Article ID: 1007－2985(2012)05－0001－08

Meson Equation of Surface Type *

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Abstract: This paper provides a meson functional equation which is extracted from counting maps (rooted) on all orientable surfaces with vertex partition vector given. The well-definedness of its solution is shown on an extension of the integral domain. Then the solution is explicitly expressed in a compact form via considering graph symmetry.

Key words: Meson functional; equation; map; surface; embedding; graph symmetry
CLC number: O177　　**Document code**: A　　**DOI**: 10.3969/j.issn.1007－2985.2012.05.001

1 Background

Let $\mathscr{V}$ and $\mathscr{F}$ be, respectively, the vector space spanned by the basis $\{1, y_1, y_2, \cdots, y_i, \cdots\}$ and the function space by the basis $\{1, y, y^2, \cdots, y^i, \cdots\}$ over the real field $\mathbf{R}$.

The transformation denoted by $\int_y$ from $\mathscr{F}$ to $\mathscr{V}$ is called a meson functional. The shadow functional used by Rota G C[1], is the case of the meson functional when $\{1, y_1, y_2, \cdots, y_i, \cdots\}$ is replaced by $\{(y)_0, (y)_1, (y)_2, \cdots, (y)_i, \cdots\}$ where

$$(y)_i = \begin{cases} 1 & \text{when } i = 0, \\ \prod_{j=0}^{i-1} (y - j) & \text{when } i \geqslant 1. \end{cases}$$

The both functionals are as a type of Blissard operator[2]. An equation involving with the meson functional is called meson equation.

Although such an equation has been used for enumerating maps with vertex partition as parameter since the 80s of last century[3], the word "meson functional" has not been distinguished from Blissard operator or shadow functional until 2010[4]. However, Tutte's enumerating maps with vertex partition appeared in literature much early without use of an equation[5]. In ref. [4,6] there are a number of meson equations, particularly in surface type, but no solution.

The purpose of this paper is to provide a new meson equation of surface type, its well-definedness in a certain domain and an explicit expression of the solution via considering related symmetry.

* **Received date**: 2011－09－12
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2 Maps and Embeddings

The concepts of polyhedra, surfaces, embeddings and maps are clarified in ref. [7] (or Appendix I in ref. [4]) with relationships among them. In this section, only results related to this paper are overlooked in certain detail.

Let $G=(V,E)$ be a graph of vertex set V and edge set E. Two Graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is called isomorphic if there exists a 1-1 mapping $\tau: V_1 \to V_2$ such that for any $u,v\in V_1$, $(u,v)\in E_1 \Leftrightarrow (\tau(u),\tau(v))\in E_2$. For such a mapping τ, both τ and τ^{-1} are called an isomorphism between $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$. An automorphism on G is an isomorphism between G and itself. The set of all automorphisms of a graph form a group called the automorphism group, denoted by $\mathrm{Aut}(G)$, of G.

For our purpose, let us see another model of a graph $G=(V,E)$. Let $B=\{0,1\}$ be the group of two elements. For any $x\in E$, let $Bx=\{x_0,x_1\}$ where $x_0=x(0)$ and $x_1=x(1)$ as in ref. [4] (firstly in ref. [8]). The graph G is seen to be such a partition κ on $\chi=\sum_{x\in E}Bx$ that $V=P$. Such a model of a graph enables us to define a $\frac{1}{2}$-automorphism of a graph $G=(V,E)$ a 1-1 mapping τ on χ itself that for $x,y\in\chi$, $x\in\kappa_y \Leftrightarrow \kappa_{\tau(x)}=\kappa_{\tau(y)}$. The set of all automorphisms of a graph forms a group as well, called the $\frac{1}{2}$-automorphism group, denoted by $\mathrm{Aut}_{1/2}(G)$, of G.

Lemma 1 Graph G is connected if and only if χ is transitive under $\Psi_{\beta,\kappa}$, the group generated by β and κ where for any $x(i)\in\chi$,

$$\beta(x(i))=\begin{cases} x(0) & \text{when } i=1, \\ x(1) & \text{when } i=0. \end{cases}$$

Proof Because of $\{1,\beta\}\simeq B$, β is the only element not the identity, similarly to what was shown in ref. [4].

