would be three possible voice-pairings and thus three values of θ; for four voices there would be six, and so on. The meaning of the metric in this case is clear: each angle calculated has the same interpretation as θ does in the case of a simple two-voice transformation.

There is another way to generalize the metric, however, that is conceptually more complicated, but might reveal different layers of structure than the others. Furthermore, it does not depend on how we divide vertical simultaneities the way the hybrid methodology does, and it does not require pairwise computations for every pair of voices as the previous method does. It is to calculate the angle between the vector difference of the two ordered n-tuplets in pitch space being compared, and each co-ordinate axis in n-dimensional space. It is worthwhile noting that in two dimensions this method reduces to the method for calculating θ discussed above.

A close study of the *Scherzo* from the String Quartet #5 reveals multiple uses for the VLC metric as applied to motivic transformation. In particular, it is shown that VLC analysis interacts profitably with the description of contour, form, motivic structure, pitch-class set and pitch centricity. Perhaps most interestingly, families of VLC values are identified, the members of which are related by reflection through the origin and by reflection about either the θ = 45°/-135° or the θ = -45°/135° axis. Two families stand out in particular because they account for the predominance of geometrically related VLC values in the movement: the one containing 53.13° and the one containing -53.13°. That is because these values represent transformations mapping minor to major thirds or vice-versa, and most of the motive forms in the movement are built in thirds. These families in at least two places play a role in articulating the form of the movement.

One direction for further research is to explore issues of perception and cognition as they pertain to voice-leading class. For example, if a listener listens to a passage of music with quasi-normal distributions centered on certain values, would they classify a transformation played afterwards as belonging or not belonging to the passage according to how close it is to the peak values? Or if one of the distributions is sharply peaked and the other is broad (as in the example presented in Section 2.4), would a listener identify values corresponding to the sharply peaked distribution more easily than values corresponding to the broader distribution?

Another direction for further research, one that might illuminate which aspects of musical structure revealed by VLC analysis apply only to Bartók and which are universal, would be to apply VLC analysis to a broader range of music. It would be interesting to see if music from the common practice period, for example, would exhibit the same type of structure corresponding to inversional symmetry as some of the music by Bartók does (the second movement of *Contrasts*, for example). Presumably it would not, since Bartók's frequent reliance on inversional symmetry is not a characteristic found in music from the common practice period. It would also be interesting to see whether or not frequency distributions which are symmetrical about their respective peaks would be found in music from the common practice period, since that too is a kind of symmetry to which Bartók might have been sensitive while the composers of the common practice period were not.

It is hoped that this work will make a useful contribution to how we think about transformations of two or more voices, offering a fresh perspective on the classic categories of two-voice counterpoint and a precise but intuitive metric that can be applied to a broad range of music.

APPENDIX A

PROPERTIES OF VOICE-LEADING CLASS

Here we explore some basic properties of voice-leading class, proving some simple theorems along the way that are useful for building one's intuition about voice-leading class. In what follows, the n^{th} VLC value for voices v_i and v_j will be denoted $\theta_n(i,j)$. This enables a third voice to be introduced without significantly altering the notational conventions used above.

To begin with, suppose that the n^{th} VLC value in a series of VLC values for voices v_1 and v_2, denoted $\theta_n(1,2)$, is known. Suppose a third voice is introduced, one that moves in purely parallel motion with v_2. Then $\theta_n(1,3) = \theta_n(1,2)$. To prove this, simply note that, from the definition of VLC value for voices 1 and 2, $\theta_n(1,2) = \arctan((v_2[n+1]-v_2[n])/(v_1[n+1]-v_1[n]))$, while from the definition of VLC value for voices 1 and 3, and from the fact that voices 2 and 3 move in parallel motion, $\theta_n(1,3) = \arctan((v_3[n+1]-v_3[n])/(v_1[n+1]-v_1[n])) = \arctan((v_2[n+1]+d-v_2[n])-d)/ (v_1[n+1]-v_1[n])) = \arctan((v_2[n+1] - v_2[n])/(v_1[n+1]-v_1[n])) = \theta_n(1,2)$, where d is the constant intervallic separation between voices 2 and 3.

