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Note on induced modules and their extensions for graded Lie algebras of Cartan type

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Abstract: In the sense of generalized restricted Lie algebras, it was proved that the modified induced modules of graded Lie algebras of Cartan types W, S, H coincide with coinduced modules. The relationship between induced modules and coinduced modules was obtained, extending the corresponding result by Rolf Farnsteiner and Helmut Strade in the case of restricted Lie algebras. Therefore, it was proved that any irreducible non-exceptional modules for graded Lie algebras of Cartan types W, S, H with generalized p-character of height not more than a precise upper-bound is a coinduced module. By applying this with some results on cohomology obtained by Rolf Farnsteiner in 1990s, we finally got some further results on extensions between simple modules of graded Lie algebras of Cartan types W, S, H.

Key words: generalized restricted Lie algebra; Cartan type Lie algebra; induced module; coinduced module; extension

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阶化Cartan型李代数的诱导模及其扩张的一点注记

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摘要: 在广义限制李代数的意义下，证明了W, S, H 型系列的阶化Cartan型李代数的“修正”诱导模为余诱导模. 得到了诱导模和余诱导模之间的关联，从而推广了 Rolf Farnsteiner 和 Helmut Strade 在限制李代数情形下关于诱导模与余诱导模之间的关联. 进而证明了W, S, H 型系列的阶化Cartan型李代数的所有具有广义特征标高度不超过某个界的不可约非例外单模均为余诱导模. 应用此结论以及Rolf Farnsteiner关于上同调的结果，最后进一步得到了一些有关W, S, H 型系列的阶化Cartan型李代数单模之间的扩张的结论.

关键词: 广义限制李代数; Cartan 型李代数; 诱导模; 余诱导模; 扩张

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0 Introduction

The cohomology theory of modular Lie algebras has received considerable attention in the last decades. As to the cohomology of graded Lie algebras of Cartan type, Qiu and Shen[1] computed some low-dimensional cohomology groups with coefficients in the mixed product modules. Shu[2] studied generalized restricted cohomology of the graded Cartan type Lie algbras. Since Farnsteiner and Strade[3] builded the affinity between induced and coinduced modules and proved that those mixed product modules defined by Shen[4] coincide with coinduced modules[5], those results in [1] can be re-derived by Farnsteiner's results.

In this note, we first prove that the previous work[6−8] of Shu, Zhang and Yao on irreducible modules for graded Lie algebras of Cartan types W, S, H can be interpreted as coinduced modules. Then we apply Farnsteiner's general results to study cohomology of those Lie algebras of Cartan types W, S and H and extensions of their simple modules.

1 Preliminaries

In this paper, we always assume that the ground field $\mathbf{F}$ is of characteristic $p > 3$, and that all vector spaces are over $\mathbf{F}$.

1.1 Graded Lie algebras of Cartan type

Fix a positive integer m and an m-tuple $\mathbf{n} = (n_1, \cdots, n_m)$ of positive integers. Denote by $A(m;\mathbf{n})$ the index set $\{\alpha = (\alpha_1, \cdots, \alpha_m) \mid 0 \leqslant \alpha_i \leqslant p^{n_i-1}, i = 1, 2, \cdots, m\}$, denote $(p^{n_1} - 1, p^{n_2} - 1, \cdots, p^{n_m} - 1)$ by τ for brevity. There are natural partial orders "$\preceq$" and "$\prec$" in $A(m;\mathbf{n})$: $\alpha \preceq \beta$, $\alpha, \beta \in A(m;\mathbf{n})$ if and only if $\alpha_i \leqslant \beta_i$ for all $i = 1, 2, \cdots, m$, and $\alpha \prec \beta$ if and only if $\alpha \preceq \beta$ and $\alpha \neq \beta$. In this setting, we can rewrite $A(m;\mathbf{n})$ as $A(m;\mathbf{n}) = \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \mid 0 \preceq \alpha \preceq \tau\}$. We use componentwise operators in $A(m;\mathbf{n})$: For any $\alpha, \beta \in A(m;\mathbf{n})$, put $\alpha+\beta := (\alpha_1+\beta_1, \alpha_2+\beta_2, \cdots, \alpha_m+\beta_m)$, $\alpha-\beta := (\alpha_1-\beta_1, \alpha_2-\beta_2, \cdots, \alpha_m-\beta_m)$, $\alpha! := \prod\limits_{i=1}^{m} \alpha_i!$, $\binom{\alpha}{\beta} := \prod\limits_{i=1}^{m} \binom{\alpha_i}{\beta_i}$, $|\alpha| := \sum\limits_{i=1}^{m} \alpha_i$.

We have a divided power algebra $\mathfrak{A}(m;\mathbf{n})$ which is by definition a commutative associative algebra with a basis $\{x^\alpha \mid \alpha \in A(m;\mathbf{n})\}$, and multiplication subject to the following rule

$$x^\alpha x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}, \quad \forall\, \alpha, \beta \in A(m;\mathbf{n}). \tag{1.1}$$

We make some conventions that

$$x^\alpha = 0 \text{ if } \alpha \notin A(m;\mathbf{n}); \quad x_i := x^{\varepsilon_i} \text{ for } \varepsilon_i = (\delta_{1,i}, \cdots, \delta_{m,i}).$$

With multiplication rule (1.1), we know $\mathfrak{A}(m;\mathbf{n})$ is a graded algebra: $\mathfrak{A}(m;\mathbf{n}) = \bigoplus\limits_{i=0}^{s} \mathfrak{A}(m;\mathbf{n})_{[i]}$, where $\mathfrak{A}(m;\mathbf{n})_{[i]} = \mathbf{F}\text{-span}\{x^\alpha \mid |\alpha| = i\}$ and $s = \sum\limits_{i=1}^{m} (p^{n_i} - 1)$.

