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Asymptotic Behavior of the Toda Equation*

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Abstract: Using the basic property of the interaction potential of exponential type, the asymptotic behavior of the solutions for the Toda equation is studied. It is proved that every solution of the Toda equation is asymptotically linear at infinity.

Key words: Toda equation; asymptotic behavior; Allen-Cahn equation
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1 Introduction

Classical Toda equation is a system of second order ordinary differential equations which describes the behavior of a set of finite many particles moving along a straight line with exponential neighboring interaction. Explicitly, it reads as:

$$\begin{cases} q_1''(t) = -e^{q_1-q_2}, \\ q_i''(t) = e^{q_{i-1}-q_i} - e^{q_{i+1}-q_i} \qquad i = 2, \cdots, n-1, \\ q_n''(t) = e^{q_{n-1}-q_n}. \end{cases} \tag{1}$$

This is an integrable Hamiltonian system and was first studied by Toda[1]. It has appeared in many areas of mathematicas and physics. Recently, its deep relation with the entire solutions of certain nonlinear partial differential equation, for example, the Allen-Cahn equation, has been revealed, see ref. [2]. It turns out that the asymptotic behavior of this system is extremely important. On the other hand, recent advances of the classification of four-end solutions to the Allen-Cahn equation[3] tells us that the understanding of the asymptotic behavior of a Toda type equation will be a first step towarding the classification of finite Morse index entire solution to the Allen-Cahn equation in $\mathbf{R}^2$. The long time scattering behavior of this Toda equation is first studied by Moser[4], based on the Flaschka transform. In this paper, we wish to consider this problem from classical ODE point of view and using elementary tool to prove that the solution is asymptotically linear based on the internal stucture of the exponential potential of the Toda equation. We remark that our method could be generalized to more general nonliearities than the one appeared in the Toda equation. Our result states as what follows

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Theorem 1 Suppose $(q_1,\cdots,q_n)$ is a solution of the Toda system (1). Then there exist constants $\alpha_i^+,\alpha_i^-,\beta_i^+,\beta_i^-,i=1,\cdots,n$, such that:

$$q_i(t)\to\alpha_i^+t+\beta_i^+\qquad t\to+\infty,$$
$$q_i(t)\to\alpha_i^-t+\beta_i^-\qquad t\to-\infty.$$

Addtionally, for each $1\leqslant i<n-1$, there holds $\alpha_i^+<\alpha_{i+1}^+,\alpha_i^->\alpha_{i+1}^-$.

2 Proof of the Main Theorem

We first remark that the equation (1) is a second order Hamiltonian system whose Hamiltonian function is given by

$$H(\boldsymbol{p},\boldsymbol{q})=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\sum_{i=1}^{n-1}\mathrm{e}^{q_i-q_{i+1}},$$

where $\boldsymbol{p}=(p_1,\cdots,p_n),\boldsymbol{q}=(q_1,\cdots,q_n)$. As a consequence, letting $p_i=q_i'$, equation (1) could also be written as a first order Hamiltonian system $\boldsymbol{x}'(t)=\boldsymbol{J}\nabla H(\boldsymbol{x})$, where $\boldsymbol{x}=(\boldsymbol{p},\boldsymbol{q})^{\mathrm{T}},\boldsymbol{J}=\begin{pmatrix}\boldsymbol{0}&-\boldsymbol{I}\\ \boldsymbol{I}&\boldsymbol{0}\end{pmatrix}$. It is easy to see that for each solution $\boldsymbol{x}=(\boldsymbol{p},\boldsymbol{q})$, there holds $H(\boldsymbol{p}(t),\boldsymbol{q}(t))=C=$constant. This implies particularly that for a solution $(q_1,\cdots,q_n)$ of equation (1), we have

$$|p_i(t)|=|q_i'(t)|\leqslant C,\tag{2}$$

where the constant C depends on this solution.

To prove the main theorem, we first of all show the following:

Proposition 1 There exist costants α_i^+ and α_i^-, such that $p_i(t)\to\alpha_i^\pm$, as $t\to\pm\infty$.

Proof Let us first of all consider the first particle q_1. From the first equation of (1), we know the velocity p_1 of this particle is monotonely decreasing. Therefore by equation (2), we find easily that there exists some constants $\alpha_1^\pm$, such that $p_1(t)\to\alpha_1^\pm,t\to\pm\infty$. In particular, this implies

$$\left|\int_0^{\pm\infty}\mathrm{e}^{q_1-q_2}\,\mathrm{d}t\right|=|\alpha_1^\pm-p_1(0)|\leqslant C.$$

Now using the second equation of (1) and the above estimate, recalling that $|p_2'|\leqslant C$, we deduce that $\left|\int_0^{\pm\infty}\mathrm{e}^{q_2-q_3}\,\mathrm{d}t\right|$ must be bounded. As a consequence, there exist constant $\alpha_2^\pm$, such that

$$p_2(t)=p_2(0)+\int_0^t\mathrm{e}^{q_1-q_2}\,\mathrm{d}t-\int_0^t\mathrm{e}^{q_2-q_3}\,\mathrm{d}t\to\alpha_2^\pm\qquad t\to\pm\infty.$$

Similar arguments lead to $p_i(t)\to\alpha_i^\pm,t\to\pm\infty$.

