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Positive Solutions of Second-Order Singular Differential Equations with Dirichlet Boundary Condition*

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Abstract: The existence and multiplicity of positive solutions are studied for the nonlinear second-order Dirichlet boundary value problem

$$u''(t)-\lambda u(t)+h(t)f(t,u(t))+g(t,u(t))=0 \qquad 0<t<1, u(0)=u(1)=0,$$

where $\lambda>-\pi^2$ is a constant and $g(t,u)$ may be singular at $u=0$. By exactly estimating the priori bound of solution and applying the Guo-Krasnoselskii fixed point theorem of cone expansion-compression type, several existence theorems are established.

Key words: nonlinear ordinary differential equation; singular boundary value problem; positive solution; existence and multiplicity

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1 Introduction

Let $\lambda>-\pi^2$ be a constant. This paper studies the existence and multiplicity of positive solutions for the following nonlinear second-order Dirichlet boundary value problem:

$$\begin{cases} u''(t)-\lambda u(t)+h(t)f(t,u(t))+g(t,u(t))=0 \qquad 0<t<1, \\ u(0)=u(1)=0, \end{cases} \tag{1}$$

where $f(t,u)$ is continuous and $g(t,u)$ may be singular at $u=0$. In other words, we allow $\limsup\limits_{u\to+0} g(t,u)=+\infty$ for any $t\in[0,1]$.

Here, a function $u^*\in C[0,1]$ is called positive solution of the problem (1) if u^* is a solution of (1) and $u^*(t)>0, 0<t<1$.

The problem (1) can model many physical phenomena, for example, the motion of a clock pendulum[1] and the motion of a particle restrained by a nonlinear spring[2]. When $g(t,u)\equiv 0$, the existence of solutions and positive solutions has been studied by many authors, for example, see ref. [3 - 9].

To this day, there is no any existence result of positive solution concerned with the problem (1) when the nonlinearity has singularity at $u=0$. Recently, there is much attention focused on the existence of positive solutions of other singular boundary value problems, for example, see ref. [10 - 16]. These studies inspire us.

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Biography: YAO Qing-liu (1946 -), male, was born in Shanghai City, professor; research area is applied differential equations.

The purpose of this paper is to study the existence and multiplicity of positive solutions for the singular problem (1). In this study, the nonlinearity $h(t)f(t,u)+g(t,u)$ may have the time and space singularities. Because the Green function of the problem (1) has precise expression, we can obtain many useful data. According to these data, we can exactly estimate the priori bound of solution.

Let $\omega=\sqrt{|\lambda|}$ and

$$q(t)=\begin{cases}\dfrac{\sin\omega}{\omega^2}\min\{\sin\omega t,\sin\omega(1-t)\} & -\pi^2<\lambda<0,\\ \min\{t,1-t\} & \lambda=0,\\ \dfrac{1}{\sinh\omega}\min\{\sinh\omega t,\sinh\omega(1-t)\} & \lambda>0.\end{cases}$$

The following assumptions will be used:

(H1) $h:(0,1)\to[0,+\infty)$ is continuous and $\int_0^1 h(t)\mathrm{d}t<+\infty$.

(H2) $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

(H3) $g:(0,1)\times(0,+\infty)\to[0,+\infty)$ is continuous.

(H4) For any $r_2>r_1>0$,

$$\int_0^1\max\{g(t,u):r_1q(t)\leqslant u\leqslant r_2\}\mathrm{d}t<+\infty.$$

Therefore, we not only allow $h(t)$ to be singular at $t=0$ and/or $t=1$, but also allow $g(t,u)$ to be singular at $u=0$ for any $t\in[0,1]$, and at $t=0$ and/or $t=1$ for any $u\in[0,+\infty)$.

This paper is based on the following Guo-Krasnoselskii fixed point theorem of cone expansion-compression type.

Lemma 1 Let X be a Banach space and K be a cone in X. Let Ω_1,Ω_2 be two bounded open subsets in K with $0\in\Omega_1\subset\bar{\Omega}_1\subset\Omega_2$. Assume that $F:\bar{\Omega}_2\backslash\Omega_1\to K$ is a completely continuous operator such that one of the following conditions is satisfied:

(1) $\|F(x)\|\leqslant\|x\|,x\in\partial\Omega_1$ and $\|F(x)\|\geqslant\|x\|,x\in\partial\Omega_2$;

(2) $\|F(x)\|\geqslant\|x\|,x\in\partial\Omega_1$ and $\|F(x)\|\leqslant\|x\|,x\in\partial\Omega_2$.