Let $D_i (1 \leqslant i \leqslant m)$ be the linear partial derivation of $\mathfrak{A}(m;\mathbf{n})$ with respect to the i-th invariant x_i such that $D_i(x^\alpha) = x^{\alpha-\varepsilon_i}$, $\forall\, \alpha \in A(m;\mathbf{n})$. In the following, we will recall the three classes of graded Cartan type Lie algebras of types W, S, H, drawing most of notations and results from [9].

(i) Let $D \in \mathrm{Der}\, \mathfrak{A}(m;\mathbf{n})$. D is called a special derivation if $D(x^{\alpha}) = \sum_{i=1}^{m} x^{\alpha-\varepsilon_i} D(x_i)$. Then by definition, the generalized Jacobson-Witt algebra $W(m;\mathbf{n})$ is the collection of all special derivations of the divided power algebra $\mathfrak{A}(m;\mathbf{n})$. Then by [9, Proposition 2.2, Chapter 4], $W(m;\mathbf{n})$ is a free module over $\mathfrak{A}(m;\mathbf{n})$ of rank m with a free basis $\{D_1,\cdots,D_m\}$, i.e. $W(m;\mathbf{n}) = \mathbf{F}\text{-span}\{x^{\alpha}D_i \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i \leqslant m\}$. In the following sequel the standard basis of $W(m;\mathbf{n})$ is always referred to $\{x^{\alpha}D_i \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i \leqslant m\}$ denoted by $\{E_i^W \mid i = 1,2,\cdots,t_W\}$ such that $E_i^W = D_i$ for $1 \leqslant i \leqslant m$, where $t_W = \dim W(m;\mathbf{n}) = mp^{\sum n_i}$. The structure of Lie algebra on $W(m;\mathbf{n})$ is defined via

$$[fD_i, gD_j] = fD_i(g)D_j - gD_j(f)D_i$$

for $f, g \in \mathfrak{A}(m;\mathbf{n})$ and $i, j \in \{1,\cdots,m\}$.

The gradation and filtration of $\mathfrak{A}(m;\mathbf{n})$ induce the corresponding ones on $W(m;\mathbf{n})$:

$$W(m;\mathbf{n}) = \bigoplus_{i=-1}^{s-1} W(m;\mathbf{n})_{[i]}, \text{ and } W(m;\mathbf{n}) = W(m;\mathbf{n})_{-1} \supset W(m;\mathbf{n})_0 \supset \cdots\cdots$$

where $W(m;\mathbf{n})_{[i]} = \mathbf{F}\text{-span}\{x^{\alpha}D_j \mid \alpha \in A(m;\mathbf{n}), |\alpha| = i+1, 1 \leqslant j \leqslant m\}, W(m;\mathbf{n})_i = \bigoplus_{j \geqslant i} W(m;\mathbf{n})_{[j]}, s = \sum_{i=1}^{m}(p^{n_i}-1)$.

It's specially worth mentioning that $W(m;\mathbf{n})_0 = \mathbf{F}\text{-span}\{x^{\alpha}D_j \,|\, |\alpha| \geqslant 1, j = 1,2,\cdots,m\}$ admits a structure of restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations. $W(m;\mathbf{n})_{[0]} \cong \mathbf{gl}(m)$ under $\varphi^W : W(m;\mathbf{n})_{[0]} \longrightarrow \mathbf{gl}(m), x_iD_j \longmapsto E_{ij}$ for all $1 \leqslant i,j \leqslant m$. $H^W := \mathbf{F}\text{-span}\{H_i^W := x_iD_i \mid i = 1,2,\cdots m\}$ is a canonical torus of $W(m;\mathbf{n})_{[0]}$.

(ii) Here in this case we assume $m \geqslant 3$. Define the divergence map div from the generalized Jacobson-Witt algebra $W(m;\mathbf{n})$ to the divided power algebra $\mathfrak{A}(m;\mathbf{n})$

$$\begin{aligned} \mathrm{div}: \quad & W(m;\mathbf{n}) \longrightarrow \mathfrak{A}(m;\mathbf{n}) \\ & \sum_{i=1}^{m} f_iD_i \longmapsto \sum_{i=1}^{m} D_i(f_i). \end{aligned}$$

Set $\widetilde{S(m;\mathbf{n})} = \{D \in W(m;\mathbf{n}) \mid \mathrm{div}D = 0\}$. Then by definition, the derived algebra of $\widetilde{S(m;\mathbf{n})}$ is called the special algebra $S(m;\mathbf{n})$, i.e. $S(m;\mathbf{n}) = \widetilde{S(m;\mathbf{n})}' = [\widetilde{S(m;\mathbf{n})}, \widetilde{S(m;\mathbf{n})}]$. By [9, Proposition 3.3, Chapter 4], $S(m;\mathbf{n}) = \mathbf{F}\text{-span}\{D_{ij}(x^{\alpha}) \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i < j \leqslant m\}$, where D_{ij} is a linear map from $\mathfrak{A}(m;\mathbf{n})$ to $W(m;\mathbf{n})$ defined as follows,

$$\begin{aligned} D_{ij}: \quad & \mathfrak{A}(m;\mathbf{n}) \longrightarrow W(m;\mathbf{n}) \\ & x^{\alpha} \longmapsto D_{ij}(x^{\alpha}) = x^{\alpha-\varepsilon_j}D_i - x^{\alpha-\varepsilon_i}D_j. \end{aligned}$$

A standard basis of $S(m;\mathbf{n})$ is the one taken from the following set:

$$\{D_{ij}(x^{\alpha}) \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i < j \leqslant m\},$$

denoted by $\{E_i^S \mid i = 1,2,\cdots,t_S\}$ such that $E_i^S = D_i$ for $1 \leqslant i \leqslant m$, where $t_S = \dim S(m;\mathbf{n}) = (m-1)(p^{\sum n_i}-1)$.