Let $\mathrm{aut}(G)=|\mathrm{Aut}(G)|$ and $\mathrm{aut}_{1/2}(G)=|\mathrm{Aut}_{1/2}(G)|$, i. e., the orders of groups, respectively, $\mathrm{Aut}(G)$ and $\mathrm{Aut}_{1/2}(G)$.

Lemma 2[9] or [4] For a graph G with l self-loops, $\mathrm{aut}_{1/2}(G)=2^l\,\mathrm{aut}(G)$.

Proof See (14.4) in ref. [4].

This lemma tells us that $\mathrm{Aut}_{1/2}(G)$ is different from $\mathrm{Aut}(G)$ if and only if G has an edge which is a self-loop.

An embedding of a graph G (connected, default without loss of generality from lemma 1) is a topological mapping from G into a surface, i. e., a compact 2-dimensional manifold without boundary. It has been shown in ref. [4] that an embedding of G is combinatorially equivalent to a permutation π on χ transitive under $\Psi_{B,\pi}$.

Lemma 3 Let $n_O(G)$ be the number of distinct embeddings on orientable surfaces, then the number of embeddings on all surfaces is

$$n_O(G)=\prod_{i\geq 2}((i-1)!)^{n_i}, \tag{1}$$

where n_i is the number of vertices of degree i in G.

Proof See (1.10) in ref. [4].

Given a graph $G=(V,E)$, let $K=\{1,\alpha,\beta,\gamma\}$ where $\gamma=\alpha\beta$, $\alpha^2=\beta^2=1$ and hence $\gamma^2=1$. In other words, K is the Klein group of order 4, denoting that $z=\sum_{x\in E}Kx$ where $Kx=\{x,\alpha,\beta,\gamma\}$ is called a quadricell. A map, or super map of G, is defined to be such a permutation π on z that z is partitioned into conjugate pairs $\{(x)_\pi,(\alpha x)_\pi\}$ of orbits for $x\in z$ and the group $\Psi_{\{\alpha,\beta,\pi\}}$ is transitive on z (because of the connectedness on G).

It is easy to see that the graph formed by each conjugate pair of orbits as a vertex and a quidricell as a edge is isomorphic to G. The map is denoted by M, or precisely, $M_G=(z,\pi)$. It is easily shown that M_G is an embedding of G.

Two maps $M_1=(z_1,\pi_1)$ and $M_2=(z_2,\pi_2)$ are said to be isomorphic if there exists a 1-1 mapping (bijection in discrete case) $\tau: z_1 \to z_2$ such that the diagrams

$$\begin{array}{ccc} z_1 & \xrightarrow{\ \tau\ } & z_2 \\ {\scriptstyle\eta_1}\downarrow & & \downarrow{\scriptstyle\eta_2} \\ z_1 & \xrightarrow{\ \tau\ } & z_2 \end{array}$$

for $\eta_1=\eta_2=\alpha$, $\eta_1=\eta_2=\beta$, and for $\eta_1=\pi_1$ and $\eta_2=\pi_2$, are all commutative. The bijection τ is an isomorphism.

If a map has an element in z is specified and called the root, then it is called a rooted map. Two rooted maps with their roots, respectively, r_1 and r_2 are said to be isomorphic if there exists an isomorphism τ between them such that $\tau(r_1)=r_2$.

Theorem 1 The number of nonisomorphic rooted super maps of a graph G on all orientable surfaces is

$$\frac{2\epsilon(G)}{\mathrm{aut}_{1/2}(G)} n_O(G), \tag{2}$$

where $\epsilon(G)$ is the size of G, $\mathrm{aut}_{1/2}(G)$ and $n_O(G)$ are, respectively, the order of 1/2-automorphism group and the number of distinct embeddings on all orientable surfaces of G.

Proof See (11.4.4) in ref. [6] (firstly in ref. [8] for the petal bundles).

3 Decomposition Via Partitions

Let $\mathscr{M}$ be the set of rooted maps on all orientable surfaces. For $M=(z,\pi)$ with its root $r=r(M)$, let $m(M)=|\{r\}_\pi|$ and $\underline{n}(M)$ be, respectively, the root-vertex valency and the partition vector of nonroot-vertices where the i-th component $n_i(M)$ of $\underline{n}(M)$ is the number of nonroot-vertices with valency i, $i\geqslant 1$.