Then F has a fixed point in $\bar{\Omega}_2\backslash\Omega_1$.

In order to apply the fixed point theorem, we construct four control functions by the priori estimation for the solution. By applying these control functions, we can describe the growth feature of the nonlinearity $h(t)f(t,u)+g(t,u)$. Main results show that the problem (1) has n positive solutions if these control functions have appropriate values. Here, n is an arbitrary positive integer.

2 Preliminaries

Let $C[0,1]$ be the Banach space of continuous functions on $[0,1]$ equipped with the norm $\|u\|=\max\limits_{0\leqslant t\leqslant 1}|u(t)|$. Let K be the following cone of nonnegative functions in $C[0,1]$:

$$K=\{u\in C[0,1]:u(t)\geqslant\|u\|q(t),0\leqslant t\leqslant 1\}.$$

Write $K[r_1,r_2]=\{u\in K:r_1\leqslant\|u\|\leqslant r_2\}$, $\Omega_r=\{u\in K:\|u\|<r\}$. Then $K[r_1,r_2]=\bar{\Omega}_{r_2}\backslash\Omega_{r_1}$.

Let $G(t,s)$ be the Green function of the homogeneous linear problem:

$$u''(t)-\lambda u(t)=0,u(0)=u(1)=0\qquad 0<t<1.$$

If $-\pi^2<\lambda<0$, then

$$G(t,s)=\begin{cases}(\omega\sin\omega)^{-1}\sin\omega s\ \sin\omega(1-t) & 0\leqslant s\leqslant t\leqslant 1,\\ (\omega\sin\omega)^{-1}\sin\omega t\ \sin\omega(1-s) & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

If $\lambda=0$, then

$$G(t,s)=\begin{cases}s(1-t) & 0\leqslant s\leqslant t\leqslant 1,\\ t(1-s) & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

If $\lambda>0$, then

$$G(t,s)=\begin{cases}(\omega\sinh\omega)^{-1}\sinh\omega s\ \sinh\omega(1-t) & 0\leqslant s\leqslant t\leqslant 1,\\ (\omega\sinh\omega)^{-1}\sinh\omega t\sinh\omega(1-s) & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

So, $G:[0,1]\times[0,1]\to[0,+\infty)$ is continuous and $G(t,s)>0, t,s\in(0,1)$.

Let the constants

$$A=\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)\mathrm{d}s,\ B=\int_\delta^{1-\delta}G(\frac{1}{2},s)h(s)\mathrm{d}s,$$
$$C=\max_{t,s\in[0,1]}G(t,s),\ D=\min_{t,s\in[\delta,1-\delta]}G(t,s),$$

where $\delta\in(0,\frac{1}{2})$ is a positive constant. In a real problem, we can choose δ by the properties of $h(t)f(t,u)+g(t,u)$, for example, $\delta=\frac{1}{4}$.

If $h(t)\equiv 1, -\pi^2<\lambda<0$, then $A=\dfrac{1-\cos\frac{1}{2}\omega}{\omega^2\cos\frac{1}{2}\omega}, B=\dfrac{2\sin\frac{1}{2}\omega(\cos\delta\omega-\cos\frac{1}{2}\omega)}{\omega^2\sin\omega}$.

If $h(t)\equiv 1, \lambda=0$, then $A=\frac{1}{8}, B=\frac{1-4\delta^2}{8}$.

If $h(t)\equiv 1, \lambda>0$, then $A=\dfrac{\cosh\frac{1}{2}\omega-1}{\omega^2\cosh\frac{1}{2}\omega}, B=\dfrac{2\sinh\frac{1}{2}\omega(\cosh\frac{1}{2}\omega-\cosh\delta\omega)}{\omega^2\sinh\omega}$.

If $-\pi^2<\lambda<0$, then $C=\dfrac{\sin\frac{\omega}{2}}{2\omega\cos\omega\frac{\omega}{2}}, D=\dfrac{\sin^2\delta\omega}{\omega\sin\omega}$.

If $\lambda=0$, then $C=\frac{1}{4}, D=\delta^2$.