Suppose now that the situation is the same as in the above paragraph, except that the third voice is inversionally symmetrical in relation to the second voice. Then $\theta_n(1,3)= - \theta_n(1,2)$. To prove this, return to the definition of VLC value for voices 1 and 3 and use the fact that $v_3[n+1] - v_3[n]= - (v_2[n+1] - v_2[n])$ by the definition of inversional symmetry. Then $\theta_n(1,3) = \arctan((v_3[n+1]-v_3[n])/(v_1[n+1]-v_1[n])) = \arctan(-(v_2[n+1]-v_2[n])/(v_1[n+1]-v_1[n]))= - \arctan((v_2[n+1] - v_2[n])/(v_1[n+1]-v_1[n]))= - \theta_n(1,2)$, where the properties of the arctan function are used in the second to last step.

Next consider a situation in which voices 1 and 2 share the same contour, but not necessarily the same intervals. Then $\theta_n(1,2)$ lies in either the first or the third quadrant for all values of n. To see this, simply note that by definition of contour equivalence, all of the motion between voices 1 and 2 must be similar (or possibly oblique, depending on whether one admits motion in one voice and stasis in another as a form of contour equivalence). Thus, referring back to Figure 4 (page 5), we find that $\theta_n(1,2)$ will always lie in the first or third quadrant (possibly including the boundaries of those quadrants if oblique motion is admitted).

The reader may have wondered how the assignment of the labels v_1 and v_2 to voices on the musical surface might affect the outcome of VLC analysis. Naturally, if this method is to be useful, arbitrary choices like these cannot affect the outcome – at least not as far as *musical* results are concerned. One might assume that v_1 should always be lower than v_2, but of course this is erroneous, as it would not allow for simple voice-crossings! In fact, all that happens when v_1 is exchanged for v_2 is that all VLC values are reflected across the axis corresponding to parallel motion (45°/-135°). In other words, the tabulated results of a VLC multiplicity analysis would *look* different, in the sense that different VLC values would have nontrivial multiplicities, but if those results were translated back to the musical surface of the passage in question, the same transformations would be singled out as having non-trivial mulitiplicity values. The effect on the VLC frequency histogram is slightly more difficult to picture, but applying the simple formula $\theta_{new} = 90° - \theta_{old}$ (mod 360°) reveals that the histogram is basically translated along the horizontal axis by 90° to the right, and then reflected through the vertical line $\theta = 90°$. Note that this will change most VLC values, so that different VLC values stand out prominently in the new histogram, but there are some important exceptions to this rule. In particular, note that the pairs {45°,-135°} and {-45°, 135°} map onto themselves using the formula given above. This means that the *musical* observation that parallel or inverionally symmetrical motion is prominent in a given passage does not depend on the choice of v_1 or v_2 on the musical surface – and neither does the symmetry or asymmetry of a distribution about the corresponding peak values.

The ambitious reader may wish to prove that exchanging voices and measuring positive integers *downward* from Middle C simply reflects all VLC values about the axis corresponding to inversionally symmetrical motion (-45°/135°), without changing the musical results of the analysis.

Finally we turn to a description of how the number of possible VLC values describing a hypothetical two-voice transformation depends on the magnitude of the difference vector pertaining to that transformation. Since every difference vector can be represented by an ordered pair of integers, it makes sense to illustrate this using concentric squares centered on the origin. In particular, it makes sense to use squares with sides of length 2n for positive integer n, since every difference vector with at least one component equal to n and no component greater than n will fall on the square of side length 2n. This is illustrated in Figure 44, where every possible difference vector between ordered pairs of pitches is classified according to the concentric square it falls on. Note that each value of n corresponds to 8n possible difference vectors and therefore, 8n possible VLC values. Thus, for small differences between pairs of notes, there are relatively few possible VLC values, but as the magnitude of the difference vectors increase, so do the number of possible VLC values. This must be borne in mind when analyzing music using voice-leading class: for relatively small difference vectors between pairs of notes, recurring VLC values may not be very significant; for larger differences, on the other hand, recurring VLC values may be more significant.

