Article ID:1007－2985(2013)03－0001－06

Positive Solution for Nonlinear Higher-Order Neutral Variable Delay Difference Equations with Continuous Arguments*

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Abstract:Positive solution for a class of nonlinear higher-order neutral variable delay difference equations with continuous arguments is studied. Using the fixed point theorem in Banach space and a lot of inequality techniques,some sufficient conditions for the existence of eventually positive solution for the equations are obtained. The examples are presented to illustrate the effects of our theorems.

Key words:eventually positive solution;continuous arguments;nonlinear;neutral delay difference equation;fixed point theorem
CLC number:O175.7　　**Document code**:A　　**DOI**:10.3969/j.issn.1007－2985.2013.03.001

1 Introduction

Recently,with the development of the natural science and the interdisciplinary subjects such as economic and financial,aerospace,bio-engineering,numerical calculation,computer science,and automation-control theory,many mathematical models need to be described by differential and difference equations,thus,it is of great theoretical significance and practical value to research the oscillatory and nonoscillatory properties,and there has been an increasing interest in studying the oscillatory and nonoscillatory behavior of differential and difference equations[1-15]. In recent years,some mathematical models of difference equations with continuous arguments appeared in the computer science research. Therefore,people pay much attention to this type of equations[5-13]. However,there are a few results for the existence of positive solution for nonlinear higher-order neutral variable delay difference equations with continuous arguments[9-12]. In this article,we consider the following nonlinear higher-order variable delay neutral difference equations with continuous arguments

* **Received date**:2012－12－15
Foundation item:Supported by Natural Science Foundation of Hunan Province (12JJ6006);Hunan Province Science and Technology Project (2012FJ3107)
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$$\Delta_\tau^d(x(t)+b(t)x(t-\tau))-(-1)^d\sum_{j=1}^m q_j(t)f_j(x(t-\sigma_j(t)))=0\qquad 0<t_0\leqslant t<+\infty,\tag{1}$$

where $d\geqslant 2, m\geqslant 1$ are integer; Δ_τ is the usual forward difference operator, $\Delta_\tau x(t)=x(t+\tau)-x(t)$, $\Delta_\tau^n x(t)=\Delta_\tau(\Delta_\tau^{n-1}x(t))$; τ is positive constant; $\sigma_j(t):[t_0,+\infty)\rightarrow(0,+\infty)$ are delay functions with $\sigma_j(t)\leqslant t$ and $\lim\limits_{t\rightarrow+\infty}(t-\sigma_j(t))=+\infty$; $b(t), q_j(t)\in C([t_0,+\infty),\mathbf{R})$; $f_j(u)\in C(\mathbf{R},\mathbf{R})$ $(j=1,2,\ldots,m$. The following is the same, so we omit it).

Obviously, eq. (1) includes a lot of neutral delay difference equations, which have a very wide range of applications in scientific research and practice. So any research results about oscillation and asymptotic are very important. Our purpose in this article is to obtain new criteria for the nonoscillation of eq. (1). With the fixed point theorem in Banach space and a lot of inequality techniques, some new sufficient conditions for the existence of eventually positive solutions of the equations are obtained. In this article, our attention is restricted to those solutions $x(t)$ for eq. (1) where $x(t)$ is not eventually identically zero. As is customary, we recall that a solution $x(t)$ for eq. (1) is said to be an eventually positive solution if $x(t)>0$ for sufficiently large $t(t\geqslant t_0)$.

We consider the following assumptions:

(H_1) $f_j(0)=0$, and there exist constants $\alpha>0$ and $L_j>0$ for $\forall\, 0\leqslant x\leqslant\alpha, 0\leqslant y\leqslant\alpha$, such that $|f_j(x)-f_j(y)|\leqslant L_j|x-y|$;

(H_2) $q_j(t)>0$, and for $t\geqslant t_0$, we have $\sum\limits_{s_{d-1}=n_0}^{+\infty}\sum\limits_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum\limits_{s_1=s_2}^{+\infty}\sum\limits_{s=s_1}^{+\infty}\Big(\sum\limits_{j=1}^m q_j(t+s\tau)\Big)<+\infty$ (where $n_0\geqslant 0$ is an integer).

2 Main Results

Theorem 1 Assume that conditions (H_1) and (H_2) hold, if there exists a constant p_0 such that $|b(t)|\leqslant p_0\leqslant\frac{1}{3}$. Then, eq. (1) has a bounded eventually positive solution.

