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Algebraic connectivity of trees with the maximum degree

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Abstract: This paper investigated how the algebraic connectivity of trees changes under some graph perturbations. Then these results were used to characterize the extremal tree which has the smallest algebraic connectivity in the set of trees given the number of vertex and the maximum degree.

Key words: algebraic connectivity; tree; Laplacian matrix; maximum degree

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给定最大度的树的代数连通度

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摘要: 研究给定最大度的树在移接变形下的代数连通度的变化. 这些结果可以用来刻画给定最大度和顶点个数的树中具有最小代数连通度的极图, 并且给出了该极图的代数连通度的一个下界.

关键词: 代数连通度; 树; 拉普拉斯矩阵; 最大度

0 Introduction

Let $G=(V,E)$ be a simple and undirected graph with vertex set $V=\{v_1,\cdots,v_n\}$ and edge set E. Let $\boldsymbol{A}(G)=(a_{ij})$ be the *adjacency matrix* of G whose entry

$$a_{ij}=\begin{cases}1 & \text{if } v_i\sim v_j,\\ 0 & \text{otherwise.}\end{cases}$$

Let $d_i=\sum_j a_{ij}$ be the *degree* of the vertex v_i. Thus $\boldsymbol{D}=\mathrm{diag}(d_1,\cdots,d_n)$ is called the *degree diagonal matrix* of G. The *Laplacian matrix* of a graph G is then defined by

$$\boldsymbol{L}(G)=\boldsymbol{D}-\boldsymbol{A}.$$

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Moreover, the eigenvalues of $\boldsymbol{L}(G)$ are denoted by $0 = \lambda_0(G) \leqslant \lambda_1(G) \cdots \leqslant \lambda_{n-1}(G)$. Fiedler[1] proved that $\lambda_0(G) = 0$ is a simple eigenvalue if and only if G is connected and called λ_1 as the algebraic connectivity of graph G, which is denoted by $\mu(G)$. So the eigenvector of G corresponding to $\lambda_1(G)$ is called *Fiedler vector* (for example, see [2]).

On one hand, the algebraic connectivity is relevant to the vertex and edge connectivity[1,3], the diameter of graphs[4], the expanding properties of graphs[5], the combinatorial optimization problems[6]. On the other hands, the algebraic connectivity also plays an important role in many applications; for example, the algebraic connectivity of graphs can be regarded as the measurement of convergence speed of solving the consensus problems in the analysis of convergence speed for the consensus problem (see [7,8]). Therefore, the algebraic connectivity has received more and more attention. Recently, there is an excellent survey on algebraic connectivity of graphs written by de Abreu[9]. By the way, the eigenvectors of weighted graphs corresponding to the second smallest Laplacian eigenvalue have been investigated[2,10−15]. Also there are some results on the third smallest and fourth smallest Laplacian eigenvalue and corresponding[16−18] as well as the largest eigenvalue of Laplacian[19−24].

In this paper, we consider the algebraic connectivity of trees with the maximum degree given. Let $\mathcal{T}(n,\Delta)$ be the set of all trees with n vertices and the maximum degree Δ. Denote by $T^*(n,\Delta)$ the tree of order n obtained by identifying a pendent vertex of the star $K_{1,\Delta}$ and a pendent vertex of a path $P_{n-\Delta}$ of order $n-\Delta$. Clearly, $T^*(n,\Delta) \in \mathcal{T}(n,\Delta)$ and its diameter is $n-\Delta+1$. The main result in this paper is as follows:

Theorem 0.1 *$T^*(n,\Delta)$ is the only extremal tree in $\mathcal{T}(n,\Delta)$ with the smallest algebraic connectivity.*

Theorem 0.2 *Let T be a tree of order n with the maximum degree Δ and $\mu(T)$ be its algebraic connectivity. Then*

$$\mu(T) \geqslant \frac{1}{1+(\Delta-1)(k^*-1)+\frac{k^*(k^*-1)}{2}},$$

where $k^* = \frac{(n-\Delta)^2+n+\Delta-4}{2(n-1)}$.

The rest of this paper is organized as follows: In Section 1, the three types of tree transformation are introduced and some Lemmas are presented. In Section 2, proofs of Theorems 0.1 and 0.2 are given.