It is obvious that $S(m;\mathbf{n})$ is a graded subalgebra of $W(m;\mathbf{n})$. The gradation and filtration of $S(m;\mathbf{n})$ inherit from $W(m;\mathbf{n})$, i.e.

$$S(m;\mathbf{n})=\bigoplus_{i=-1}^{s-2}S(m;\mathbf{n})_{[i]},\ \text{ and }\ S(m;\mathbf{n})=S(m;\mathbf{n})_{-1}\supset S(m;\mathbf{n})_0\supset\cdots\cdots$$

where $S(m;\mathbf{n})_{[i]}=S(m;\mathbf{n})\bigcap W(m;\mathbf{n})_{[i]}=\mathbf{F}\text{-span}\{D_{kl}(x^\alpha)\mid \alpha\in A(m;\mathbf{n}),|\alpha|=i+2,1\leqslant k<l\leqslant m\}$, $S(m;\mathbf{n})_i=S(m;\mathbf{n})\bigcap W(m;\mathbf{n})_i=\bigoplus\limits_{j\geqslant i}S(m;\mathbf{n})_{[j]}$, $s=\sum\limits_{i=1}^{m}(p^{n_i}-1)$.

It's specially worth mentioning that $S(m;\mathbf{n})_0=\mathbf{F}\text{-span}\{D_{ij}(x^\alpha)\mid |\alpha|\geqslant 2,1\leqslant i<j\leqslant m\}$ admits a structure of restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations. $S(m;\mathbf{n})_{[0]}\cong \mathbf{sl}(m)$ under $\varphi^S: S(m;\mathbf{n})_{[0]}\longrightarrow \mathbf{sl}(m), x^{\varepsilon_i}D_j\longmapsto E_{ij}$ for all $1\leqslant i\neq j\leqslant m$, and $x^{\varepsilon_i}D_i-x^{\varepsilon_j}D_j\longmapsto E_{ii}-E_{jj}$ for all $1\leqslant i,j\leqslant m$. $H^S:=\mathbf{F}\text{-span}\{H_i^S:=x_iD_i-x_{i+1}D_{i+1}\mid i=1,2,\cdots m-1\}$ is a canonical torus of $S(m;\mathbf{n})_{[0]}$.

(3) Here in this case we assume $m=2r$ is even. Define the Hamiltonian operator D_H from $\mathfrak{A}(2r;\mathbf{n})$ to $W(2r;\mathbf{n})$ as follows

$$\begin{aligned} D_H: \quad & \mathfrak{A}(2r;\mathbf{n})\longrightarrow W(2r;\mathbf{n})\\ & f\longmapsto D_H(f)=\sum_{i=1}^{2r}\sigma(i)D_i(f)D_{i'}\end{aligned}$$

where $\sigma(i):=\begin{cases}1, & \text{if}\quad 1\leqslant i\leqslant r,\\ -1, & \text{if}\quad r+1\leqslant i\leqslant 2r,\end{cases}$ and $i':=\begin{cases}i+r, & \text{if}\quad 1\leqslant i\leqslant r,\\ i-r, & \text{if}\quad r+1\leqslant i\leqslant 2r.\end{cases}$

Then by definition $H(2r;\mathbf{n})=\mathbf{F}\text{-span}\{D_H(x^\alpha)\mid 0\prec\alpha\prec\tau\}$ is the Hamiltonian algebra. The standard basis is always referred to $\{D_H(x^\alpha)\mid\alpha\in A(2r;\mathbf{n})\}$ denoted by $\{E_i^H\mid i=1,2,\cdots,t_H\}$ such that $E_i^H=D_i$ for $1\leqslant i\leqslant 2r$, where $t_H=\dim H(m;\mathbf{n})=p^{\sum n_i}-2$.

It is obvious that $H(2r;\mathbf{n})$ is a graded subalgebra of $W(2r;\mathbf{n})$. The gradation and filtration of $H(2r;\mathbf{n})$ inherit from $W(2r;\mathbf{n})$, i.e.

$$H(2r;\mathbf{n})=\bigoplus_{i=-1}^{s-3}H(2r;\mathbf{n})_{[i]},\ \text{ and }\ H(2r;\mathbf{n})=H(2r;\mathbf{n})_{-1}\supset H(2r;\mathbf{n})_0\supset\cdots\cdots$$

where $H(2r;\mathbf{n})_{[i]}=H(2r;\mathbf{n})\bigcap W(2r;\mathbf{n})_{[i]}=\mathbf{F}\text{-span}\{D_H(x^\alpha)\mid 0\prec\alpha\prec\tau,|\alpha|=i+2\}$, $H(2r;\mathbf{n})_i=H(2r;\mathbf{n})\bigcap W(2r;\mathbf{n})_i=\bigoplus\limits_{j\geqslant i}H(2r;\mathbf{n})_{[j]}$, $s=\sum\limits_{i=1}^{2r}(p^{n_i}-1)$.

It's specially worth mentioning that $H(2r;\mathbf{n})_0=\mathbf{F}\text{-span}\{D_H(x^\alpha)\mid|\alpha|\geqslant 2\}$ admits a structure of restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations. $H(2r;\mathbf{n})_{[0]}\cong\mathfrak{sp}(2r)$ under $\varphi^H: H(2r;\mathbf{n})_{[0]}\to\mathfrak{sp}(2r)$, $D_H(x^{2\varepsilon_i})\mapsto\sigma(i)E_{ii'}$ and $D_H(x^{\varepsilon_i+\varepsilon_j})\mapsto\sigma(j)E_{ij'}+\sigma(i)E_{ji'}$. $H^H:=\mathbf{F}\text{-span}\{H_i^H:=x_iD_i-x_{i+r}D_{i+r}\mid i=1,2,\cdots r\}$ is a canonical torus of $H(2r;\mathbf{n})_{[0]}$.