Proof From condition (H_2), choose $n_1\geqslant n_0$ sufficiently large such that

$$\sum_{s_{d-1}=n_1}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^m q_j(t+s\tau)\Big)<\frac{1-2p_0}{6L},$$

where $L=\max\limits_{1\leqslant j\leqslant m}\{L_j\}$. Let B be the set of all bounded real sequences $x=\{x(t+n\tau)\}+\infty_{n=0}$ with the norm $\|x\|=\sup\limits_{n\geqslant 0}|x(t+n\tau)|$. Then B is a Banach space. Define a subset B_1 in B as $B_1=\{x\in B\,\Big|\,(\frac{1}{2}-p_0)\alpha\leqslant x(t+n\tau)\leqslant\alpha\}$, and then it is easy to see that B_1 is a bounded, closed and convex subset of B. We define an operator $T:B_1\rightarrow B$ as the following

$$(Tx)(t+n\tau)=\begin{cases}\dfrac{\alpha}{2}-b(t+n\tau)x(t+n\tau-\tau)+\sum\limits_{s_{d-1}=n}^{+\infty}\sum\limits_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\\ \sum\limits_{s_1=s_2}^{+\infty}\sum\limits_{s=s_1}^{+\infty}\Big(\sum\limits_{j=1}^m q_j(t+s\tau)\cdot f_j(x(t+s\tau-\sigma_j(t+s\tau)))\Big) & n\geqslant n_1,\\ (Tx)(t+n_1\tau) & n\leqslant n_1.\end{cases}\tag{2}$$

Clearly, T is continuous. For every $x\in B_1$ and $n\geqslant n_1$, we have

$$(Tx)(t+n\tau)\leqslant\frac{\alpha}{2}+p_0\alpha+\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^m L_jq_j(t+s\tau)|x(t+s\tau-\sigma_j(t+s\tau))|\Big)\leqslant$$

$$\frac{\alpha}{2}+p_0\alpha+L\alpha\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^m q_j(t+s\tau)\Big)\leqslant$$

$$\frac{\alpha}{2}+p_0\alpha+L\alpha\cdot\frac{1-2p_0}{6L}\leqslant\alpha,$$

and $(Tx)(t+n\tau)\geqslant\frac{\alpha}{2}-p_0\alpha=(\frac{1}{2}-p_0)\alpha$. Hence, $(\frac{1}{2}-p_0)\alpha\leqslant(Tx)(t+n\tau)\leqslant\alpha$. Thus we have proved that $TB_1\subseteq B_1$.

Now we show that T is a contraction mapping on B_1. In fact, for $\forall x^{(1)},x^{(2)}\in B_1$ and $n\geqslant n_1$, we have

$$|(Tx^{(1)})(t+n\tau)-(Tx^{(2)})(t+n\tau)|\leqslant|b(t+n\tau)||x^{(1)}(t+n\tau)-x^{(2)}(t+n\tau)|+\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots$$
$$\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}q_j(t+s\tau)|f_j(x^{(1)}(t+s\tau-\sigma_j(t+s\tau)))-f_j(x^{(2)}(t+s\tau-\sigma_j(t+s\tau)))|\Big)\leqslant$$
$$p_0\|x^{(1)}-x^{(2)}\|+\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots$$
$$\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}L_jq_j(t+s\tau)|x^{(1)}(t+s\tau-\sigma_j(t+s\tau))-x^{(2)}(t+s\tau-\sigma_j(t+s\tau))|\Big)\leqslant$$
$$p_0\|x^{(1)}-x^{(2)}\|+L\cdot\frac{1-2p_0}{6L}\|x^{(1)}-x^{(2)}\|=\frac{4p_0+1}{6}\|x^{(1)}-x^{(2)}\|.$$

It follows that $\|Tx^{(1)}-Tx^{(2)}\|\leqslant\frac{4p_0+1}{6}\|x^{(1)}-x^{(2)}\|$. Since $0<\frac{4p_0+1}{6}<1$, it is easy to see that T is a contraction mapping on B_1. Thus, by the Banach contraction principle, T has an unique fixed point $x\in B_1$, that is $Tx=x$. From eq. (2), for sufficiently large n, this fixed point x satisfies:

$$x(t+n\tau)=\begin{cases}\frac{\alpha}{2}-b(t+n\tau)x(t+n\tau-\tau)+\sum\limits_{s_{d-1}=n}^{+\infty}\sum\limits_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\\ \sum\limits_{s_1=s_2}^{+\infty}\sum\limits_{s=s_1}^{+\infty}\Big(\sum\limits_{j=1}^{m}q_j(t+s\tau)f_j(x(t+s\tau-\sigma_j(t+s\tau)))\Big) & n\geqslant n_1,\\ (Tx)(t+n_1\tau) & n\leqslant n_1.\end{cases}$$

Furthermore, we can obtain, for sufficiently large n,

$$\Delta_\tau^d(x(t+n\tau)+b(t+n\tau)x(t+n\tau-\tau))=(-1)^d\sum_{j=1}^{m}q_j(t+n\tau)f_j(x(t+n\tau-\sigma_j(t+n\tau)));$$

hence, $x(t)$ is a bounded eventually positive solution for eq. (1). This completes the proof.