On the basis of availability which will be seen in what follows, $\mathscr{M}$ can be partitioned into three classes $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ as

$$\mathscr{M}=\mathscr{M}_0+\mathscr{M}_1+\mathscr{M}_2, \tag{3}$$

where $\mathscr{M}_0$ has the single vertex map ϑ, $\mathscr{M}_1$ is the set of all maps with their root-edges a selfloop and $\mathscr{M}_2$ is the set of all maps with their root-edges a link.

Lemma 4 Let $\mathscr{M}_{\langle 1\rangle}=\{M-Kr \mid \forall M\in\mathscr{M}_1\}$, then $\mathscr{M}_{\langle 1\rangle}=\mathscr{M}$.

Proof For any $M=(z,\pi)\in\mathscr{M}_1$, because of Kr as a selfloop, $M'=M-Kr=(z-Kr,\pi')\in\mathscr{M}_{\langle 1\rangle}$ with π' different from π in $\{r'\}_{\pi'}=\{r\}_\pi-\{r,\gamma r\}$ and its conjugate where $r'=\pi r$. Therefore, $M'\in\mathscr{M}$. This implies $\mathscr{M}_{\langle 1\rangle}\subseteq\mathscr{M}$.

Conversely, for any $M(z,\pi)\in\mathscr{M}$, we may construct $M'=(z+Kr',\pi')$ where $r'\notin z$ and π' are different from π only in $(r')_{\pi'}=(r',\langle r\rangle_\pi,\gamma r')$ and its conjugate. It is easy to check that M' is a map and hence $M=M'-Kr'$. Because of Kr' as a selfloop, $M'\in\mathscr{M}_1$ and hence $M\in\mathscr{M}_{\langle 1\rangle}$. Therefore, $\mathscr{M}\subseteq\mathscr{M}_{\langle 1\rangle}$.

It is seen from the converse part in the proof that all $M_i=(z+Kr_i,\pi_i)\in\mathscr{M}_1$, $0\leqslant i\leqslant m(M)-1$, constructed from M are nonisomorphic where π_i is different from π only in

$$(r_i)_{\pi_i}=\begin{cases}(r_i,r,\pi r,\cdots,\pi^i r,\gamma r_i,\pi^{i+1}r,\cdots,\pi^{m(M)-1}r) & \text{when } 0\leqslant i\leqslant m(M)-1,\\ (r_i,\gamma r_i,\langle r\rangle_\pi) & \text{when } i=m(M)\end{cases} \tag{4}$$

and their conjugates.

Lemma 5 For $M\in\mathscr{M}$, let $\mathsf{S}_M=\{M_i \mid 0\leqslant i\leqslant m(M)\}$, then we have

$$\mathscr{M}_1=\sum_{M\in\mathscr{M}}\mathsf{S}_M. \tag{5}$$

Proof From (4), $\mathsf{S}_M\cap\mathsf{S}_N=\emptyset\Leftrightarrow M\not\cong N$. From lemma 4, (5) is then obtained.

Lemma 6 Let $\mathscr{M}_{(2)}=\{M\cdot Kr \mid \forall M\in\mathscr{M}_2\}$, then $\mathscr{M}_{(2)}=\mathscr{M}$.

Proof For any $M=(\mathscr{X},\pi)\in\mathscr{M}_2$, because of Kr as a link, $M'=M\cdot Kr=(\mathscr{X}-Kr,\pi')\in\mathscr{M}_{(2)}$ with π' different from π only in

$$(r')_{\pi'}=(\pi\gamma r,\pi^2\gamma r,\cdots,\pi^{-1}\gamma r,\pi r,\pi^2 r,\cdots,\pi^{-1}r)$$

and its conjugate where $r'=\pi\gamma r$. Therefore, $M'\in\mathscr{M}$. This implies $\mathscr{M}_{(2)}\subseteq\mathscr{M}$.

Conversely, for any $M=(z,\pi)\in\mathscr{M}$, we may construct $M'=(z+Kr',\pi')$ where $r'\notin z$ and π' is different from π in $(r')_{\pi'}=(r')$ and $(\gamma r')_{\pi'}=(\gamma r',\langle r\rangle_\pi)$ with their conjugates. Easy to check that M' is a map and hence $M=M'\cdot Kr'$. Because of Kr' as a link, $M'\in\mathscr{M}_2$ and hence $M\in\mathscr{M}_{(2)}$. Therefore, $\mathscr{M}\subseteq\mathscr{M}_{(2)}$.