If $\lambda>0$, then $C=\dfrac{\sinh\frac{\omega}{2}}{2\omega\cosh\frac{\omega}{2}}, D=\dfrac{\sinh^2\delta\omega}{\omega\sinh\omega}$.

Additionally, let

$$H(s)=\begin{cases}\dfrac{s(1-s)}{\omega\sin\omega} & -\pi^2<\lambda<0,\\ s(1-s) & \lambda=0,\\ \dfrac{\sinh\omega s\sinh\omega(1-s)}{\omega\sinh\omega} & \lambda>0.\end{cases}$$

For $u\in K\backslash\{0\}$, define the operator T as follows:

$$(Tu)(t)=\int_0^1 G(t,s)[h(s)f(s,u(s))+g(s,u(s))]\mathrm{d}s\qquad 0\leqslant t\leqslant 1.$$

Lemma 2　$q(t)H(s)\leqslant G(t,s)\leqslant H(s)$ for any $t,s\in[0,1]$.

Proof　$G(t,s)\leqslant H(s)$ is immediate. We only prove the another inequality.

Let $-\pi^2<\lambda<0$, then $0<\omega<\pi$. Since $\frac{\sin x}{x}$ is a decreasing function on $(0,\pi]$, one has

$$\frac{\sin \omega t}{\omega t} \geqslant \frac{\sin \omega}{\omega}, \frac{\sin \omega(1-t)}{\omega(1-t)} \geqslant \frac{\sin \omega}{\omega} \qquad 0<t<1.$$

If $0<s\leqslant t<1$, then

$$\frac{G(t,s)}{H(s)}=\frac{\sin \omega s}{\omega s} \cdot \frac{\sin \omega(1-t)}{\omega(1-s)} \geqslant \frac{\sin \omega}{\omega} \cdot \frac{\sin \omega(1-t)}{\omega(1-s)} > \frac{\sin \omega}{\omega^2}\sin \omega(1-t) \geqslant q(t).$$

If $0<t\leqslant s<1$, then

$$\frac{G(t,s)}{H(s)}=\frac{\sin \omega t}{\omega s} \cdot \frac{\sin \omega(1-s)}{\omega(1-s)} \geqslant \frac{\sin \omega t}{\omega s} \cdot \frac{\sin \omega}{\omega} > \frac{\sin \omega}{\omega^2}\sin \omega t \geqslant q(t).$$

By the continuity, then $G(t,s)\geqslant q(t)H(s), t,s\in[0,1]$.

For $\lambda=0$ and $\lambda>0$, the proofs are similar.

Lemma 3 Suppose that (H1)-(H4) hold. Then $T:K[r_1,r_2]\to K$ is a completely continuous operator for any $r_2>r_1>0$.

Proof If $u\in K[r_1,r_2]$, then

$$r_1 q(t) \leqslant \| u \| q(t) \leqslant u(t) \leqslant \| u \| \leqslant r_2 \qquad 0\leqslant t\leqslant 1.$$

By (H1)-(H4), then $(Tu)(t)$ is well defined and $Tu\in C[0,1]$.

For $r_2>r_1>0$, let

$$j_{r_1}^{r_2}(t)=\max\{g(t,u): r_1 q(t)\leqslant u\leqslant r_2\} \qquad 0<t<1.$$

By (H3) and (H4), $j_{r_1}^{r_2}\in C(0,1)$ and $\int_0^1 j_{r_1}^{r_2}(t)\mathrm{d}t<+\infty$.

For $n\geqslant 3$, let

$$[j_{r_1}^{r_2}]_n(t)=\begin{cases}\min\{j_{r_1}^{r_2}(t), ntj_{r_1}^{r_2}(\frac{1}{n})\} & 0\leqslant t\leqslant \frac{1}{n},\\ j_{r_1}^{r_2}(t) & \frac{1}{n}\leqslant t\leqslant \frac{n-1}{n},\\ \min\{j_{r_1}^{r_2}(t), n(1-t)j_{r_1}^{r_2}(\frac{n-1}{n})\} & \frac{n-1}{n}\leqslant t\leqslant 1,\end{cases}$$

then $[j_{r_1}^{r_2}]_n\in C[0,1]$, $[j_{r_1}^{r_2}]_n(0)=[j_{r_1}^{r_2}]_n(1)=0$ and $\lim\limits_{n\to\infty}\int_0^1[j_{r_1}^{r_2}(t)-[j_{r_1}^{r_2}(t)]_n(t)]\mathrm{d}t=0$.