Theorem 2　Assume that conditions (H_1) and (H_2) hold, if there exists a constant p such that $-1<p\leqslant b(t)<0$. Then, the eq. (1) has a bounded eventually positive solution.

Proof　Similar to the proof of theorem 1. Choose $n_2\geqslant n_0$ sufficiently large such that

$$\sum_{s_{d-1}=n_2}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}q_j(t+s\tau)\Big)<\frac{1+p}{4L}.$$

Define a subset B_2 in B as

$$B_2=\left\{x\in B\,\middle|\,\frac{(1+p)\alpha}{4}\leqslant x(t+n\tau)\leqslant\alpha\right\},$$

and then it is easy to see that B_2 is a bounded, closed and convex subset of B. We define an operator T: $B_2\rightarrow B$ as the following

$$(Tx)(t+n\tau)=\begin{cases}\frac{1+p}{2}\alpha-b(t+n\tau)x(t+n\tau-\tau)+\sum\limits_{s_{d-1}=n}^{+\infty}\sum\limits_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\\ \sum\limits_{s_1=s_2}^{+\infty}\sum\limits_{s=s_1}^{+\infty}\Big(\sum\limits_{j=1}^{m}q_j(t+s\tau)f_j(x(t+s\tau-\sigma_j(t+s\tau)))\Big) & n\geqslant n_2,\\ (Tx)(t+n_2\tau) & n\leqslant n_2.\end{cases}\tag{3}$$

Clearly, T is continuous. For $\forall x\in B_2$ and $n\geqslant n_2$, we have

$$(Tx)(t+n\tau)\leqslant\frac{1+p}{2}\alpha-p\alpha+L\alpha\cdot\frac{1+p}{4L}\leqslant\alpha,$$

and

$$(Tx)(t+n\tau)\geqslant\frac{1+p}{2}\alpha-L\alpha\cdot\frac{1+p}{4L}=\frac{1+p}{4}\alpha.$$

Hence, $\frac{1+p}{4}\alpha\leqslant(Tx)(t+n\tau)\leqslant\alpha$. Thus we have proved that $TB_2\subseteq B_2$.

By similar argument, for $\forall x^{(1)},x^{(2)}\in B_2$ and $n\geqslant n_2$, we have

$$|(Tx^{(1)})(t+n\tau)-(Tx^{(2)})(t+n\tau)|\leqslant-p\|x^{(1)}-x^{(2)}\|+L\cdot\frac{1+p}{4L}\|x^{(1)}-x^{(2)}\|=\frac{1-3p}{4}\|x^{(1)}-x^{(2)}\|.$$

It follows that

$$\|Tx^{(1)}-Tx^{(2)}\|\leqslant\frac{1-3p}{4}\|x^{(1)}-x^{(2)}\|.$$

Since $0<\frac{1-3p}{4}<1$, it is not difficult to see that T is a contraction mapping on B_2. Thus, by the Banach contraction principle, T has an unique fixed point $x\in B_2$, that is $Tx=x$. From eq. (3), for sufficiently large n, this fixed point x satisfies:

$$x(t+n\tau)+b(t+n\tau)x(t+n\tau-\tau)=\frac{1+p}{2}\alpha+\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}q_j(t+s\tau)f_j(x(t+s\tau-\sigma_j(t+s\tau)))\Big).$$

Furthermore, we can obtain, for sufficiently large n,

$$\Delta_\tau^d(x(t+n\tau)+b(t+n\tau)x(t+n\tau-\tau))=(-1)^d\sum_{j=1}^{m}q_j(t+n\tau)f_j(x(t+n\tau-\sigma_j(t+n\tau)));$$

hence, $x(t)$ is a bounded eventually positive solution of eq. (1). The proof of the theorem is completed.

Theorem 3 Assume that conditions (H_1) and (H_2) hold, if there exist constants p_1 and p_2 such that $-\infty<p_1\leqslant b(t)\leqslant p_2<-1$. Then, the eq. (1) has a bounded eventually positive solution.