1.2 Generalized restricted Lie algebras and their generalized reduced enveloping algebras

As is well known that not all of Cartan type Lie algebras are restricted Lie algebras, but those algebras are generalized restricted Lie algebras in the following sense.

Definition 1.1[10] A generalized restricted Lie algebra L is a Lie algebra associated with an ordered basis $E = (e_i)_{i\in I}$ and a so-called generalized restricted map $\varphi_{\mathbf{s}} : E \to L$ sending $e_i \longmapsto e_i^{\varphi_{\mathbf{s}}}$ with $\mathbf{s} = (\mathbf{s_i})_{\mathbf{i}\in\mathbf{I}}, \mathbf{s_i} \in \mathbb{Z}_+$ such that $\mathrm{ad}e_i^{\varphi_{\mathbf{s}}} = (\mathrm{ad}e_i)^{p^{s_i}}$ for all $i \in I$.

Let us demonstrate how the three graded Cartan type Lie algebras $X(m;\mathbf{n})$, $X \in \{W, S, H\}$, are endowed with generalized restricted structures.

Example 1.2 (i) For any restricted Lie algebra $(\mathfrak{g}, [p])$, $\mathfrak{g}$ is obviously a generalized restricted Lie algebra associated with an arbitrary given basis E, $\mathbf{s} = \mathbf{1} := (1, \cdots, 1)$ and $\varphi_{\mathbf{s}} = [p]|_E$. Conversely, if a generalized restricted Lie algebra $(\mathfrak{g}, \varphi_{\mathbf{s}})$ is associated with a basis E and $\mathbf{s} = \mathbf{1}$, then $\mathfrak{g}$ is a restricted Lie algebra in usual sense by a Jacobson's result (cf. [9](2.2.3)), with p-mapping $[p]$ coinciding with $\varphi_{\mathbf{s}}$ on E.

(ii) In $L = X(m;\mathbf{n}), X \in \{W, S, H\}$, there is a standard basis $\{e_i := E_i^X \mid i = 1, 2, \cdots t^X\}$ of L (see 1.1). Then, associated with this basis and $\mathbf{s} := (n_1, n_2, \cdots, n_m, 1, 1, \cdots, 1)$, L is a generalized restricted Lie algebra with a generalized restricted mapping $\varphi_{\mathbf{s}}$: $e_i^{\varphi_{\mathbf{s}}} = 0$ if $i = 1, \cdots, m$ and $e_i^{\varphi_{\mathbf{s}}} = e_i^{[p]}$. This is because L_0 is a restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations, as well as ad $(e_i)^{p^{n_i}} = 0$ for $i = 1, \cdots, m$.

For restricted Lie algebras and generalized restricted Lie algebras over an algebraically closed field, we have the following basic fact directly by Schur lemma.

Lemma 1.3 Let $\mathbf{F}$ be an algebraically closed field of characteristic $p > 0$.

(i) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over $\mathbf{F}$ and (V, ρ) is an irreducible representation of $\mathfrak{g}$, then there exists a unique $\chi \in \mathfrak{g}^*$ such that

$$\rho(x)^p - \rho(x^{[p]}) = \chi(x)^p \,\mathrm{id}_V, \ \ \forall\, x \in \mathfrak{g}. \tag{1.2}$$

Here the function χ is called a p-character of V. A $\mathfrak{g}$-representation (V, ρ) (module V) satisfying (1.2) is called a χ-reduced representation (module).

(ii) Let $(L, \varphi_{\mathbf{s}})$ be a generalized restricted Lie algebra over F associated with a basis $E = (e_i)_{i\in I}$ and $\varphi_{\mathbf{s}}$ with $\mathbf{s} = (s_i)_{i\in I}$. If (V, ρ) is an irreducible representation of L, then there exists a unique $\chi \in L^*$ such that

$$\rho(e_i)^{p^{s_i}} - \rho(e_i^{\varphi_{\mathbf{s}}}) = \chi(e_i)^{p^{s_i}} \,\mathbf{i}d_V, \ \ \forall\, x \in L. \tag{1.3}$$

Here the function χ is also called a (generalized) p-character of V. A representation (module) of L satisfying (1.3) is called a generalized χ-reduced representation (module), all of which constitute a full subcategory of the Lie algebra representation category.

Let's continue to recall some facts. Assume as above, that $\mathfrak{g}$ is a restricted Lie algebra and that L is a generalized restricted Lie algebra. For $\chi \in \mathfrak{g}^*$ or $\chi \in L^*$, we define $U(\mathfrak{g}, \chi) := U(\mathfrak{g})/\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\rangle$, $U_{p^{\mathbf{s}}}(L, \chi) := U(L)/\langle e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}} \mid i \in I\rangle$ where $\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\rangle$ means the ideal in $U(\mathfrak{g})$ generated by these central elements $x^p - x^{[p]} - \chi(x)^p$ for $x \in \mathfrak{g}$, and where $\langle e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}} \mid i \in I\rangle$ means the ideal in $U(L)$ generated by those central elements $e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}}$ for all $e_i \in E$. Call $U(\mathfrak{g}, \chi)$ and $U_{p^{\mathbf{s}}}(L, \chi)$ the χ-reduced enveloping algebra of $\mathfrak{g}$ and the generalized χ-reduced enveloping algebra of L respectively. A χ-reduced module category of $\mathfrak{g}$ coincides with the unitary $U(\mathfrak{g}, \chi)$-module category; and a generalized χ-reduced module category of L coincides with the unitary $U_{p^{\mathbf{s}}}(L, \chi)$-module category. Especially, in the case when $\chi = 0$ we have the restricted enveloping algebra $U_p(\mathfrak{g}) := U(\mathfrak{g}, 0)$ and the generalized restricted enveloping algebra $U_{p^{\mathbf{s}}}(L) := U_{p^{\mathbf{s}}}(L, 0)$ respectively(cf. 10, 11).