By observing the converse part in the proof of lemma 6, it is seen that that all $M_i=(z+Kr_i,\pi_i)\in\mathscr{M}_2$, $1\leqslant i\leqslant m(M)+1$, constructed from M are nonisomorphic where π_i is different from π in

$$\begin{cases}(r_1)_{\pi_1}=(r_1),(\gamma r_1)_{\pi_1}=(\gamma r_1,\langle r\rangle_\pi) & \text{when } i=1,\\ (r_2)_{\pi_2}=(r_2,\pi^{m(M)-1}r),(\gamma r_2)_{\pi_2}=(\gamma r_2,r,\pi r,\cdots,\pi^{m(M)-2}r) & \text{when } i=2,\\ (r_i)_{\pi_i}=(r_i,\pi^{m(M)-i+1}r,\cdots,\pi^{m(M)-1}r),(\gamma r_i)_{\pi_i}=(\gamma r_1,r,\cdots,\pi^{m(M)-i}r) & \\ & \text{when } 3\leqslant i\leqslant m(M)+1\end{cases} \tag{6}$$

with their conjugates for $1\leqslant i\leqslant m(M)+1$.

Lemma 7 For $M\in\mathscr{M}$, let $\mathscr{T}_M=\{M_i \mid 1\leqslant i\leqslant m(M)+1\}$, then we have

$$\mathscr{M}_2=\sum_{M\in\mathscr{M}}\mathscr{T}_M. \tag{7}$$

Proof From (6), $\mathscr{T}_M\cap\mathscr{T}_N=\emptyset\Leftrightarrow M\not\cong N$. From lemma 6, (7) is then obtained.

Nothing should be done for the decomposition of $\mathscr{M}_0$ because of only one map ϑ considered.

4 Meson Equation

Given the set $\mathscr{M}$ of rooted maps on all orientable surfaces. For a map $M\in\mathscr{M}$, let $m(M)$ and $\underline{n}(M)=(n_1(M),n_2(M),\cdots,n_i(M),\cdots)$ be, respectively, the root-vertex valency and the vertex partition vector of M where $n_i(M)$ is the number of nonroot-vertices with valency i, $i\geqslant 1$.

The function $f_{\mathscr{M}}$ determined by

$$f_{\mathscr{M}}(x,\underline{y})=\sum_{M\in\mathscr{M}}x^{m(M)}\underline{y}^{\underline{n}(M)} \tag{8}$$

is called the enumerating function, or in short enufunction of $\mathscr{M}$ where $\underline{y}^{n(M)}=\prod_{i\geqslant 1}y_i^{n_i(M)}$.

For the integral domain $\mathscr{R}$ of all integers, i. e., the integral ring with the cancelation law considered, let $\mathscr{R}\{x,\underline{y}\}$ be the extension of $\mathscr{R}$ with x and $\underline{y}$ (called undeterminate) involved. Because of $\mathscr{R}$ infinity, $\mathscr{R}\{x,\underline{y}\}$ is still a domain. Denote by $\mathscr{R}_+\{x,\underline{y}\}$ the set of all functions in $\mathscr{R}\{x,\underline{y}\}$ with coefficients in $\mathscr{R}_+$, the set of all nonnegative integers. Apparently, $f_{\mathscr{M}}(x,\underline{y})\in\mathscr{R}_+\{x,\underline{y}\}\subset\mathscr{R}\{x,\underline{y}\}$.

Lemma 8(Theorem 1.3.5 in ref. [6]) Let S and $\mathscr{T}$ be two sets of maps. If there exists a mapping λ from $\mathscr{T}$ to S such that $\lambda(T)\subseteq\mathsf{S}$ for any $T\in\mathscr{T}$ with the properties: (i) $|\lambda(T)|=am(T)+b$ where $m=m(T)$ is an isomorphic invariant on maps and both a and b are constants; (ii) $\mathsf{S}=\sum_{T\in\mathscr{T}}\lambda(T)$; (iii) $m(S)=m(T)+c$ where c is a constant for any $S\in\mathsf{S}$, then the enufunction of S with parameter m, $g_{\mathsf{S}}(x)=$

$x^c(bg_{\mathscr{T}}+ax\dfrac{\mathrm{d}g_{\mathscr{T}}}{\mathrm{d}x})$.