1 Preliminary

First we introduce the notion bottleneck matrix developed by Kirkland and Neumann[2]. Let T be a tree of order n. A *branch* at a vertex v of T is one of the connected components in the graph obtained from T by deleting v and all edges incident with v. For a branch B at v (vertices labelled $1,\ldots,k$) the *bottleneck matrix for B based at v*, $\boldsymbol{M}_v(B)$, is a $k\times k$ matrix such that for $1 \leqslant i,j \leqslant k$, the entry in position (i,j) is the number of edges in T which are on both the path from i to v and the path from j to v. Denote by $\rho(B)$ the spectral radius of a nonnegative matrix $\boldsymbol{M}_v(B)$. Thus, $\rho(B)$ is the *Perron eigenvalue.* A branch B at v of T is called a *Perron branch* if the Perron eigenvalue of B is the largest amongst all branches at v (note that there may be several Perron branches at a particular vertex). In the following passages, $\boldsymbol{J}$ denote the all one matrix. Let y be a Fiedler eigenvector of T. Then exact one of the following two cases occurs (see [2]).

(a) No entry of y is 0. In this case, there is a unique pair of vertices i and j such that they are adjacent in T with $y_i > 0$ and $y_j < 0$. Furthermore, let $\boldsymbol{M}_1$ be the bottleneck matrix for the branch at j containing i and $\boldsymbol{M}_2$ be the bottleneck matrix for the branch at i containing j. Then there exists a $0 < \gamma < 1$ such that

$$\frac{1}{\mu(T)} = \rho(\boldsymbol{M}_1 - \gamma \boldsymbol{J}) = \rho(\boldsymbol{M}_2 - (1-\gamma)\boldsymbol{J}).$$

This case is called a *type II tree*. Moreover, the special vertices i, j are called the *characteristic vertices* of T.

(b) Some entry of y is 0. Then there is a unique vertex k that $y_k = 0$, and there are two or more Perron branches of T at k. Moreover $\mu(T) = 1/\rho(B)$, where B is a Perron branch at k. This is called a *type I tree*. Moreover, the special vertex k is called the *characteristic vertex* of T.

Clearly, if T is a type II tree, there are exactly two adjacent characteristic vertices; while if T is a type I tree, there is only one characteristic vertex. Let $T(k,l,d)$ be the tree of order $n = k+l+d-1$ by identifying the center of stars $K_{1,k}$ and one pendent vertex of a path P_{d-1} of order d and by identifying the center of $K_{1,l}$, and the other pendent vertex of P_{d-1}. Kirkland and Neumann[2] proved that for all trees on n vertices with fixed diameter d, the algebraic connectivity is minimized by $T(k,l,d)$, for some $0 \leqslant k \leqslant n-d$. Further, Fallat and Kirland (see [13] p60) proved that

Lemma 1.1[13] (1) *Let $\mu(T(k,l,d))$ denote the algebraic connectivity of tree $T(k,l,d)$. Then*

$$\mu(T(k,l,d)) < \mu(T(k-1,l+1,d)), \quad 1 \leqslant k < l+1.$$

(2) *Let T' be any tree of order n with diameter $d+1$. Then the algebraic connectivity $\mu(T')$ of T' has*

$$\mu(T') \geqslant \mu(T(\lceil\frac{n-d+1}{2}\rceil, \lfloor\frac{n-d+1}{2}\rfloor, d))$$

with equality if and only if T' is $T(\lceil\frac{n-d+1}{2}\rceil, \lfloor\frac{n-d+1}{2}\rfloor, d)$.

We also need the following Lemmas which are from [2,15].

Lemma 1.2[2] *Let T be a tree of order n. Then T is a type I tree if and only if there exists a vertex, which is called the characteristic vertex of T, at which there are two or more Perron branches. T is a type II one if and only if T has adjacent vertices i and j, which are called the characteristic vertices of T, such that the branch at vertex i containing vertex j is the unique Perron branch at i, which the branch at vertex j containing vertex i is the unique Perron branch at j.*

Lemma 1.3[15] *Let T be a tree of order n and v_m is not a characteristic vertex of T. Then the unique Perron branch of T at v_m is the branch which contains the characteristic vertex (or vertices) of T.*

2 Proofs of Theorems 0.1 and 0.2

In order to prove the main results, we also need the following lemma.

Lemma 2.1 *Let T be a tree of order n with the maximum degree Δ. If T is not the type $T(k,l,d)$, then there exists a tree $T(k,l,d)$ of order n such that $\mu(T)>\mu(T(k,l,d))$ and* $\max\{k,l\}\geqslant\Delta-1$.

Proof If $\Delta=n-1$ and $\Delta=2$, then the assertion clearly holds. Hence assume $3\leqslant\Delta\leqslant n-2$. We consider the following two cases.