Remark 1.4 (i) In Example 1.2, we know that a restricted Lie algebra $(\mathfrak{g}, [p])$ can be a generalized restricted Lie algebra associated with an arbitrary given basis E and $\mathbf{s} = \mathbf{1}$. Furthermore, it's easily seen that in this sense, a generalized χ-reduced module category and a generalized χ-reduced enveloping algebra coincide with the ones arising from a restricted Lie algebra.

(ii) The invariance of filtration for $L = X(m; \mathbf{n})$ under $\mathrm{Aut}(L)$, $X \in \{W, S, H\}$, enables us to define the height of a nonzero $\chi \in L^*$ via $\mathrm{ht}(\chi) := \max\{i \mid \chi(L_{i-1}) \neq 0\}$, and $\mathrm{ht}(0) := -1$. Then the height function on L^* is invariant under the action of $\mathrm{Aut}(L)$ defined by $\sigma \cdot \chi = \chi \circ \sigma^{-1}$ for $\sigma \in \mathrm{Aut}(L)$ and $\chi \in L^*$.

1.3 Induced and coinduced modules

Let $L = X(m; \mathbf{n}), X \in \{W, S, H\}$, $\chi \in L^*$. Set $z_i = D_i^{p^{m_i}} - \chi(D_i)^{p^{m_i}} \in C(U(L)), 1 \leqslant i \leqslant m$. Denote by $\mathcal{O}(L, L_0)$ the subalgebra of $U(L)$ generated by L_0 and those central elements $z_i, 1 \leqslant i \leqslant m$. Then by PBW Theorem, $U(L)$ is a free $\mathcal{O}(L, L_0)$-module with basis $\{E^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_m^{\alpha_m} \mid 0 \preceq \alpha \preceq \tau\}$.

Let $\sigma : L_0 \longrightarrow F$ be the Lie algebra homomorphism given by $\sigma(x) := \mathrm{tr}(\mathrm{ad}_{L/L_0}(x))$, $\forall\, x \in L_0$. Note that the correspondence $x \longmapsto x + \sigma(x)$ is a homomorphism from L_0 to $U(L_0)^-$, then it extends uniquely to an algebra homomorphism $\Psi : U(L_0) \longrightarrow U(L_0)$. Ψ is indeed an isomorphism with inverse $\Psi^{-1} : x \longmapsto x - \sigma(x), \forall\, x \in L_0$. Note that $\mathcal{O}(L, L_0) \cong F[z_1, z_2, \cdots, z_m] \bigotimes U(L_0)$, then $\rho = 1 \bigotimes \Psi$ defines an isomorphism of $\mathcal{O}(L, L_0)$.

For any L_0-module V, the action of $U(L_0)$ can be extended to $\mathcal{O}(L, L_0)$ by letting the polynomial algebra $F[z_1, z_2, \cdots, z_m]$ operate via canonical supplementation. Henceforth all L_0-modules will be considered as $\mathcal{O}(L, L_0)$-modules in this fashion. A twisted action on V can be introduced by setting $x \circ v = x \cdot v + \sigma(x)v$, $\forall\, x \in L_0, v \in V$. This new L_0-module will be denoted as V_σ. One can easily see that $x \circ v = \rho(x)v$, $\forall\, x \in L_1, v \in V$.

Definition 1.5 The induced L-module from a module V of the subalgebra L_0 denoted by $\mathbf{Ind}_{L_0}(V)$ is by definition $U(L) \bigotimes_{\mathcal{O}(L,L_0)} V$ with L-action defined by left multiplication. The coinduced module from a module V of the subalgebra L_0 denoted by $\mathbf{Coind}_{L_0}(V)$ is by definition $\mathrm{Hom}_{\mathcal{O}(L,L_0)}(U(L), V)$ with L-action given by $(x \cdot f)(u) = f(ux)$, $\forall\, u \in U(L), x \in L$.

The following theorem is from [5, Theorem 1.4] which gives the relationship between induced modules and coinduced modules.

Theorem 1.6[5] Let V be an L_0-module, then $\mathbf{Ind}_{L_0}(V_\sigma) \cong \mathbf{Coind}_{L_0}(V)$ as L-modules.

2 Irreducible modules of graded Cartan type Lie algebras

In the sequel, let $L = X(m; \mathbf{n})$, $X \in \{W, S, H\}$. Any irreducible L-module is a generalized χ-reduced L-module for some $\chi \in L^*$.

Definition 2.1 An irreducible L-module M with generalized p-character $\chi \in L^*$ is called an exceptional module if $\mathrm{ht}(\chi) \leqslant 0$ and M contains an irreducible $L_{[0]}$-module which is a highest weight module with a fundamental weight as the highest weight. Any irreducible L-module which is not exceptional is called non-exceptional.

We first have the following key observation.

Proposition 2.2 Let $L = X(m;\mathbf{n})$, $X \in \{W,S,H\}$, and $\chi \in L^*$. Assume V is a $\chi|_{L_0}$-reduced L_0-module. Let $M = U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_{p^{\mathbf{s}}}(L_0,\chi|_{L_0})} V$, which is a generalized χ-reduced L-module. Denote by **Ind** the induced module structure on M. Define another L-module structure $\rho_{_L}$ on M by $\rho_{_L} = \mathbf{Ind} - \rho_{_R}\circ \mathrm{div}$, where $\rho_{_R}(x^\alpha)(E^\beta\otimes v) = (-1)^{|\alpha|}\binom{\beta}{\alpha}E^{\beta-\alpha}\otimes v$, $0\preceq \alpha,\beta\preceq\tau, v\in V, E^\beta = \prod_{i=1}^m D_i^{\beta_i}$. Then $(M,\rho_L)\cong \mathbf{Ind}_{L_0}^L(V_\sigma)$.

