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Lower semicontinuity to parametric lexicographic vector equilibrium problems

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Abstract: In this paper, a parametric vector equilibrium problem in a lexicographic order was first introduced. Then, by using an auxiliary problem, the lower semicontinuity of the solution set map was established based on the density of the solution set mapping for a parametric lexicographic vector equilibrium problem.

Key words: lexicographic vector equilibrium problem; lower semicontinuity; lexicographic cone

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参数字典序向量平衡问题的下半连续性

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摘要: 首先介绍由字典锥所构造的参数向量平衡问题. 进而, 基于稠密性结果, 通过构造一个辅助问题的方法得到了参数字典序意义下向量平衡问题解集映射的下半连续性.

关键词: 字典序向量平衡问题; 下半连续; 字典锥

0 Introduction

As a unified model of vector optimization problems, vector variational inequality problems, variational inclusion problems and vector complementarity problems, vector equilibrium problems have been intensively studied. The stability analysis of the solution mapping for these problems is an important topic in vector optimization theory. Recently, a great deal of research has been denoted to the semicontinuity of the solution mapping for a parametric vector equilibrium problem.

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Based on the assumption of (strong) C-inclusion property of a function, Anh and Khanh[1] obtained the upper and lower semicontinuity of the solution set map of parametric multivalued (strong) vector quasiequilibrium problems. By virtue of a scalarization technique, Chen et al.[2,3] discussed the semicontinuity of the solution set map of parametric (weak) vector equilibrium problems. Especially, they investigated the lower semicontinuity and continuity of the solution set map of a parametric generalized vector equilibrium problem by a scalarization method and a property of the union of a family of set-valued mapping. Anh and Khanh[4] obtained the semicontinuity of a class of parametric quasiequilibrium problems by a generalized concavity assumption and a closedness of the level set of functions.

It is well known that partial order plays an important role in vector optimization theory. The vector optimization problems in the previous references are studied in the partial order induced by a closed or open cone. But in some situations, the cone is neither open nor closed, such as the lexicographic cone. On the other hand, since the lexicographic order induced by the lexicographic cone is a total order, it can refine the optimal solution points to make it smaller in the theory of vector optimization. Thus, it is valuable to investigate the vector optimization problems in the lexicographic order. In fact, there has been some literature in this respect, such as [5, 6]. But, there is no paper studing the stability of the problems.

Motivated by the work of [2, 3, 5, 7-9], this paper aims to establish the lower semicontinuity of the solution set map to a parametric lexicographic vector equilibrium problem. Since the lexicographic cone is neither open nor closed, the direct investigation of the problem is not an easy task. For the particularity of lexicographic cone, by using an auxiliary problem and a density result, we first obtain the lower semicontinuity of the solution mapping to parametric lexicographic vector equilibrium problem.

The rest of the paper is organized as follows. In Section 1, we introduce the parametric lexicographic vector equilibrium problems, and recall some concepts of semicontinuity. In Section 2, we investigate the lower semicontinuity of the solution set mapping to parametric lexicographic vector equilibrium problems.

1 Preliminaries

Throughout this paper, let X and Λ be Hausdorff topological vector spaces. The lexicographic cone of $\mathbf{R}^2$ is defined as the set of all vectors whose first nonzero coordinate (if any) is positive:

$$S_{\rm lex} := \{(s_1, s_2) \in \mathbf{R}^2 : s_1 > 0)\} \cup \{(s_1, s_2) \in \mathbf{R}^2 : s_1 = 0 \text{ and } s_2 \geqslant 0\}.$$

Note that the lexicographic cone $S_{\rm lex}$ is neither closed nor open. Remark also that $S_{\rm lex}$ is convex, pointed and $S_{\rm lex} \cup (-S_{\rm lex}) = \mathbf{R}^2$. The binary relation defined for any $u, v \in \mathbf{R}^2$ by

$$u \leqslant_{\rm lex} v \Leftrightarrow u \in v - S_{\rm lex}$$

is a total order on $\mathbf{R}^2$. The binary relation induced by $S_{\rm lex}$ is called a lexicographic order.

Let $f(\cdot,\cdot)=(f_1(\cdot,\cdot),f_2(\cdot,\cdot)):A\times A\to\mathbf{R}^2$ be a vector-valued function, where A is a subset of X and $f_i:A\times A\to\mathbf{R}$, for $i=1,2$. Now, we introduce the lexicographic vector equilibrium problem: find $x\in A$ such that

$$f(x,y)\in S_{\rm lex},\ \ \forall y\in A.$$

If $f(x,y)=g(y)-g(x),\ x,y\in A$, and if $\bar{x}\in A$ is a lexicographic vector equilibrium solution, then $\bar{x}\in A$ is called a lexicographic optimization solution[12] of g.