Case 1 T is a type I tree. By Lemma 1.2, there is a characteristic vertex v such that there are $s\geqslant 2$ Perron branches $B_1,\cdots,B_s$. If there are two Perron branches, let us say $B_1=B_2=T(k_1,1,d_1)$, in $B_1,\cdots,B_s$, and v is adjacent to the pendent vertex of B_1 and B_2 whose neighbor vertex with degree 2; let T_1 be the tree from T by deleting the branches $B_3,\cdots,B_s$. Then by Theorem 2 in [15], $\mu(T)=\mu(T_1)$. Further, let T_2 be the tree of order n from T_1 by adding $n-|V(T_1)|>0$ pendent vertices to a vertex with degree k_1+1. Then by Corollary 1.1 and its proof in [2], $\mu(T_1)>\mu(T_2)$. Similarly for an arbitrary type I tree one can make the vertices in $B_3,\cdots,B_s$ become pendant vertices of B_1. Now without loss of generality, there are only two branches at v and exists a Perron branch $B_1\neq T(k_1,1,d_1)$ of order $r_1\geqslant 2$ with diameter d_1 at vertex v. Let $\boldsymbol{M}_v(B_1)$ be the bottleneck matrix of branch B_1 at v. Let $\widehat{T}$ be the tree of order n obtained from T replacing the branch B_1 by the branch $\widehat{B}_1$ which is the tree $T(r_1-d_1,1,d_1)$ and joining both v and the pendent vertex of $T(r_1-d_1,1,d_1)$ whose neighbor vertex with degree 2. It is easy to see that $\boldsymbol{M}_v(\widehat{B}_1)\gg\boldsymbol{M}_v(B_1)$ (see [2], pp.199) with equality if and only if $\widehat{B}_1=B_1$. Moreover, the maximum degree of $\widehat{T}$ is at least Δ. By Theorem 1 and its proof in [2], we have $\mu(\widehat{T})<\mu(T)$. Now we consider the following two subcases.

Subcase 1.1 v is a characteristic vertex of $\widehat{T}$. By the construction of $\widehat{T}$, it is easy to see that $\widehat{B}_1$ is the unique Perron branch of $\widehat{T}$ at vertex v. Hence by Lemma 1.2, $\widehat{T}$ is a type II tree. Moreover, let u be the other characteristic vertex with adjacent to v. By Lemma 1.2, u must belong to the branch $\widehat{B}_1$. Let $\widehat{B}_2$ be the unique branch of $\widehat{T}$ at vertex u containing v. Let T' be the tree of order n obtained from $\widehat{T}$ replacing the branch $\widehat{B}_2$ of order $r_2\geqslant 2$ with diameter d_2 by the branch B_2' which is the tree $T(r_2-d_2,1,d_2)$ and joining both u and the pendent vertex of $T(r_2-d_2,1,d_2)$ whose neighbor vertex with degree 2. It is easy to see that the maximum degree of T' at least Δ and T' is the type $T(k,l,d)$ with $\max\{k,l\}\geqslant\Delta-1$. Further by Theorem 1 in [2], $\mu(T')\leqslant\mu(\widehat{T})<\mu(T)$. The assertion holds.

Subcase 1.2 v is not a characteristic vertex of $\widehat{T}$. Since $\widehat{B}_1$ is the unique Perron branch of $\widehat{T}$ at vertex v. Hence by Lemma 1.3, $\widehat{B}_1$ contains the characteristic vertex (or vertices) of $\widehat{T}$. Let $u\in V(\widehat{B}_1)$ is adjacent to v. Then there are exactly two branches C_1 which contains vertex v and C_2 of $\widehat{T}$ at vertex u, here C_2 is the same branch correspondent to B_2 in T. Let T' be the tree of order n from $\widehat{T}$ replacing C_1 of order r_3 with diameter d_3 by $T(r_3-d_3,1,d_3)$ and joining both vertex u and the pendent vertex of $T(r_3-d_3,1,d_3)$ whose neighbor vertex with degree 2. Clearly, the maximum degree of T' at least Δ. Therefore $\max\{k,l\}\geqslant\Delta-1$. Further by Theorem 1 in [2], $\mu(T')\leqslant\mu(\widehat{T})<\mu(T)$. The assertion holds.

Case 2 T is a type II tree. There exist two adjacent characteristic vertices u and v.

By Lemma 1.2, the branch B_1 of tree T at vertex u containing vertex v is the unique Perron branch at u. Let $\widehat{T}$ be the tree of order n from T replacing B_1 of order $r_1 \geqslant 2$ with diameter d_1 by $\widehat{B}_1 = T(r_1-d_1, 1, d_1)$ and joining both vertex u and the pendent vertex of $T(r_1-d_1,1,d_1)$ the degree of whose neighbor vertex is 2. It is easy to see that $\boldsymbol{M}_u(\widehat{B}_1) \gg \boldsymbol{M}_u(B_1)$ (see [2], pp.199) with equality if and only if $\widehat{B}_1 = B_1$. By Theorem 1 in [2] and its proof, $\mu(\widehat{T}) < \mu(T)$. Clearly, the maximum degree of $\widehat{T}$ is at least Δ. Now we consider the following two subcases.