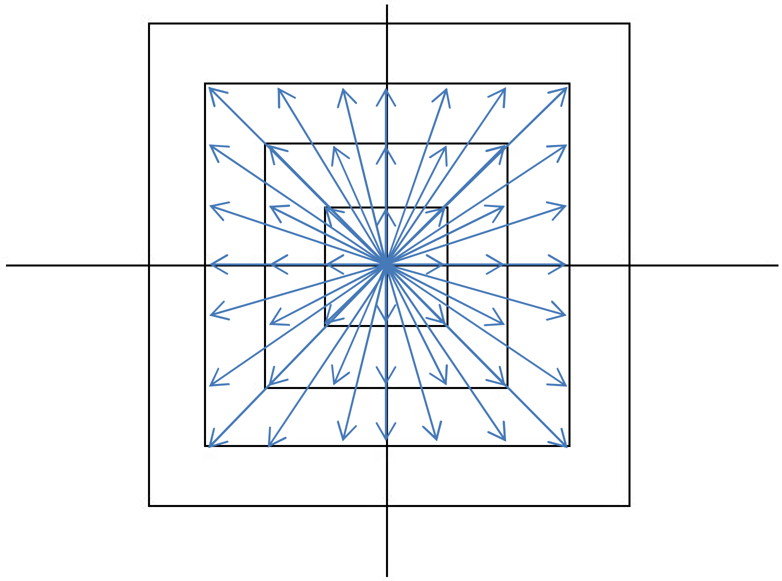

Figure 44. Conceptual diagram showing how the number of possible VLC values increases with the magnitude of the difference vector. For a difference vector that falls on a concentric square of side length 2n there are 8n possible VLC values. The squares shown have side lengths 2, 4, 6, and 8.

APPENDIX B

NUMERICAL DATA

Section B-1. *Music for String Instruments, Percussion and Celesta* 4th mvt. Mm. 28—43

(outer voices only)

(Read down first column, then down second column)

violin 1	cello 2	VLC value	violin 1	cello 2	VLC value
-3					
1	-11	120.96	15	-5	116.57
6	-14	45.00	17	-6	-26.57
11	-9	45.00	16	-4	-135.00
16	-4	45.00	15	-5	116.57
21	1	-135.00	17	-6	-26.57
16	-4	45.00	16	-4	-135.00
18	-2	-143.13	15	-5	116.57
15	-6	-126.87	17	-6	-26.57
11	-9	-143.13	16	-4	-135.00
8	-13	-126.87	15	-5	116.57
4	-16	45.00	17	-6	-26.57
6	-14	-78.69	16	-4	-135.00
-4	-12	78.69	15	-5	116.57
1	-11	120.96	17	-6	
6	-14	59.04	17	-6	
11	-11	35.54	17	-6	
16	-4	45.00	17	-6	
21	1	45.00	17	-6	168.69
26	6	-135.00	19	-16	
24	4	-135.00	19	-16	
23	3	-135.00	19	-16	
20	0	-135.00	19	-16	
16	-4	-135.00	19	-16	
			19	-16	
			19	-16	-49.90

Section B-2. *Sonata For Two Pianos and Percussion*

(1st mvt. Mm. 57–60, piano 2 only)

VLC values appear in right-most column, labeled “hybrid/gravitational”

LH	LH	RH	RH	LH_avg	RH_avg	hybrid/gravitational
5	11	20	25	8	22.5	143.13
0	8	23	28	4	25.5	-9.46
7	13	22	27	10	24.5	143.13
2	10	25	30	6	27.5	-21.80
8	14	23	28	11	25.5	143.13
3	11	26	31	7	28.5	-80.54
5	11	20	25	8	22.5	143.13
0	8	23	28	4	25.5	-9.46
7	13	22	27	10	24.5	143.13
2	10	25	30	6	27.5	-21.80
8	14	23	28	11	25.5	143.13
3	11	26	31	7	28.5	-9.46
10	16	25	30	13	27.5	143.13
5	13	28	33	9	30.5	-112.62
1	7	16	21	4	18.5	143.13
-4	4	19	24	0	21.5	-9.46
3	9	18	23	6	20.5	143.13
-2	6	21	26	2	23.5	-21.80
4	10	19	24	7	21.5	143.13
-1	7	22	27	3	24.5	-9.46
6	12	21	26	9	23.5	143.13
1	9	24	29	5	26.5	-9.46
8	14	23	28	11	25.5	143.13
3	11	26	31	7	28.5	-80.54
5	11	20	25	8	22.5	143.13
0	8	23	28	4	25.5	-9.46
7	13	22	27	10	24.5	143.13
2	10	25	30	6	27.5	-9.46
9	15	24	29	12	26.5	143.13
4	12	27	32	8	29.5	-9.46
11	17	26	31	14	28.5	143.13
6	14	29	34	10	31.5	