The above proposition tells us that the particles q_i have limiting velocities at infinity. The next proposition states that the velocities are indeed ordered.

Proposition 2 $\alpha_i^+<\alpha_{i+1}^+$ and $\alpha_i^->\alpha_{i+1}^-,i=1,\cdots,n-1$.

Proof We first claim

$$q_{i+1}-q_i\to+\infty\qquad t\to+\infty.\tag{3}$$

We only prove this for the case $i=1$, the other case is quite similar. Suppose on the contrary that there exist $\{t_i\}_{i=1}^{+\infty}$ and $M>0$, with $t_i\to+\infty$, such that $q_2(t_i)-q_1(t_i)\leqslant M$, then using the fact that $|p_i(t)|\leqslant C$ and that $q_1''=-\mathrm{e}^{q_1-q_2}$, we easily find that $\alpha_1^+-q_1(t_0)\leqslant-\sum_{i=1}^{+\infty}\mathrm{e}^{-2M}\frac{M}{C}$, contradicting with the fact that α_1^+ is a real number. Therefore the claim holds.

The above claim in particular implies that $\alpha_i^+\leqslant\alpha_{i+1}^+$. We now proceed to show that $\alpha_1^+<\alpha_n^+$. In fact, by equation (3), there exists t_0 such that $p_1(t_0)<p_n(t_0)$. Then from the identity $q_1''-q_n''=-\mathrm{e}^{q_1-q_2}-\mathrm{e}^{q_{n-1}-q_n}<0$, we infer that $\alpha_1^+<\alpha_n^+$.

Now to prove that all α_i^+ are strictly ordered, without loss of generality, we could suppose to the contrary that

$$a_1^+ < a_2^+ = \cdots = a_{n-1}^+ < a_n^+. \quad (4)$$

From equation (1), we have

$$q_2'' - q_{n-1}'' = -e^{q_2-q_3} - e^{q_{n-2}-q_{n-1}} + e^{q_1-q_2} + e^{q_{n-1}-q_n}.$$

By equation (4), we know that $q_1 - q_2$ and $q_{n-1} - q_n$ are much smaller than $q_2 - q_3$ and $q_{n-2} - q_{n-1}$ for t large. Hence there must exists $T > 0$ such that

$$-e^{q_2-q_3} - e^{q_{n-2}-q_{n-1}} + e^{q_1-q_2} + e^{q_{n-1}-q_n} < 0 \qquad t > T.$$

But this will lead to $a_2^+ < a_{n-1}^+$, contradicting with equation (4).

Similarly, one could also show that for $1 \leqslant i < n$, $\alpha_i^- > \alpha_{i+1}^-$. This finishes the proof.

Now we are ready to prove theorem 1, which we restate as the following:

Theorem 2 There exists $\delta > 0$, such that $q_i(t) = \alpha_i^\pm t + \beta_i^\pm + O(e^{-\delta|t|})$, $t \to \pm\infty$.

Proof Consider the function $\varphi_i(t) = q_i(t) - \alpha_i^+ t$. First of all we wish to show that there exists β_i^+, such that $\varphi_i(t) \to \beta_i^+$ as $t \to +\infty$. Obviously,

$$\phi_i'(t) = p_i(t) - \alpha_i^+ = -\int_t^{+\infty} (e^{q_{i-1}-q_i} - e^{q_i-q_{i+1}})\,ds.$$

Here we have denoted $q_0 = -\infty$, $q_{n+1} = +\infty$. Using proposition 2, we know that

$$q_i(t) - q_{i-1}(t) = \int_0^t (p_i - p_{i-1})\,ds \geqslant \delta t \qquad \text{for } t \text{ large enough.}$$

It follows that for t large, $|\phi_i'(t)| \leqslant Ce^{-\delta|t|}$, which implies $\int_0^{+\infty} |\phi_i'(s)|\,ds \leqslant C$. As a consequence, $\phi_i(t) = \phi_i(0) + \int_0^t \phi_i'(s)\,ds \to \beta_i^+$, $t \to +\infty$.

Now with this understood, we consider the function $g_i(t) := q_i(t) - \alpha_i^+ t - \beta_i^+$, which satisfies $|g_i'(t)| < Ce^{-\delta t}$ for $t > 0$ and $g_i(t) \to 0$ as $t \to +\infty$; therefore, one could deduce $q_i(t) = \alpha_i^+ t + \beta_i^+ + O(e^{-\delta|t|})$, $t \to +\infty$.

Analogously, one could show $q_i(t) = \alpha_i^- t + \beta_i^- + O(e^{-\delta|t|})$, $t \to -\infty$, for certain constants β_i^-. This completes the proof.

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Toda 方程的渐近行为

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摘　要:利用 Toda 方程中的指数势函数的性质,讨论了 Toda 方程解在无穷远处的渐近行为,证明了 Toda 方程的任何一个解在无穷远处是渐近线性的.

关键词:Toda 方程;渐近行为;Allen-Cahn 方程

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