Proof Because λ is a 1 to $an(T)+b$ correspondence for $T\in\mathscr{T}$, properties (i - iii) yield $g_{\mathsf{S}}(x)=x^c\sum_{T\in\mathscr{T}}(am(T)+b)x^{m(T)}$.

Further more, by considering that $\sum_{T\in\mathscr{T}}m(T)x^{m(T)}=x\dfrac{\mathrm{d}g_{\mathscr{T}}}{\mathrm{d}x}$, the theorem is soon found.

Theorem 2 For $\mathscr{M}_1$, we have

$$f_{\mathscr{M}_1}=x^3\frac{\partial f_{\mathscr{M}}}{\partial x}+x^2f_{\mathscr{M}}. \tag{9}$$

Proof On account of

$$f_{\mathscr{M}_1}=\sum_{M\in\mathscr{M}_1}x^{m(M)}\underline{y}^{\underline{n}(M)}\underline{\underline{(\text{by lemma 4})}}\,x^2\sum_{M-Kr\in\mathscr{M}_{\langle 1\rangle}}x^{m(M)}\underline{y}^{\underline{n}(M)}\underline{\underline{(\text{by lemma 5 and lemma 8)}}}$$
$$x^2(x\frac{\partial f_{\mathscr{M}}}{\partial x}+f_{\mathscr{M}}),$$

the theorem is obtained.

Lemma 9(Theorem 1. 6. 3 in ref. [6]) Let S and $\mathscr{T}$ be two sets of maps. If there exists a mapping λ from $\mathscr{T}$ to S, $\lambda(T)=\{S_1,S_2,\cdots,S_{m(T)+1}\}$ for any $T\in\mathscr{T}$ such that S_i 1 - 1 corresponds to $\{i,m(T)+2-i\}$ where i and $m(T)+2-i$ are, respectively, the contributions to the first and the second parameters, $i=1,2,\cdots,m(T)+1$, with the condition that $\mathsf{S}=\sum_{T\in\mathscr{T}}\lambda(T)$, then

$$F_{\mathsf{S}}(x,y)=xy\delta_{x,y}(zf_T), \tag{10}$$

where $f_T=f_T(z,\underline{y})$.

Proof By virtue of the way determining λ, we have

$$F_{\mathsf{S}}(x,y)=\sum_{T\in\mathscr{T}}\sum_{i=1}^{m(T)+1}x^iy^{m(T)-i+2}\underline{y}^{\underline{n}(T)}=xy\sum_{T\in\mathscr{T}}\frac{x^{m(T)+1}-y^{m(T)+1}}{x-y}\underline{y}^{\underline{n}(T)}=xy\delta_{x,y}(zf_T).$$

This is (10).

Theorem 3 For $\mathscr{M}_2$, we have

$$f_{\mathscr{M}_2}=\int_y y\delta_{x,y}(zf_{\mathscr{M}}). \tag{11}$$

Proof On account of

$$f_{\mathscr{M}_2}=\sum_{M\in\mathscr{M}_2}x^{m(M)}\underline{y}^{\underline{n}(M)}\underline{\underline{(\text{by lemma 6})}}x\sum_{M\cdot Kr\in\mathscr{M}_{(2)}}x^{m(M)}\underline{y}^{\underline{n}(M)}\underline{\underline{(\text{by lemma 7 and lemma 9)}}}$$
$$(x\int_y y\delta_{x,y}(zf_{\mathscr{M}})),$$

the theorem is obtained.

Theorem 4 The enufunction $f_{\mathscr{M}}$ of general rooted maps on all orientable surfaces with x for root-vertex valency and $\underline{y}$ for partition vector of nonroot-vertices satisfies the following meson equation about f as

$$f=1+x^3\frac{\partial f}{\partial x}+x^2f+x\int_y y\delta_{x,y}(zf). \tag{12}$$

Proof Because of ϑ with no edge, $f_{\mathscr{M}_0}=1$. On the basis of (3), from (9) and (11), The theorem is soon obtained.

5 Well-Definedness

Now, we discuss the well-definedness of the meson equation as

$$\begin{cases} x^3 \dfrac{\partial f}{\partial x} = -1 + (1-x^2) f - x \displaystyle\int_y y\delta_{x,y}(zf), \\ f_{x=0,\underline{y}=\underline{0}} = 1 \end{cases} \tag{13}$$

on $\mathscr{R}\{x,\underline{y}\}$.