Section B-3. *Concerto for Orchestra*, 2^{nd} mvt., Mm. 28–43

(Alternative VLC Values for Voices 1—5 are given in rightmost columns)

tpt 1	tpt 2	trb 1	trb 2	Tuba	1	2	3	4	5
6	3	-1	-6	-13					
4	1	-4	-4	-11	113.58	113.58	126.87	66.42	66.42
6	3	-1	-6	-13	66.42	66.42	53.13	113.58	113.58
8	3	-1	-9	-16	64.76	90.00	90.00	129.76	129.76
9	4	1	-11	-18	74.50	74.50	57.69	122.31	122.31
4	1	-3	-4	-11	114.27	104.28	109.20	54.87	54.87
6	3	-1	-6	-13	63.43	63.43	63.43	116.57	116.57
8	3	-1	-9	-16	64.76	90.00	90.00	129.76	129.76
10	6	1	-11	-18	66.42	53.13	66.42	113.58	113.58
11	6	-1	-10	-17	67.79	90.00	139.11	67.79	67.79
12	4	-5	-12	-24	83.32	103.44	117.71	103.44	144.46
11	4	-5	-8	-23	103.63	90.00	90.00	19.47	76.37
10	2	-6	-9	-21	107.55	127.09	107.55	107.55	52.91
8	3	1	-9	-16	103.00	83.54	38.04	90.00	55.77
8	3	-1	-9	-16	90.00	90.00	180.00	90.00	90.00
13	4	-4	-11	-23	57.79	83.88	108.65	102.31	138.26
11	3	-4	-11	-20	122.31	105.50	90.00	90.00	36.70
10	1	-6	-11	-18	106.10	123.69	123.69	90.00	56.31
8	-1	-1	-8	-16	107.15	107.15	42.51	63.75	72.85
6	3	-1	-6	-13	110.37	45.87	90.00	69.63	58.52
8	3	-1	-8	-20	74.64	90.00	90.00	105.36	158.00
4	1	-4	-8	-15	122.98	105.79	114.09	90.00	47.12
6	4	1	-8	-15	71.07	60.88	35.80	90.00	90.00
8	4	1	-8	-16	26.57	90.00	90.00	90.00	116.57
9	4	1	-8	-18	63.43	90.00	90.00	90.00	153.43
8	4	-1	-8	-11	97.82	90.00	105.79	90.00	17.72
8	3	-1	-4	-8	90.00	101.31	90.00	38.33	53.96
6	1	1	-2	-6	116.57	116.57	63.43	63.43	63.43
6	1	1	-2	-6					

Section B-4. *Scherzo* of String Quartet #5

1	1	-4	33.69	5	15	8	45.00
	4	-2	14.04		18	11	45.00
	8	-1	-18.43		22	15	45.00
	11	-2	-126.87		25	18	-135.00
	8	-6	153.43		22	15	-135.00
	4	-4	-126.87		18	11	-135.00
	1	-8	146.31		15	8	-135.00
	-2	-6	-59.04		11	4	-151.39
	1	-11				-2	
2	-4	4	63.43	5	8	15	-59.04
	-2	8	71.57		11	10	-26.57
	-1	11	104.04		15	8	-53.13
	-2	15	-135.00		18	4	-135.00
	-6	11	45.00		15	1	-153.43
	-4	13	-143.13		11	-1	126.87
	-8	10	-63.43		8	3	143.13
	-6	6	158.20		4	6	-140.19
	-11	8			-2	1	
3	4	-4	51.34	7	15	-4	158.20
	8	1	33.69		10	-2	153.43
	11	3	14.04		8	-1	153.43
	15	4	-143.13		4	1	-146.31
	11	1	45.00		1	-1	116.57
	13	3	-126.87		-1	3	-26.57
	10	-1	153.43		3	1	-59.04
	6	1	-68.20		6	-4	141.34
	8	-4			1		
4	-4	15	30.96		fermata		
	1	18	63.43	8	-11	23	-161.57
	3	22	71.57		-14	22	-146.31
	4	25	-135.00		-17	20	-143.13
	1	22	-63.43		-21	17	33.69
	3	18	-143.13		-18	19	-26.57
	-1	15	-63.43		-12	16	-161.57
	1	11	-114.44		-15	15	-15.95
	-4				-8	13	-168.69
					-13	12	