Define the function $g_n(t,u)$ as follows:

$$g_n(t,u)=\min\{g(t,u),[j_{r_1}^{r_2}]_n(t)\} \qquad (t,u)\in[0,1]\times[0,+\infty).$$

Then $g_n:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

Define the operator T_n as follows:

$$(T_n u)(t)=\int_0^1 G(t,s)(h(s)f(s,u(s))+g_n(s,u(s)))\mathrm{d}s \qquad 0\leqslant t\leqslant 1.$$

Then $T_n:K[r_1,r_2]\to C[0,1]$ is completely continuous by the Arzela-Ascoli theorem. Since for any $0<t<1$, $r_1 q(t)\leqslant u\leqslant r_2$,

$$0\leqslant g(t,u)-g_n(t,u)\leqslant j_{r_1}^{r_2}(t)-[j_{r_1}^{r_2}]_n(t),$$

we get that

$$\sup_{u\in K[r_1,r_2]}\| Tu-T_n u\| = \sup_{u\in K[r_1,r_2]}\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)(g(s,u(s))-g_n(s,u(s)))\mathrm{d}s \leqslant$$

$$\sup_{u\in K[r_1,r_2]}\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)[j_{r_1}^{r_2}(t)-[j_{r_1}^{r_2}]_n(t)]\mathrm{d}s \leqslant$$

$$\max_{t,s\in[0,1]} G(t,s)\int_0^1[j_{r_1}^{r_2}(t)-[j_{r_1}^{r_2}]_n(t)]\mathrm{d}s \to 0.$$

Therefore, $T:K[r_1,r_2]\to C[0,1]$ is a completely continuous operator.

On the other hand, by lemma 2, for $0\leqslant t\leqslant 1$,

$$(Tu)(t) \geqslant q(t)\int_0^1 H(s)[h(s)f(s,u(s))+g(s,u(s))]\mathrm{d}s \geqslant q(t)\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)[h(s)f(s,u(s))+ g(s,u(s))]\mathrm{d}s = \|Tu\|q(t).$$

This shows $T:K[r_1,r_2]\to K$.

3 Local Existence Theorems

In order to prove the main results, we introduce the following four control functions:

$$\varphi(r)=\max\{f(t,u):0\leqslant t\leqslant 1, rq(t)\leqslant u\leqslant r\},$$
$$\psi(r)=\min\{f(t,u):\delta\leqslant t\leqslant 1-\delta, rq(t)\leqslant u\leqslant r\},$$
$$\mu(r)=\int_0^1\max\{g(t,u):rq(t)\leqslant u\leqslant r\}\mathrm{d}t,$$
$$\upsilon(r)=\int_\delta^{1-\delta}\min\{g(t,u):rq(t)\leqslant u\leqslant r\}\mathrm{d}t.$$

If (H3) and (H4) hold, then $\upsilon(r)\leqslant\mu(r)<+\infty$ for any $r>0$.

The main results are the following local existence theorems.

Theorem 1 Suppose that (H1)-(H4) hold and there exist two positive numbers $a<b$ such that one of the following conditions is satisfied:

(ⅰ) $A\varphi(a)+C\mu(a)\leqslant a$, $B\psi(b)+D\upsilon(b)\geqslant b$;

(ⅱ) $B\psi(a)+D\upsilon(a)\geqslant a$, $A\varphi(b)+C\mu(b)\leqslant b$.

Then the problem (1) has at least a positive solution $u^*\in K$ and $a\leqslant\|u^*\|\leqslant b$.

Proof We only prove the case (ⅰ). The proof of the case (ⅱ) is similar.