Proof Let $L = W(m;\mathbf{n})$. Take $x^\alpha D_i\in L$, then $\rho_{_L}(x^\alpha D_i) = (\mathbf{Ind}-\rho_{_R}\circ\mathrm{div})(x^\alpha D_i)$. So, for any $E^\beta\otimes v\in M$, we have

$$\begin{aligned}
\rho_{_L}(x^\alpha D_i)(E^\beta\otimes v) =&(x^\alpha D_i)E^\beta\otimes v - \rho_{_R}(x^{\alpha-\varepsilon_i})(E^\beta\otimes v)\\
=&\sum_\gamma(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}(x^{\alpha-\gamma}D_i)\otimes v - (-1)^{|\alpha|-1}\binom{\beta}{\alpha-\varepsilon_i}E^{\beta-\alpha+\varepsilon_i}\otimes v\\
=&\sum_{0\preceq\gamma\prec\alpha}(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}\otimes x^{\alpha-\gamma}D_i\cdot v + (-1)^{|\alpha|}\binom{\beta}{\alpha}E^{\beta-\alpha+\varepsilon_i}\otimes v\\
&-(-1)^{|\alpha|-1}\binom{\beta}{\alpha-\varepsilon_i}E^{\beta-\alpha+\varepsilon_i}\otimes v\\
=&\sum_{0\preceq\gamma\prec\alpha}(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}\otimes x^{\alpha-\gamma}D_i\cdot v + (-1)^{|\alpha|}\binom{\beta+\varepsilon_i}{\alpha}E^{\beta-\alpha+\varepsilon_i}\otimes v.
\end{aligned}$$

On the other hand, the action of $x^\alpha D_i$ on $E^\beta\otimes v$ in the module $\mathbf{Ind}_{L_0}^L V_\sigma$ is computed as follows:

$$\begin{aligned}
x^\alpha D_i\cdot(E^\beta\otimes v) =&(x^\alpha D_i)E^\beta\otimes v\\
=&\sum_\gamma(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}(x^{\alpha-\gamma}D_i)\otimes v\\
=&\sum_{0\preceq\gamma\prec\alpha}(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}\otimes x^{\alpha-\gamma}D_i\circ v + (-1)^{|\alpha|}\binom{\beta}{\alpha}E^{\beta-\alpha+\varepsilon_i}\otimes v\\
=&\sum_{0\preceq\gamma\prec\alpha}(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}\otimes x^{\alpha-\gamma}D_i\cdot v - (-1)^{|\alpha|-1}\binom{\beta}{\alpha-\varepsilon_i}E^{\beta-\alpha+\varepsilon_i}\otimes v\\
&+(-1)^{|\alpha|}\binom{\beta}{\alpha}E^{\beta-\alpha+\varepsilon_i}\otimes v\\
=&\sum_{0\preceq\gamma\prec\alpha}(-1)^{|\gamma|}\binom{\beta}{\gamma}E^{\beta-\gamma}\otimes x^{\alpha-\gamma}D_i\cdot v + (-1)^{|\alpha|}\binom{\beta+\varepsilon_i}{\alpha}E^{\beta-\alpha+\varepsilon_i}\otimes v.
\end{aligned}$$

as desired.

Let $L = S(m;\mathbf{n})$ or $H(m;\mathbf{n})$. Note that in this case, $\sigma = 0$. So $M\cong \mathbf{Ind}_{L_0}^L(V)\cong \mathbf{Ind}_{L_0}^L(V_\sigma)$.

Summing up, we complete the proof.

Remark 2.3 Keep notations as in Proposition 2.2. Moreover assume $\mathrm{ht}(\chi) < \min\{p^{n_i} - p^{n_i-1} \mid 1\leqslant i\leqslant m\} - 1 + \delta_{XW}$. Then by [6-8], any non-exceptional irreducible L-module with generalized p-character $\chi \in L^*$ is of the form (M, ρ_L). Furthermore, any two irreducible L_0-submodules of M are isomorphic. And any two irreducible non-exceptional generalized χ-reduced L-modules are isomorphic if and only if their irreducible L_0-submodules are isomorphic. Combining this with the above theorem, any irreducible generalized χ-reduced non-exceptional L-module with $\mathrm{ht}(\chi) < \min\{p^{n_i} - p^{n_i-1} \mid 1 \leqslant i \leqslant m\} - 1 + \delta_{XW}$ is induced from some irreducible $\chi|_{L_0}$-reduced L_0-module.

Theorem 2.4 Let $\chi \in L^*$ and V be an L_0-module with p-character $\chi|_{L_0}$, then there exists isomorphism of $U(L)$-modules

$$\varphi: \quad U(L)\bigotimes\nolimits_{\mathcal{O}(L,\,L_0)} V \longrightarrow U_{p^{\mathbf{s}}}(L,\chi)\bigotimes\nolimits_{U_{p^{\mathbf{s}}}(L_0,\,\chi|_{L_0})} V.$$

Proof Let $\pi: U(L) \longrightarrow U_{p^{\mathbf{s}}}(L,\chi)$ be the canonical projection, then π maps $U(L_0)$ onto $U_{p^{\mathbf{s}}}(L_0,\,\chi|_{L_0})$. Consider now the following mapping

$$\begin{aligned} \Gamma: \qquad U(L)\times V &\longrightarrow U_{p^{\mathbf{s}}}(L,\chi)\bigotimes\nolimits_{U_{p^{\mathbf{s}}}(L_0,\,\chi|_{L_0})} V \\ (u,v) &\longmapsto \pi(u)\otimes v,\ \ \forall\, u\in U(L),\, v\in V. \end{aligned}$$