9	23	-4	-108.43	13	8	-1	36.87
	22	-7	-123.69		12	2	156.80
	20	-10	-126.87		5	5	135.00
	17	-14	56.31		1	9	45.00
	19	-11	116.57		4	12	18.43
	16	-5	-108.43		7	13	-51.34
	15	-8	105.95		11	8	-26.57
	13	-1	135.00		15	6	-161.57
	12					1	
10	-4	4	-126.87	14	-1	-7	18.43
	-7	0	146.31		2	-6	33.69
	-10	2	-135.00		5	-4	14.04
	-14	-2	-18.43		9	-3	-18.43
	-11	-3	-18.43		12	-4	-63.43
	-5	-5	180.00		13	-6	158.20
	-8	-5	-8.13		8	-4	-116.57
	-1	-6	-45.00		6	-8	135.00
		-7			1	-3	
11	4	15	-123.69	15	-7	0	45.00
	0	9	56.31		-6	1	45.00
	2	12	-119.74		-4	3	45.00
	-2	5	108.43		-3	4	-135.00
	-3	8	123.69		-4	3	-135.00
	-5	11	90.00		-6	1	45.00
	-5	14	104.04		-4	3	-153.43
	-6	18	-93.18		-8	1	-21.80
	-7				-3	-1	
12	15	8	146.31	16	0	-4	45.00
	9	12	-66.80		1	-3	45.00
	12	5	-150.26		3	-1	45.00
	5	1	45.00		4	0	-135.00
	8	4	45.00		3	-1	-123.69
	11	7	53.13		1	-4	-63.43
	14	11	45.00		3	-8	135.00
	18	15	#REF!		1	-6	-111.80
					-1	-11	

	letter A						
17	-3	3	-53.13	21	-1	-13	45.00
	0	-1	-53.13		2	-10	45.00
	3	-5	-36.87		5	-7	45.00
	7	-8	-135.00		8	-4	-135.00
	4	-11	-161.57		4	-8	-135.00
	1	-12	104.04		1	-11	-135.00
	-2				-2	-14	-135.00
	1				-5	-17	
	3						
				22	-13	11	45.00
18	3	-1	143.13		-10	14	45.00
	-1	2	143.13		-7	17	45.00
	-5	5	135.00		-4	20	-135.00
	-8	8	-126.87		-8	16	-135.00
	-11	4	-108.43		-11	13	-135.00
	-12	1	-14.04		-14	10	-126.87
		-2	-90.00		-17	6	-3.37
		-5	90.00			5	
		8					
				23	11	-1	45.00
19	-1	-13	45.00		14	2	45.00
	2	-10	45.00		17	5	45.00
	5	-7	45.00		20	8	-135.00
	8	-4	-135.00		16	4	-135.00
	4	-8	-135.00		13	1	-116.57
	1	-11	-135.00		10	-5	
	-2	-14	-135.00		6		
	-5	-17			5		
	8				m. 30		
				24	11	4	45.00
20	-13	-1	45.00		14	7	45.00
	-10	2	45.00		17	10	-45.00
	-7	5	45.00		20	7	-143.13
	-4	8	-135.00		16	4	-146.31
	-8	4	-135.00		13	2	-135.00
	-11	1	-135.00		10	-1	-135.00
	-14	-2	-135.00		6	-5	101.31
	-17	-5			5		

	m. 30		
25	11	4	45.00
	14	7	45.00
	17	10	-45.00
	20	7	-143.13
	16	4	-146.31
	13	2	-135.00
	10	-1	-135.00
	6	-5	101.31
	5		
26	4	0	53.13
	7	4	45.00
	10	7	-135.00
	7	4	-126.87
	4	0	-123.69
	2	-3	-126.87
	-1	-7	-143.13
	-5	-10	63.43
27	0	0	45.00
	4	4	45.00
	7	7	135.00
	4	10	-143.13
	0	7	-146.31
	-3	5	-143.13
	-7	2	-135.00
	-10	-1	-21.80
		-5	
28	0	1	36.87
	4	4	53.13
	7	8	45.00
	10	11	-126.87
	7	7	-123.69
	5	4	-135.00
	2	1	-135.00