If $u\in\partial\Omega_a$, then $\|u\|=a$ and $aq(t)\leqslant u(t)\leqslant a$, $0\leqslant t\leqslant 1$. By the definitions of $\varphi(a)$ and $\mu(a)$, then

$$\max_{0\leqslant t\leqslant 1} f(t,u(t))\leqslant\varphi(a), \int_0^1 g(t,u(t))\mathrm{d}t\leqslant\mu(a).$$

Applying $A\varphi(a)+C\mu(a)\leqslant a$, we get that

$$\|Tu\| = \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)[h(s)f(s,u(s))+g(s,u(s))]\mathrm{d}s \leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)f(s,u(s))\mathrm{d}s + \max_{t,s\in[0,1]} G(t,s)\int_0^1 g(s,u(s))\mathrm{d}s \leqslant \varphi(a)\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)\mathrm{d}s + \mu(a)\max_{t,s\in[0,1]} G(t,s) = A\varphi(a)+C\mu(a)\leqslant a=\|u\|.$$

If $u\in\partial\Omega(b)$, then $\|u\|=b$ and $bq(t)\leqslant u(t)\leqslant b$, $0\leqslant t\leqslant 1$. So,

$$\min_{\delta\leqslant t\leqslant 1-\delta} f(t,u(t))\geqslant\psi(b), \int_\delta^{1-\delta} g(t,u(t))\mathrm{d}t\geqslant\upsilon(b).$$

Applying $B\psi(b)+D\upsilon(b)\geqslant b$, we get that

$$\|Tu\| \geqslant \max_{\delta\leqslant t\leqslant 1-\delta}\int_\delta^{1-\delta} G(t,s)[h(s)f(s,u(s))+g(s,u(s))]\mathrm{d}s \geqslant \max_{\delta\leqslant t\leqslant 1-\delta}\int_\delta^{1-\delta} G(t,s)h(s)f(s,u(s))\mathrm{d}s + \min_{\delta\leqslant t\leqslant 1-\delta}\int_\delta^{1-\delta} G(t,s)g(s,u(s))\mathrm{d}s \geqslant \int_\delta^{1-\delta} G(\frac{1}{2},s)h(s)f(s,u(s))\mathrm{d}s + \min_{t,s\in[\delta,1-\delta]} G(t,s)\int_\delta^{1-\delta} g(s,u(s))\mathrm{d}s \geqslant \psi(b)\int_\delta^{1-\delta} G(\frac{1}{2},s)h(s)\mathrm{d}s + \upsilon(b)\min_{t,s\in[\delta,1-\delta]} G(t,s) = B\psi(b)+D\upsilon(b)\geqslant b=\|u\|.$$

According to lemmas 1 and lemmas 3, there exists $u^*\in K[a,b]$ such that $Tu^*=u^*$. In other words, $u^*\in K$, $a\leqslant\|u^*\|\leqslant b$ and

$$u^*(t)=\int_0^1 G(t,s)[h(s)f(s,u^*(s))+g(s,u^*(s))]\mathrm{d}s \qquad 0\leqslant t\leqslant 1.$$

Twice differentiating the both ends of the equality on (0,1), we get that

$$(u^*)''(t)-\lambda u^*(t)+h(t)f(t,u^*(t))+g(t,u^*(t))=0 \qquad 0<t<1.$$

And since $G(0,s)=G(1,s)=0, 0\leqslant s\leqslant 1$, one has $u^*(0)=u^*(1)=0$. This shows that u^* is a solution of the problem (1). Since $u^*(t)\geqslant aq(t)>0, 0<t<1$, the solution u^* is positive.

Theorem 2 Suppose that (H1)-(H4) hold and there exist three positive numbers $a<b<c$ such that one of the following conditions is satisfied:

(ⅰ) $A\varphi(a)+C\mu(a)\leqslant a, B\psi(b)+D\upsilon(b)>b, A\varphi(c)+C\mu(c)\leqslant c$;

(ⅱ) $B\psi(a)+D\upsilon(a)\geqslant a, A\varphi(b)+C\mu(b)<b, B\psi(c)+D\upsilon(c)\geqslant c$.

Then the problem (1) has at least two positive solutions $u_1^*, u_2^*\in K$ and $a\leqslant\|u_1^*\|<b<\|u_2^*\|\leqslant c$.

Proof By applying $A\varphi(a)+C\mu(a)\leqslant a, B\psi(b)+D\upsilon(b)>b$ and imitating the proof of theorem 1 (ⅰ), the problem (1) has a positive solution $u_1^*\in K$ such that $a\leqslant\|u_1^*\|<b$. Since $B\psi(b)+D\upsilon(b)>b, A\varphi(c)+C\mu(c)\leqslant c$, the problem (1) has another positive solution $u_2^*\in K$ such that $b<\|u_2^*\|\leqslant c$. The case (ⅰ) is proved. The proof of (ⅱ) is similar.