Since $\Gamma(uz_i, v) = \pi(uz_i)\otimes v = \pi(u)\pi(z_i)\otimes v = 0$ and $z_i\cdot v = 0\,(1\leqslant i\leqslant m)$, as well as

$$\Gamma(ux,v) = \pi(u)\pi(x)\otimes v = \pi(u)\otimes\pi(x)v = \pi(u)\otimes x\cdot v = \Gamma(u, x\cdot v), \quad \forall\, x\in L_0.$$

Γ is $\mathcal{O}(L,L_0)$-balanced. Therefore, Γ induces an F-linear mapping

$$\begin{aligned} \varphi: \qquad U(L)\bigotimes\nolimits_{\mathcal{O}(L,\,L_0)} V &\longrightarrow U_{p^{\mathbf{s}}}(L,\chi)\bigotimes\nolimits_{U_{p^{\mathbf{s}}}(L_0,\,\chi|_{L_0})} V \\ u\otimes v &\longmapsto \pi(u)\otimes v,\ \ \forall\, u\in U(L),\, v\in V. \end{aligned}$$

φ is obvious a $U(L)$-module homomorphism. Assume that $\{v_i \mid 1\leqslant i\leqslant t\}$ is a basis of V, then $\{E^\beta\otimes v_i := D_1^{\beta_1}D_2^{\beta_2}\cdots D_m^{\beta_m}\otimes v_i \mid 0\preceq\beta\preceq\tau, 1\leqslant i\leqslant t\}$ and $\{\pi(E^\beta\otimes v_i) := \pi(D_1)^{\beta_1}\pi(D_2)^{\beta_2}\cdots\pi(D_m)^{\beta_m}\otimes v_i \mid 0\preceq\beta\preceq\tau, 1\leqslant i\leqslant t\}$ are basis of $U(L)\bigotimes_{\mathcal{O}(L,\,L_0)} V$ and $U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_{p^{\mathbf{s}}}(L_0,\,\chi|_{L_0})} V$ respectively. So φ is an isomorphism. We complete the proof.

Combining Theorem 1.6, Remark 2.3 and Theorem 2.4, we obtain the following.

Corollary 2.5 Let $\chi \in L^*$ and V be an L_0-module with p-character $\chi|_{L_0}$. Then $U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_{p^{\mathbf{s}}}(L_0,\,\chi|_{L_0})} V_\sigma \cong \mathbf{Coind}_{L_0}(V)$ as L-modules. In particular, all irreducible non-exceptional L-modules with generalized p-character $\chi \in L^*$ satisfying $\mathrm{ht}(\chi) < \min\{p^{n_i} - p^{n_i-1} \mid 1\leqslant i\leqslant m\} - 1 + \delta_{XW}$ are coinduced modules.

3 Extensions and cohomology

Let $L = X(m;\mathbf{n}), X\in\{W,S,H\}$. We know in the previous section that all irreducible non-exceptional L-modules with generalized p-character $\chi\in L^*$ satisfying $\mathrm{ht}(\chi) < \min\{p^{n_i} - p^{n_i-1} \mid 1\leqslant i\leqslant m\} - 1 + \delta_{XW}$ are coinduced modules. Furthermore, in this section, we assume in addition $\chi|_{L_{[-1]}} = 0$ so that we can apply some results of Rolf. Farnsteiner to our case to study extensions between irreducible L-modules as well as cohomology of L.

Theorem 3.1 Let $L = X(m;\mathbf{n})$, $X \in \{W, S, H\}$. For any two irreducible non-exceptional L-modules $M = U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_{p^{\mathbf{s}}}(L_0,\chi|_{L_0})} V$ and $N = U_{p^{\mathbf{s}}}(L,\chi')\bigotimes_{U_{p^{\mathbf{s}}}(L_0,\chi'|_{L_0})} W$ with generalized p-characters $\chi, \chi' \in L^*$ satisfying $\mathrm{ht}(\chi), \mathrm{ht}(\chi') < \min\{p^{n_i} - p^{n_i-1} \mid 1 \leqslant i \leqslant m\} - 1 + \delta_{XW}$, where V and W are irreducible L_0-submodules of M and N respectively, we have

$$\begin{aligned}\mathrm{Ext}^n_{U(L)}(M,N) &\cong \bigoplus_{p+q=n} \bigwedge^p (L/L_0) \bigotimes_F \mathrm{Ext}^q_{U(L_0)}(M,W) \\ &\cong \bigoplus_{p+q=n} \bigwedge^p (L/L_0) \bigotimes_F \mathrm{Ext}^q_{U(L_0)}(V_\sigma,N).\end{aligned}$$

Proof Note that by Proposition 2.2 and Remark 2.3, $M \cong \mathbf{Ind}_{L_0}(V_\sigma), N \cong \mathbf{Ind}_{L_0}(W_\sigma)$. Let $z_i = D_i^{p^{m_i}}, 1 \leqslant i \leqslant m$, then $z_i \cdot M = z_i \cdot N = 0$, for all i. By [5, Corollary 3.3],

$$\begin{aligned}\mathrm{Ext}^n_{U(L)}(M,N) &= \mathrm{Ext}^n_{U(L)}(\mathbf{Ind}_{L_0}(V_\sigma), \mathbf{Ind}_{L_0}(W_\sigma)) \\ &\cong \bigoplus_{p+q=n} \bigwedge^p (L/L_0) \bigotimes_F \mathrm{Ext}^q_{U(L_0)}(M,W) \\ &\cong \bigoplus_{p+q=n} \bigwedge^p (L/L_0) \bigotimes_F \mathrm{Ext}^q_{U(L_0)}(V_\sigma,N).\end{aligned}$$

Theorem 3.2 Keep assumptions as in Theorem 3.1 and in addition assume that $\chi(H_i^X) \neq \chi'(H_i^X)$ for some $i \in \Xi_X$, where Ξ_X is defined as follows:

$$\Xi_X := \begin{cases} \{1,2,\cdots,m\}, & \text{if} \quad X = W, \\ \{1,2,\cdots,m-1\}, & \text{if} \quad X = S, \\ \{1,2,\cdots,m/2\}, & \text{if} \quad X = H. \end{cases}$$

Then $\mathrm{Ext}^n_{U(L)}(M,N) = 0, \forall\, n \geqslant 0$.