29	-11	1	45.00
	-8	4	45.00
	-4	8	45.00
	-1	11	-135.00
	-5	7	-135.00
	-8	4	-135.00
	-11	1	-135.00
	-14	-2	
30	1	-11	45.00
	4	-8	45.00
	8	-4	45.00
	11	-1	-135.00
	7	-5	-135.00
	4	-8	-135.00
	1	-11	-135.00
	-2	-14	
31	-11	1	45.00
	-8	4	45.00
	-4	8	45.00
	-1	11	-135.00
	-5	7	-135.00
	-8	4	-135.00
	-11	1	-135.00
	-14	-2	
32	0	1	36.87
	4	4	53.13
	7	8	45.00
	10	11	-126.87
	7	7	-123.69
	5	4	-135.00
	2	1	-135.00
	-1	-2	-143.13
	-5	-5	

Syncopated Quarter Notes

B, m. 50: transition to trio

33	1	-4	26.57
	5	-2	18.43
	8	-1	-18.43
	11	-2	-153.43
	7	-4	-146.31
	1	-8	126.87
	-2	-4	-33.69
	1	-6	-108.43
		-9	
34	-4	16	-56.31
	-2	13	-71.57
	-1	10	-104.04
	-2	6	123.69
	-4	9	135.00
	-8	13	36.87
	-4	16	116.57
	-6	20	-126.87
	-9	16	
35	16	7	146.31
	13	9	146.31
	10	11	165.96
	6	12	-45.00
	9	9	26.57
	13	11	-53.13
	16	7	26.57
	20	9	-128.66
	16	4	
36	7	12	63.43
	9	16	56.31
	11	19	75.96
	12	23	-126.87
	9	19	-56.31
	11	16	-135.00
	7	12	-56.31
	9	9	-119.05
	4		

37	12	9	36.87
	16	12	53.13
	19	16	36.87
	23	19	-143.13
	19	16	-126.87
	16	12	-143.13
	12	9	-126.87
	9	5	-150.95
38	9	-3	45.00
	12	0	45.00
	16	4	45.00
	19	7	-135.00
	16	4	-135.00
	12	0	-135.00
	9	-3	-135.00
	5	-7	168.69
		-6	
39	-3	13	-45.00
	0	10	-45.00
	4	6	-53.13
	7	2	135.00
	4	5	135.00
	0	9	135.00
	-3	12	135.00
	-7	16	-86.42
	-6		
40	13	2	-161.57
	10	1	-153.43
	6	-1	-153.43
	2	-3	-33.69
	5	-5	-14.04
	9	-6	-33.69
	12	-8	26.57
	16	-6	-169.38
		-9	

41	2	14	-135.00		Trio		
					Da		
	1	13	-135.00		Capo		
	-1	11	-135.00	45	2	-12	-53.13
	-3	9			5	-16	-36.87
					9	-19	-53.13
					12	-23	126.87
42	-5	7	-135.00		9	-19	143.13
	-6	6	-135.00		5	-16	126.87
	-8	4	-45.00		2	-12	143.13
	-6	2	-161.57		-2	-9	-53.13
	-9	1			1	-13	
		-1					
		-2		46	-12	13	165.96
					-16	14	146.31
43	14	26	-135.00		-19	16	165.96
	13	25	-135.00		-23	17	0.00
	11	23	-135.00		-19	17	-45.00
	9	21	-135.00		-16	14	-36.87
	7	19	-135.00		-12	11	33.69
	6	18	-135.00		-9	13	
	4	16	135.00		-13		
	2	18	-101.31				
	1	13	-135.00	47	13	6	45.00
	-1	11			14	7	45.00
	-2				16	9	45.00
					17	10	
44	26	-10	-135.00		17	10	-135.00
	25	-11	-135.00		14	7	-135.00
	23	-13	-135.00		11	4	45.00
	21	-15	-135.00		13	6	-155.22
	19	-17	-135.00			0	
	18	-18	-135.00				
	16	-20	86.99	48	6	16	-75.96
	18	18	-96.95		7	12	-56.31
	13	-23	94.97		9	9	-71.57
	11				10	6	135.00
					7	9	135.00
					4	12	75.96
					5	16	149.04
					0	19	
						16	