Generally, we have the following existence theorem concerned with n positive solutions, where $[c]$ is the integer part of c.

Theorem 3 Suppose that (H1)-(H4) hold and there exist $n+1$ positive numbers $a_1<a_2<\cdots<a_{n+1}$ such that one of the following conditions is satisfied:

(ⅰ) $A\varphi(a_{2k-1})+C\mu(a_{2k-1})<a_{2k-1}, k=1,2,\cdots,[\frac{n+2}{2}]$ and $B\psi(a_{2k})+D\upsilon(a_{2k})>a_{2k}, k=1,2,\cdots,[\frac{n+1}{2}]$;

(ⅱ) $B\psi(a_{2k-1})+D\upsilon(a_{2k-1})>a_{2k-1}, k=1,2,\cdots,[\frac{n+2}{2}]$ and $A\varphi(a_{2k})+C\mu(a_{2k})<a_{2k}, k=1,2,\cdots,[\frac{n+1}{2}]$.

Then the problem (1) has at least n positive solutions $u_k^*\in K, k=1,2,\cdots,n$ and $a_k<\|u_k^*\|<a_{k+1}$.

4 Several Interesting Results

In this section, we consider some specific cases of theorem 1.

Theorem 4 Suppose that (H1)-(H4) hold and:

(ⅰ) $\lim\limits_{r\to+0}\sup[B\psi(r)/r+D\upsilon(r)/r]>1$;

(ⅱ) $\lim\limits_{r\to+\infty}\inf[A\varphi(r)/r+C\mu(r)/r]<1$.

Then the problem (1) has at least a positive solution $u^*\in K$.

Proof By (ⅰ), there exists $a>0$ such that $B\psi(a)+D\upsilon(a)\geqslant a$. By (ⅱ), there exists $b>a$ such that $A\varphi(b)+C\mu(b)\leqslant b$. By theorem 1 (ⅱ), the proof is completed.

Theorem 5 Suppose that (H1)-(H4) hold and:

(ⅰ) $\lim\limits_{u\to+0}\min\limits_{\delta\leqslant t\leqslant 1-\delta} g(t,u)>0$;

(ⅱ) $\lim\limits_{u\to+\infty}\max\limits_{0\leqslant t\leqslant 1} f(t,u)/u<A^{-1}$;

(ⅲ) there exist $\xi\in L^1[0,1], 0\leqslant\theta<1$ and $\bar{r}>0$ such that $g(t,u)\leqslant\xi(t)u^\theta$ for any $(t,u)\in(0,1)\times[\bar{r},+\infty)$.

Then the problem (1) has at least a positive solution $u^*\in K$.

Proof By (ⅰ), there exist two positive numbers L and $\bar{r}_1$ such that

$$g(t,u)\geqslant L \qquad (t,u)\in[\delta,1-\delta]\times(0,\bar{r}_1].$$

Let $a=\min\{DL(1-2\delta),\bar{r}_1\}$, then $a>0$ and

$$B\psi(a)+D\upsilon(a)\geqslant D\upsilon(a)=D\int_\delta^{1-\delta} g(t,a)\mathrm{d}t\geqslant DL(1-2\delta)\geqslant a.$$

By (ⅲ), for any $r>\bar{r}$, one has

$$\mu(r)\leqslant\mu(\bar{r})+\int_0^1\max\{g(t,u):\bar{r}\leqslant u\leqslant r\}\mathrm{d}t\leqslant\mu(\bar{r})+r^{\theta}\int_0^1\xi(t)\mathrm{d}t.$$

Since $0\leqslant\theta<1$, it follows that

$$\lim_{r\to+\infty}\mu(r)/r\leqslant\lim_{r\to+\infty}\frac{1}{r}(\mu(\bar{r})+r^{\theta}\int_0^1\xi(t)\mathrm{d}t)=0.$$

Let $A^{-1}-\lim_{u\to+\infty}\max_{0\leqslant t\leqslant 1}f(t,u)/u=3\varepsilon$. By (ⅱ), $\varepsilon>0$. So, there exists a positive number $\bar{r}_2$ such that $\bar{r}_2>\bar{r}_1$ and

$$f(t,u)/u\leqslant A^{-1}-2\varepsilon\qquad(t,u)\in[0,1]\times[\bar{r}_2,+\infty).$$

Since $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, then

$$M=\max\{f(t,u):(t,u)\in[0,1]\times[0,\bar{r}_2]\}<+\infty.$$

Let $\bar{r}_3>\bar{r}_2$ such that $\bar{r}_3\varepsilon>M$. Since $\lim_{r\to+\infty}\mu(r)/r=0$, there exists $\bar{r}_4>0$ such that $C\mu(r)/r<A\varepsilon$ for any $r\in[\bar{r}_4,+\infty)$.