By observing the limitation of domain considered, f enables us to write in form as

$$f = \sum_{m\geqslant 0} F_m(\underline{y}) x^m, \tag{14}$$

where

$$F_m(\underline{y}) = \sum_{n\geqslant 0} \Big(\sum_{\substack{|\underline{n}|=n \\ \underline{n}\geqslant\underline{0}}} A_{m,\underline{n}} \underline{y}^{\underline{n}}\Big), \tag{15}$$

and $|\underline{n}| = n_1 + 2n_2 + 3n_3 + \cdots + in_i + \cdots$.

Then, we can get

$$\frac{\partial f}{\partial x} = \sum_{m\geqslant 1} mF_m(\underline{y}) x^{m-1} = \sum_{m\geqslant 0} (m+1) F_{m+1}(\underline{y}) x^m, \tag{16}$$

and

$$\delta_{x,y}(zf) = \sum_{m\geqslant 0} F_m(\underline{y}) \delta_{x,y} z^{m+1} = \sum_{m\geqslant 0} F_m(\underline{y}) \sum_{i=1}^{m+1} x^i y^{m-i+2}. \tag{17}$$

On the basis of (17), we have

$$\int_y y\delta_{x,y}(zf) = \sum_{m\geqslant 0} F_m(\underline{y}) \sum_{i=1}^{m+1} x^i y_{m-i+3} = \sum_{m\geqslant 1} \Big(\sum_{i\geqslant m-1} F_i(\underline{y}) y_{i-m+2}\Big) x^m. \tag{18}$$

From the initial condition of (13), we have

$$F_0(\underline{y}) = 1. \tag{19}$$

For the convenience of logical reasoning, the equivalent form (12) of (13) is employed in what follows. On the basis of (14), (15), (16) and (18), we have

$$F_m(\underline{y}) = \begin{cases} \displaystyle\sum_{i\geqslant 0} F_i(\underline{y}) y_{i+1} & \text{when } m = 1, \\ 1 + \displaystyle\sum_{i\geqslant 1} F_i(\underline{y}) y_i & \text{when } m = 2, \\ (m-1) F_{m-2}(\underline{y}) + \displaystyle\sum_{i\geqslant m-1} F_i(\underline{y}) y_{i-m+2} & \text{when } m \geqslant 3. \end{cases} \tag{20}$$

Lemma 10 for any integer $m \geqslant 0$ and integral vector $\underline{n} \geqslant \underline{0}$, $m + n = 0 \pmod 2$.

Proof On the basis of (15), let

$$F_{m,n}(\underline{y}) = \sum_{\substack{|\underline{n}|=n \\ \underline{n}\geqslant\underline{0}}} A_{m,\underline{n}} \underline{y}^{n}, \tag{21}$$

then we have

$$F_m(\underline{y}) = \sum_{n\geqslant 0} F_{m,\underline{n}}(\underline{y}). \tag{22}$$

By induction on $m+n \geqslant 0$ and in virtue of (9) and (11), from m=n=0 on, for any $m+n \geqslant 0$, the pair $(m,\underline{n})$ is obtained in two available possibilities: from a pair $(m',\underline{n}')$ such that $m'+n' \leqslant m+n$ by $m=m'+2$ while $n=n'$ as in (9) or by $m=m'+1$ while $n=n'+1$. In the both cases, $m+n=m'+n'+2$. Moreover, from (19), the initial condition, the lemma is done.

This lemma enables us to take $s=(m+n)/2$, a nonnegative integer. For $s=0,1,2,\cdots$, do our procedure to determine all $F_m(\underline{y})$ via (20).

Theorem 5 The meson functional equation in form as (13) does have, and only have a solution in $\mathscr{R}\{x,\underline{y}\}$.

Proof The does case of the theorem is from theorem 4. The only case is from the uniqueness of the procedure for determining all $F_m(\underline{y})$ via $A_{m,\underline{n}}$ shown in (21) and (22) step by step from (19) on the basis of (20) in $\mathscr{R}\{x, \underline{y}\}$.

6 Solution

Theorem 4 and theorem 5 enable us to extract the solution of equation (13) by either solving (13) directly in $\mathscr{R}\{x, \underline{y}\}$, or enumerating all general rooted map on orientable surfaces with the root-vertex valency m and the partition vector $\underline{n}$ of nonroot-vertices. Because the former is not a suitable way founded yet, we have to consider the later first. This is what the paper is concentrated on.