49	16	-6	143.13
	12	-3	135.00
	9	0	126.87
	6	4	-53.13
	9	0	-45.00
	12	-3	-36.87
	16	-6	-45.00
	19	-9	135.00
	16	-6	
50	-6	9	33.69
	-3	11	18.43
	0	12	26.57
	4	14	180.00
	0	14	-135.00
	-3	11	-126.87
	-6	7	146.31
	-9	9	
	-6		
51	9	4	45.00
	11	6	45.00
	12	7	45.00
	14	9	
	14	9	-135.00
	11	6	-135.00
	7	2	45.00
	9	4	
52	4	14	56.31
	6	17	75.96
	7	21	56.31
	9	24	-90.00
	9	21	-126.87
	6	17	-143.13
	2	14	-56.31
	4	11	

53	14	-1	45.00
	17	2	36.87
	21	5	53.13
	24	9	-126.87
	21	5	-143.13
	17	2	-135.00
	14	-1	-135.00
	11	-4	
54	-1	2	45.00
	2	5	53.13
	5	9	36.87
	9	12	-143.13
	5	9	-126.87
	2	5	-135.00
	-1	2	-135.00
	-4	-1	
55	2	-13	45.00
	5	-10	36.87
	9	-7	53.13
	12	-3	-126.87
	9	-7	-143.13
	5	-10	-135.00
	2	-13	-135.00
	-1	-16	
56	-13	17	18.43
	-10	18	-18.43
	-7	17	14.04
	-3	18	-153.43
	-7	16	-135.00
	-10	13	146.31
	-13	15	-126.87
	-16	11	

57	17	-10	81.87	61	3	-1	53.13
	18	-3	108.43		6	3	33.69
	17	0	71.57		9	5	53.13
	18	3	-116.57		12	9	-104.04
	16	-1	-135.00		11	5	36.87
	13	-4	-56.31		15	8	45.00
	15	-7	-143.13		18	11	53.13
	11	-10			21	15	-119.74
					17	8	
58	-10	5	23.20				
	-3	8	45.00	62	-1	1	36.87
	0	11	53.13		3	4	56.31
	3	15	-153.43		5	7	36.87
	-1	13	126.87		9	10	-165.96
	-4	17	135.00		5	9	53.13
	-7	20	135.00		8	13	45.00
	-10	23			11	16	36.87
		19			15	19	156.80
					8	22	
59	5	-11	53.13				
	8	-7	45.00	63	1	-4	53.13
	11	-4	36.87		4	0	45.00
	15	-1	-116.57		7	3	45.00
	13	-5	36.87		10	6	-104.04
	17	-2	45.00		9	2	36.87
	20	1	53.13		13	5	53.13
	23	5	-135.00		16	9	45.00
	19	1			19	12	
					22		
60	-11	3	36.87				
	-7	6	45.00	64	-4	1	45.00
	-4	9	45.00		0	5	45.00
	-1	12	-165.96		3	8	45.00
	-5	11	53.13		6	11	-135.00
	-2	15	45.00		2	7	-45.00
	1	18	36.87		5	4	-45.00
	5	21	-135.00		9	0	-33.69
	1	17			12	-2	

65	1	16	36.87
	5	19	53.13
	8	23	45.00
	11	26	-143.13
	7	23	-135.00
	4	20	-143.13
	0	17	-116.57
	-2	13	
66	16	1	53.13
	19	5	36.87
	23	8	45.00
	26	11	-126.87
	23	7	-135.00
	20	4	-135.00
	17	1	-143.13
	13	-2	
67	1	4	36.87
	5	7	53.13
	8	11	45.00
	11	14	-143.13
	7	11	-135.00
	4	8	-135.00
	1	5	-126.87
	-2	1	
68	4	1	53.13
	7	5	36.87
	11	8	45.00
	14	11	-126.87
	11	7	-135.00
	8	4	-135.00
	5	1	-143.13
	1	-2	

69	1	4	36.87
	5	7	53.13
	8	11	45.00
	11	14	-143.13
	7	11	-135.00
	4	8	-135.00
	1	5	-126.87
	-2	1	
70	4	-11	53.13
	7	-7	36.87
	11	-4	45.00
	14	-1	-126.87
	11	-5	-135.00
	8	-8	-126.87
	5	-12	
	1		
71	-11	-11	56.31
	-7	-5	53.13
	-4	-1	45.00
	-1	2	-143.13
	-5	-1	-135.00
	-8	-4	-135.00
	-11	-7	
	-14		
72	-11	-11	33.69
	-5	-7	36.87
	-1	-4	45.00
	2	-1	-126.87
	-1	-5	-135.00
	-4	-8	-135.00
	-7	-11	
		-14	