Let $b=\max\{\bar{r}_3,\bar{r}_4\}$, then $b\varepsilon>M$, $C\mu(b)<A\varepsilon b$. So,

$$f(t,u)\leqslant(A^{-1}-2\varepsilon)u<(A^{-1}-2\varepsilon)b\qquad\forall(t,u)\in[0,1]\times[\bar{r}_2,b],$$
$$f(t,u)\leqslant M+(A^{-1}-2\varepsilon)b<(A^{-1}-\varepsilon)b\qquad\forall(t,u)\in[0,1]\times[0,b].$$

It follows that $A\varphi(b)<(1-A\varepsilon)b$ and

$$A\varphi(b)+C\mu(b)<(1-A\varepsilon)b+A\varepsilon b=b.$$

By theorem 1 (ⅱ), (1) has a positive solution $u^*\in K$.

The following problem is a special form of the problem (1):

$$\begin{cases}u''(t)-\lambda u(t)+h(t)u^{\rho}(t)+\dfrac{\eta(t)}{u^{\tau}(t)}=0\qquad 0<t<1,\\ u(0)=u(1)=0,\end{cases}\tag{2}$$

where $\eta(t)$ is a function and ρ,τ are constants.

Theorem 6　Suppose that (H1) holds and:

(ⅰ) $\eta\in C[0,1]$ is nonnegative and $\int_\delta^{1-\delta}\eta(t)\mathrm{d}t>0$;

(ⅱ) $0<\rho<1$, $0<\tau<1$.

Then the problem (2) has a positive solution $u^*\in K$.

Proof　Let $f(t,u)=f(u)=u^{\rho}$, $g(t,u)=\dfrac{\eta(t)}{u^{\tau}}$, then (H2) and (H3) naturally hold. It is easy to see that

$$q(t)\geqslant\begin{cases}\dfrac{2\sin\omega\sin\dfrac{\omega}{2}}{\omega^2}\min\{t,1-t\}\qquad-\pi^2<\lambda<0,\\ \min\{t,1-t\}\qquad\lambda=0,\\ \dfrac{1}{\sinh\omega}\min\{t,1-t\}\qquad\lambda>0.\end{cases}$$

Since $0<\tau<1$, one has $\int_0^1\dfrac{\mathrm{d}t}{[\min\{t,1-t\}]^{\tau}}<+\infty$. So, $\int_0^1\dfrac{\mathrm{d}t}{[rq(t)]^{\tau}}<+\infty$ for any $r>0$. Since $\eta\in C[0,1]$ is a nonnegative function, one has $\int_0^1\dfrac{\eta(t)\mathrm{d}t}{[rq(t)]^{\tau}}<+\infty$ for any $r>0$. Hence, (H4) is satisfied.

Let $\sigma=\min_{\delta\leqslant t\leqslant 1-\delta}q(t)$, then

$$\lim_{r\to+0}\frac{\upsilon(r)}{r}=\lim_{r\to+0}\frac{\int_\delta^{1-\delta}\min\{\frac{\eta(t)}{u^\tau}:rq(t)\leqslant u\leqslant r\}\mathrm{d}t}{r}=\lim_{r\to+0}\frac{\int_\delta^{1-\delta}\eta(t)\mathrm{d}t}{r^{1+\tau}}=+\infty,$$

$$\lim_{r\to+\infty}\frac{\mu(r)}{r}=\lim_{r\to+\infty}\frac{\int_0^1\max\{\frac{\eta(t)}{u^\tau}:rq(t)\leqslant u\leqslant r\}\mathrm{d}t}{r}=\lim_{r\to+\infty}\frac{1}{r^{1+\tau}}\int_0^1\frac{\eta(t)}{q^\tau(t)}\mathrm{d}t=0,$$

$$\lim_{r\to+0}\frac{\psi(r)}{r}=\lim_{r\to+0}\frac{\min\{u^\rho:\delta\leqslant t\leqslant 1-\delta,rq(t)\leqslant u\leqslant r\}}{r}=\lim_{r\to+0}\frac{\sigma^\tau}{r^{1-\rho}}=+\infty,$$

$$\lim_{r\to+\infty}\frac{\varphi(r)}{r}=\lim_{r\to+\infty}\frac{\max\{u^\rho:0\leqslant t\leqslant 1,rq(t)\leqslant u\leqslant r\}}{r}=\lim_{r\to+\infty}\frac{1}{r^{1-\rho}}=0.$$