Subcase 2.1 u is a characteristic vertex of $\widehat{T}$. Then $\widehat{B}_1$ is the unique Perron branch of $\widehat{T}$ at vertex u. Hence by Lemma 1.2, $\widehat{T}$ is a type II tree and the other characteristic vertex w is in branch $\widehat{B}_1$. Clearly the degree of vertex w in $\widehat{T}$ is 2. Moreover, let $\widehat{C}_1$ be the unique Perron branch of $\widehat{T}$ at vertex w containing vertex u. Let T' be the tree from $\widehat{T}$ replacing $\widehat{C}_1$ by $T(r_1-d_1,1,d_1)$ and joining both vertex w and the pendent vertex whose neighbor vertex with degree 2, where $\widehat{C}_1$ is the branch of order r_1 with diameter d_1. Hence by Theorem 1 in [2], $\mu(T) > \mu(\widehat{T}) \geqslant \mu(T')$. Moreover the maximum degree of T' is at least Δ and T' is already the type $T(k,l,d)$. The assertion holds.

Subcase 2.2 u is not a characteristic vertex of $\widehat{T}$. By Lemma 1.3, it is easy to show that the assertion holds by the method of similar to Subcase 1.2. Therefore we finish our proof.

Now we are ready to prove Theorem 0.1.

Proof of Theorem 0.1 Let T be any tree of order n with the maximum degree Δ. If T is not the type $T(k,l,d)$, then by Lemma 2.1, there exists a tree $T(k,l,d)$ such that $\mu(T(k,l,d)) < \mu(T)$ and $k \geqslant \Delta-1$, where $k \geqslant l$. By Example 2.6 in [13], $\mu(T(k,l,d)) \geqslant \mu(T(k,1,d+l-1))$ with equality if and only if $l=1$. By the above method together with the part (1) of Lemma 1.1, it is easy to prove that

$$\mu(T(k,1,d+l-1)) \geqslant \mu(T(k-1,1,d+l)) \cdots \geqslant \mu(T(\Delta-1,1,n-\Delta+1)) = \mu(T^*(n,\Delta)).$$

Hence $\mu(T) \geqslant \mu(T^*(n,\Delta))$ with equality if and only if T is $T^*(n,\Delta)$.

Proof of Theorem 0.2 Let $A_{k-1,\Delta-1}$ be the bottleneck matrix for the branch of $T^*(n,\Delta)$ at vertex k containing $k-1$ which is a path of length $k-1$ and $\Delta-1$ vertices pendant to one end of the path. Let $B_{n-\Delta-k}$ be the bottleneck matrix for the branch of $T^*(n,\Delta)$ at k containing $k+1$ with a path of order $n-\Delta-k$. Then by the argument in Theorem 6 in [2], we have

$$\frac{1}{\mu(T^*(n,\Delta))} \leqslant \min\{\rho(A_{k-1,\Delta-1}), \rho(B_{n-\Delta-k})\}.$$

Since $f(k) = \frac{k(k-1)}{2} + (\Delta-1)(k-1)+1$ and $g(k) = \frac{(n-\Delta-k)(n-\Delta-k+1)}{2}$ are the maximum row sum of matrix $A_{k-1,\Delta-1}$ and $B_{n-\Delta-k}$ respectively, we have

$$\frac{1}{\mu(T^*(n,\Delta))} \leqslant \max_k \min\{f(k), g(k)\}.$$

Moreover, let $k^* = \frac{(n-\Delta)^2+n+\Delta-4}{2(n-1)}$. Then $f(k^*) = g(k^*)$, and for $k > k^*$, we have $f(k) > g(k)$ while for $k < k^*$, $f(k) < g(k)$. Then by direct computation $f(k) \leqslant f(k^*)$ for all $k \leqslant k^*$, while $g(k) \leqslant g(k^*)$ for all $k \geqslant k^*$. Hence it follows that for all $k \geqslant k^*$,

$$\frac{1}{\mu(T^*(n,\Delta))} \leqslant \min\{f(k), g(k)\} = g(k) \leqslant g(k^*) = f(k^*),$$

and for all $k < k^*$,

$$\frac{1}{\mu(T^*(n,\Delta))} \leqslant \min\{f(k), g(k)\} = f(k) \leqslant f(k^*).$$

Therefore

$$\frac{1}{\mu(T^*(n,\Delta))} \leqslant \max_k \min\{f(k), g(k)\} \leqslant f(k^*).$$

So we finish our proof.

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