73	-11	-11	56.31	77	10	-3	116.57
	-7	-5	53.13		9	-1	11.31
	-4	-1	45.00		19	1	153.43
	-1	2	-143.13		17	2	116.57
	-5	-1	-135.00		16	4	153.43
	-8	-4	-135.00		14	5	135.00
	-11	-7			12	7	-45.00
	-14				14	5	
					9		
74	-11	-11	33.69				
	-5	-7	36.87	78	-3	7	45.00
	-1	-4	45.00		-1	9	45.00
	2	-1	-126.87		1	11	63.43
	-1	-5	-135.00		2	13	45.00
	-4	-8	-135.00		4	15	45.00
	-7	-11			5	16	26.57
					7	17	-153.43
					5	16	
75	-11	14	-36.87				
	-7	11	-53.13	79	7	-24	71.57
	-4	7	-53.13		9	-18	26.57
	-1	3	143.13		11	-17	45.00
	-5	6	135.00		13	-15	45.00
	-8	9	135.00		15	-13	63.43
	-11	12	19.98		16	-11	-63.43
		16			17	-13	116.57
		19			16	-11	
	letters A through C					-20	
76	24	22	-94.76				
	23	10	102.53	80	-24	17	-9.46
	21	19	-135.00		-18	16	-63.43
	19	17	-153.43		-17	14	-45.00
	17	16	-116.57		-15	12	-26.57
	16	14	-135.00		-13	11	-45.00
	14	12	45.00		-11	9	135.00
	16	14			-13	11	-45.00
		9			-11	9	-150.95
					-20	4	

81	17	0	-135.00	85	2	16	-53.13
	16	-1	-135.00		5	12	-36.87
	14	-3	-153.43		9	9	-33.69
	12	-4	-108.43		12	7	146.31
	11	-7	180.00		9	9	143.13
	9	-7	74.05		5	12	126.87
	11	0	-100.30		2	16	135.00
	9	-11			-1	19	
	4				2		
82	0	2	-116.57	86	16	1	143.13
	-1	0	-153.43		12	4	126.87
	-3	-1	-116.57		9	8	123.69
	-4	-3	-161.57		7	11	-56.31
	-7	-4	-90.00		9	8	-53.13
	-7	-6	-8.13		12	4	-36.87
	0	-7	-169.70		16	1	-45.00
	-11	-9			19	-2	
83	2	19	-123.69	87	1	-3	45.00
	0	16	-104.04		4	0	45.00
	-1	12	-123.69		8	4	45.00
	-3	9	108.43		11	7	-135.00
	-4	12	116.57		8	4	-135.00
	-6	16	108.43		4	0	-135.00
	-7	19	116.57		1	-3	-146.31
	-9	23			-2	-5	
84	19	2	-126.87	88	-3	-7	45.00
	16	5	-143.13		0	-4	45.00
	12	9	-146.31		4	0	53.13
	9	12	33.69		7	4	-126.87
	12	9	36.87		4	0	-135.00
	16	5	53.13		0	-4	-135.00
	19	2	36.87		-3	-7	-153.43
	23	-1			-5	-8	
	19	2					

89	-7	-9	45.00
	-4	-6	36.87
	0	-3	36.87
	4	0	-143.13
	0	-3	-143.13
	-4	-6	-135.00
	-7	-9	97.13
	-8	-1	
90	-9	1	45.00
	-6	4	45.00
	-3	7	53.13
	0	11	-126.87
	-3	7	-135.00
	-6	4	-126.87
	-9	0	-14.04
	-1	-2	
91	1	21	-45.00
	4	18	-45.00
	7	15	-45.00
	11	11	143.13
	7	14	135.00
	4	17	135.00
	0	21	116.57
	-2	25	
92	21	13	146.31
	18	15	146.31
	15	17	165.96
	11	18	-18.43
	14	17	-53.13
	17	13	26.57
	21	15	-51.34
	25	10	

93	13	-3	26.57
	15	-2	45.00
	17	0	45.00
	18	1	-135.00
	17	0	-135.00
	13	-4	45.00
	15	-2	-129.81
	10	-8	
94	-3	13	-71.57
	-2	10	-56.31
	0	7	-75.96
	1	3	108.43
	0	6	143.13
	-4	9	56.31
	-2	12	146.31
	-8	16	
		13	

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