It follows that

$$\lim_{r\to+0}(B\psi(r)/r+D\upsilon(r)/r)=+\infty,\ \lim_{r\to+\infty}(A\varphi(r)/r+C\mu(r)/r)=0.$$

By theorem 4, the proof is completed.

5 Illustration

Consider the nonlinear Dirichlet boundary value problem

$$\begin{cases}u''(t)-u(t)+\dfrac{u^3(t)-t5u^2(t)+3u(t)+9}{1+2t+u^2(t)}+\dfrac{1+\sin u(t)}{\sqrt[3]{t(1-t)u(t)}}=0 \qquad 0<t<1,\\ u(0)=u(1)=0,\end{cases}$$

here, $\lambda=1$, $h(t)\equiv1$, $f(t,u)=\dfrac{u^3-t5u^2+3u+9}{1+2t+u^2}$ and $g(t,u)=\dfrac{1+\sin u}{\sqrt[3]{t(1-t)u}}$. So, $g(t,u)$ is singular at $u=0$ for any $0\leqslant t\leqslant1$, and at $t=0$, $t=1$ for any $u\in[0,+\infty)$. Since, for any $u\geqslant0$,

$$u^3-t5u^2+3u+9\geqslant u^3-5u^2+3u+9=(u-3)^2(u+1)>0,$$

we see that $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

In such a case, $A^{-1}=\dfrac{\cosh\frac{1}{2}}{\cosh\frac{1}{2}-1}\approx8.835\ 4$.

Let $\gamma=\dfrac{1}{\sinh 1}=\dfrac{2}{\mathrm{e}-\mathrm{e}^{-1}}$, then

$$q(t)=\gamma\min\{\sinh t,\sinh(1-t)\}\geqslant\gamma t(1-t)\qquad\forall\ 0\leqslant t\leqslant 1.$$

So, for any $r_2>r_1>0$ and $0<t<1$,

$$\max\{g(t,u):r_1q(t)\leqslant u\leqslant r_2\}=\frac{2}{\sqrt[3]{r_1t(1-t)q(t)}}\leqslant\frac{2}{\sqrt[3]{\gamma r_1t^2(1-t)^2}}.$$

Since $\int_0^1\dfrac{\mathrm{d}t}{\sqrt[3]{t^2(1-t)^2}}<+\infty$, we see that the assumption (H4) holds.

Obviously, $\lim\limits_{u\to+0}\min\limits_{0\leqslant t\leqslant1}g(t,u)=+\infty$, $\lim\limits_{u\to+\infty}\max\limits_{0\leqslant t\leqslant1}f(t,u)/u=1<A^{-1}$ and $g(t,u)\leqslant\dfrac{2}{\sqrt[3]{t(1-t)}}$ for any $(t,u)\in[0,1]\times[1,+\infty)$.

By theorem 5, the problem has a positive solution $u^*\in K$.

The conclusion can not be derived from the existing results because the function $g(t,u)$ is singular at $u=0$.

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满足 Dirichlet 边界条件的 2 阶奇异微分方程的正解

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摘　要:研究了非线性 2 阶 Dirichlet 边值问题

$$u''(t) - \lambda u(t) + h(t) f(t, u(t)) + g(t, u(t)) = 0 \qquad 0 < t < 1, u(0) = u(1) = 0$$

的正解存在性与多解性,其中 $\lambda > -\pi^2$ 是常数,而 $g(t, u)$ 可以在 $u = 0$ 处奇异. 通过精确估计解的先验界并且利用锥拉伸-压缩的 Guo-Krasnoselskii 不动点定理,建立了几个存在定理.

关键词:非线性常微分方程;奇异边值问题;正解;存在性与多解性

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