According to (14), (15), (21) and (22), in order to determine the solution of equation (13), it is only necessary to find $A_{m,\underline{n}}$ for all available integer $m \geqslant 0$ and integral vector $\underline{n} \geqslant \underline{0}$.

Lemma 11　Let $\mathscr{G}_t[m, \underline{n}]$ be the set of all graphs whose super maps are with root-vertex valency m, vertex partition vector of nonroot-vertices $\underline{n}$ and $t = \mathrm{aut}_{1/2}(G)$, then the number of nonisomorphic rooted maps with $(m, \underline{n})$ is

$$\sum_{t \in I_{m,\underline{n}}} \frac{m + |\underline{n}|}{t} |\mathscr{G}_t[m, \underline{n}]| (m-1)! \, (\underline{l} - \underline{1})!^{\underline{n}},$$

where $I_{m,\underline{n}}$ is the set of all orders of $\frac{1}{2}$-automorphism groups in $\mathscr{G}_t[m, \underline{n}]$, $\underline{l} = (1,2,3,4,\cdots,l,\cdots)$ and $\underline{1} = (1,1,1,1,\cdots,1,\cdots)$. The factorial of a vector is defined to be the product of its component factorials.

Proof　From (1) and (2), the lemma is soon found whenever notifying that $m + |\underline{n}| = 2\epsilon$ where ϵ is the size of graphs in $\mathscr{G}_t[m, \underline{n}] \subseteq \mathscr{G}_{m,\underline{n}}$.

This lemma enables us to get an explicit expression of the solution of equation (13).

Theorem 6　The solution of equation (13) is

$$f = \sum_{\substack{m \geqslant 0 \\ \underline{n} \geqslant \underline{0}}} \Big(\sum_{t \in I_{m,\underline{n}}} \frac{m + |\underline{n}|}{t} |\mathscr{G}_t[m, \underline{n}]| (m-1)! \, (\underline{l} - \underline{1})!^{\underline{n}} \Big) x^m \underline{y}^{\underline{n}}. \tag{23}$$

Proof　Because of

$$f = \sum_{M \in \mathscr{M}} x^{m(M)} \underline{y}^{\underline{n}(M)} = \sum_{\substack{\underline{n} \geqslant \underline{0} \\ m \geqslant 0}} |\mathscr{M}_{m,\underline{n}}| x^m \underline{y}^{\underline{n}},$$

we see that $A_{m,\underline{n}} = |\mathscr{M}_{m,\underline{n}}|$ is the number of nonisomorphic rooted maps with $(m, \underline{n})$. Hence from lemma 11, the proof is done.

7 Conclusions

(1) On lemma 5 and lemma 7, (20), (21) and (22) have already presented a recursive formula for determining all $A_{m,\underline{n}}$ and hence the solution of meson functional equation (13), it is still necessary to make the formula more simple, particularly more compact.

(2) The solution of equation (13) obtained in §6 enables us to access other surface types of meson functional equations appeared in ref. [6].

(3) Although all meson functional equations of planar type shown in ref. [6] can also be done in a similar way, their solutions look more complicate in company with (23) because the number of distinct planar embeddings depending on splitting pairs of the graph.

(4) The solution of equation (13) shown in (23) is not hard to calculate for the order of maps not too big by using the computing program mentioned in ref. [8] because the order of graph $\frac{1}{2}$-automorphism groups can be done by a program.

However, it is absolutely not easy yet for determining all $\frac{1}{2}$-automorphism groups of a graph in general even in a number of manners suggested in ref. [4] to access.

(5) A further important and difficult case is for $\mathscr{M}$ and a surface of genus not zero given, to establish a meson functional equation satisfied by the enufunction (8) restricted and then to get the solution.

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曲面型介子泛函方程

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摘 要: 提出一个介子泛函方程.它是在以根点次和非根顶点剖分向量为参数,数所有可定向曲面上的地图时萃取出来的.在一个整域扩张中,证明了它的解存在且唯一.进而,通过图的对称性,给出这个解的一个紧凑显式.

关键词: 介子泛函;方程;地图;曲面;嵌入;对称性

中图分类号: O177 **文献标识码:** A

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更正说明

本刊2012年第3期第21页图2的纵坐标标目应为"lg(分数)",特此更正,并向作者、读者致歉。

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