Proof Similar to the proof of theorem 1. Choose $n_3\geqslant n_0$ sufficiently large such that

$$\sum_{s_{d-1}=n_3}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}q_j(t+s\tau)\Big)<\frac{-(1+p_2)}{16L}.$$

Define a subset B_3 in B as $B_3=\left\{x\in B\,\middle|\,\frac{(p_1-1)\alpha}{2p_2}\leqslant x(t+n\tau)\leqslant\alpha\right\}$, and then it is easy to see that B_3 is a bounded, closed and convex subset of B. We define an operator $T:B_3\to B$ as following:

$$(Tx)(t+n\tau-\tau)=\begin{cases}\frac{(1+p_2)\alpha}{8b(t+n\tau)}-\frac{1}{b(t+n\tau)}\Big(x(t+n\tau)-\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots\\ \quad\sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}q_j(t+s\tau)f_j(x(t+s\tau-\sigma_j(t+s\tau)))\Big)\Big) & n\geqslant n_3,\\ (Tx)(t+n_3\tau-\tau) & n\leqslant n_3.\end{cases}\tag{4}$$

Obviously, T is continuous. For $\forall x\in B_3$ and $n\geqslant n_3$, we have

$$(Tx)(t+n\tau-\tau)\leqslant\frac{(1+p_2)\alpha}{8b(t+n\tau)}-\frac{\alpha}{p_2}\leqslant\frac{(1+p_2)\alpha}{8p_2}-\frac{\alpha}{p_2}<\alpha,$$

and

$$(Tx)(t+n\tau-\tau)\geqslant\frac{(1+p_2)\alpha}{8b(t+n\tau)}+\frac{1}{b(t+n\tau)}\cdot\frac{-(1+p_2)}{16L}\cdot L\alpha=$$

$$\frac{(1+p_2)\alpha}{16b(t+n\tau)} \geqslant \frac{(1+p_2)\alpha}{16p_1}.$$

Hence, $\frac{(1+p_2)\alpha}{16p_1} \leqslant (Tx)(t+n\tau-\tau) \leqslant \alpha$. Thus we have proved that $TB_3 \subseteq B_3$.

By similar argument, for $\forall x^{(1)}, x^{(2)} \in B_3$ and $n \geqslant n_5$, we have $\| Tx^{(1)} - Tx^{(2)} \| \leqslant \frac{p_2-15}{16p_2} \| x^{(1)} - x^{(2)} \|$. Since $0 < \frac{p_2-15}{16p_2} < 1$, it is not difficult to see that T is a contraction mapping on B_3. Thus, by the Banach contraction principle, T has an unique fixed point $x \in B_3$, that is $Tx = x$. From eq. (4), for sufficiently large n, this fixed point x satisfies:

$$x(t+n\tau)+b(t+n\tau)x(t+n\tau-\tau)=\frac{(1+p_2)\alpha}{8}+\sum_{s_{d-1}=n}^{+\infty}\sum_{s_{d-2}=s_{d-1}}^{+\infty}\cdots \sum_{s_1=s_2}^{+\infty}\sum_{s=s_1}^{+\infty}\Big(\sum_{j=1}^{m}q_j(t+s\tau)f_j(x(t+s\tau-\sigma_j(t+s\tau)))\Big).$$

Furthermore, we can obtain, for sufficiently large n,

$$\Delta_\tau^d(x(t+n\tau)+b(t+n\tau)x(t+n\tau-\tau))=(-1)^d\sum_{j=1}^{m}q_j(t+n\tau)f_j(x(t+n\tau-\sigma_j(t+n\tau)));$$

hence, $x(t)$ is a bounded eventually positive solution for eq. (1). This completes the proof.

Example 1　Consider the following second-order neutral difference equation with continuous arguments:

$$\Delta_\tau^2(x(t)+p(t)x(t-\tau))-q(t)f(x(t-2))=0 \qquad 7 \leqslant t < +\infty.$$

(1) If take

$$\tau=1, p(t)=-\frac{1}{t}, q(t)=\frac{2(t-7)(t-2)}{t^2(t-1)^2(t+2)}, f(x)=x.$$

Then, one can see that the conditions of theorem 1 and theorem 2 are satisfied. Hence, by theorem 1 or theorem 2, this equation exists a bounded eventually positive solution. In fact, it is easy to verify that $x(t)=\frac{1}{2}+\frac{1}{t}$ is such a solution.

(2) If take

$$\tau=\frac{3}{2}, p(t)=-2+\frac{1}{t}, q(t)=\frac{9}{t(t+3)(2t+3)}, f(x)=x.$$

Then, one can see that the conditions of theorem 3 are satisfied. So, by theorem 3, this equation exists a bounded eventually positive solution. In fact, it is easy to verify that $x(t)=1$ is such a solution.

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具连续变量的高阶非线性变时滞差分方程的正解

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摘 要:研究了一类具有连续变量的高阶非线性变时滞中立型差分方程,利用 Banach 空间的不动点原理和一些分析技巧,得到了这类方程存在最终正解的几个新的充分条件,同时给出实例验证其有效性.

关键词:最终正解;连续变量;非线性;中立型时滞差分方程;不动点原理
中图分类号:O175.7 **文献标识码**:A

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