Proof Note that V and W are H-weight modules with weights $P = \{\gamma \mid V_\gamma \neq 0\}, Q = \{\gamma' \mid W_{\gamma'} \neq 0\}$, where $V_\gamma = \{v \in V \mid h \cdot v = \gamma(h)v, \forall\, h \in H^X\}, W_{\gamma'} = \{w \in W \mid h \cdot w = \gamma'(h)w, \forall\, h \in H^X\}$. For any $\gamma \in P$, as

$$H_i^p v - H_i v = \chi(H_i)^p v, \quad \forall\, v \in V_\gamma.$$

Then $\gamma(H_i)^p - \gamma(H_i) = \chi(H_i)^p$. Similarly, for any $\gamma' \in Q$

$$\gamma'(H_i)^p - \gamma'(H_i) = \chi'(H_i)^p.$$

So $(\gamma - \gamma')(H_i) \notin \mathbb{F}_p$. Then the statement is a consequence of [5, Corollary 3.5].

The following result is a direct consequence of Theorem 5.1 in [5].

Proposition 3.3 Keep notations and assumptions as in Theorem 1.3, then

(i) $H^n(L_0, V) \cong H^n(L, L_{[-1]}, M), \forall\, n \geqslant 0$.

(ii) $H^n(L, M) \cong \bigoplus_{p+q=n} \bigwedge^p L_{[-1]} \bigotimes_F H^q(L, L_{[-1]}, M), \forall\, n \geqslant 0$.

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REFERENCES

[1] QIU S, SHEN G Y. Cohomology of graded Lie algebras of Cartan type of characteristic p[J]. Abh Math Semin Univ Hamb, 1987, 57: 139-156.

[2] SHU B. On the cohomology of generalized restricted Lie algebras[J]. Chinese Ann Math Ser B, 1998, 19: 421-432.

[3] FARNSTEINER R, STRADE H. Shapiro's lemma and its consequences in the cohomology theory of modular Lie algebras[J]. Math Z, 1991, 206: 153-168.

[4] SHEN G Y. Graded modules of graded Lie algebras of Cartan type, I[J]. Scientica Sinica, 1986, 29: 570-581.

[5] FARNSTEINER R. Extension functors of modular Lie algebras[J]. Math Ann, 1990, 288: 713-730.

[6] ZHANG C W. Representations of the restricted Lie algebras of the Cartan type[J]. J Algebra, 2005, 290(2): 408-432.

[7] YAO Y F, SHU B. Irreducible representations of the special algebras in prime characteristic[J]. Contemp Math, 2009, 478: 273-295.

[8] SHU B, ZHANT C W. Restricted representations of the witt superalgebras[J]. J Algebra, 2010, 324: 652-672.

[9] STRADE H, FARNSTEINER R. Modular Lie Algebras and Their Representations[M]. New York: Marcel Dekker, 1988.

[10] SHU B. The generalized restricted representations of graded Lie algebras of cartan type[J]. J Algebra, 1997, 194: 157-177.

[11] SHU B. Generalized restricted Lie algebras and representations of the Zassenhaus algebra[J]. J Algebra, 1998, 204: 549-572.

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[8] WANG Q L, LI S J. Generalized higher-order optimality conditions for set-valued optimization under Henig efficiency[J]. Numer Funct Anal Optim, 2009, 30(7-8): 849-869.

[9] LI S J, YANG X Q, CHEN G Y. Nonconvex vector optimization of set-valued mappings[J]. J Math Anal Appl, 2003, 283: 337-350.

[10] CERTH C, WEIDNER P. Nonconvex separation theorems and some applications in vector optimization[J]. J Optim Theory Appl, 1990, 67: 297-320.

[11] CHEN G Y, GOH C J, YANG X Q. Vector network equilibrium problems and nonlinear scalarization methods[J]. Math Meth Oper Res , 1999, 49: 239-253.

(上接第 84 页)

[11] FAN Y Z, TAM B S, ZHOU J. Maximizing spectral radius of unoriented Laplacian matrix over bicyclic graphs of a given order[J]. Linear Multilinear Algebra, 2008, 56(4): 381-397.

[12] FENG L H, GUI G. The signless laplacian spectral radius of unicyclic graphs with graph constraints[J]. KYUNG-POOK Math J, 2009, 49: 123-131.

[13] FENG L H. The signless laplacian spectral radius for bicyclic graphs with k Pendant Vertices[J]. KYUNGPOOK Math J, 2010, 50: 109-116.

[14] DAS K C. A characterization on graphs which achieve the upper bound for the largest Laplacian eigenvalue of graphs[J]. Linear Algebra and its Applications, 2004, 376: 173-186.

[15] OLIVEIRA C S, DE LIMA L S, DE ABRREU N M M, et al. Bounds on the index of the signless Laplacian of a graph[J]. Discrete Appl Math, 2010, 158(4): 355-360.

[16] HONG Y, ZHANG X D. Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees[J]. Discrete Math, 2005, 296: 187-197.