When the function f is perturbed by a parameter $\lambda\in\Lambda$, we can define the parametric lexicographic vector equilibrium problem: find $x\in E(\lambda)$ such that

$$f(x,y,\lambda)\in S_{\rm lex},\ \forall y\in E(\lambda),$$

where $E:\Lambda\to 2^X\backslash\{\emptyset\}$ is a set-valued mapping with $E(\Lambda)=\bigcup\limits_{\lambda\in\Lambda}E(\lambda)\subset A$. For each $\lambda\in\Lambda$, we denote by $S(\lambda)$ the solution set mapping of the parametric lexicographic vector equilibrium problem, i.e.,

$$S(\lambda):=\{x\in E(\lambda):f(x,y,\lambda)\in S_{\rm lex},\ \ \forall y\in E(\lambda)\}.$$

Next we recall some basic definitions. Let $F:\Lambda\to 2^X$ be a set-valued mapping, and given $\lambda_0\in\Lambda$.

Definition 1.1[10]

(i) F is said to be upper semicontinuous (u.s.c., in short) at λ_0 if for any open set $N\subset X$ with $F(\lambda_0)\subset N$, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that for every $\lambda\in N(\lambda_0)$, $F(\lambda)\subset N$.

(ii) F is said to be lower semicontinuous (l.s.c., in short) at λ_0 if for any open set $N\subset X$ with $F(\lambda_0)\cap N\neq\emptyset$, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that for every $\lambda\in N(\lambda_0)$, $F(\lambda)\cap N\neq\emptyset$.

We say F is u.s.c.(resp. u.s.c.) on Λ if it is u.s.c.(resp. l.s.c.) at each $\lambda_0\in X$. F is said to be continuous on Λ if it is both u.s.c. and l.s.c. on Λ.

Proposition 1.2[10,11] If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda\in\Lambda$), then F is u.s.c. at $\lambda_0\in\Lambda$ if and only if for any net $\{\lambda_\alpha\}\subset\Lambda$ with $\lambda_\alpha\to\lambda_0$ and for any $y_\alpha\in F(\lambda_\alpha)$, there exist $y_0\in F(\lambda_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$, such that $y_\beta\to y_0$.

The lower limit of F is defined as

$$\liminf_{\lambda\to\lambda_0}F(\lambda):=\{x\in X:\forall\lambda_\gamma\to\lambda_0,\exists x_\gamma\in F(\lambda_\gamma),{\rm s.t.}x_\gamma\to x)\}.$$

Proposition 1.3[7]

(i) $\liminf_{\lambda\to\lambda_0}F(\lambda)$ is a closed set;

(ii) F is l.s.c at $\lambda_0\in{\rm dom}\,F:=\{\lambda|F(\lambda)\neq\emptyset\}$ if and only if $F(\lambda_0)\subseteq\liminf\limits_{\lambda\to\lambda_0}F(\lambda)$.

Now, we introduce the following concept of concavity in the sense of lexicographic order.

Definition 1.4 Let A be a convex subset of X and $f=(f_1,f_2):A\to\mathbf{R}^2$ be a vector-valued mapping. f is said to be $S_{\rm lex}$-concave if, for all $x,y\in A$ and $l\in(0,1)$ such that

$$f(lx+(1-l)y)-lf(x)-(1-l)f(y)\in S_{\rm lex}.$$

If $f_1 : A \to \mathbf{R}$ is a strict concave function on A, i.e., for any $x, y \in A$ with $x \neq y$ and $l \in (0,1)$,

$$f_1(lx + (1-l)y) > lf_1(x) + (1-l)f_1(y),$$

then f is S_{lex}-concave on A. However, the converse is not valid. The following example illustrates this case.

Example 1.5 Let $A = [0,1]$ and $f(x,y) = (f_1(x,y), f_2(x,y)) : A \times A \to \mathbf{R}^2$, where $f_1(x,y) = f_2(x,y) = x + y$. For each $x_1, x_2 \in A$ and $l \in (0,1)$,

$$f(x_1 + (1-l)x_2, y) - lf(x_1, y) - (1-l)f(x_2, y) = (0,0) \in S_{\text{lex}},$$

namely, for each $y \in A$, $f(\cdot, y)$ is S_{lex}-concave on A. However, it is obvious that $f_1(\cdot, y)$ is not a strict concave function on A, for each $y \in A$.

2 Main results

In order to investigate the lower semicontinuity of the solution mapping $S(\cdot)$ to the parametric lexicographic vector equilibrium problem, we consider the following auxiliary problem: find $x \in E(\lambda)$ such that

$$f(x, y, \lambda) \in S, \quad \forall y \in E(\lambda),$$

where $S = \{(s_1, s_2) \in \mathbf{R}^2 : s_1 > 0\}$. For each $\lambda \in \Lambda$, we denote by $S_1(\lambda)$ the solution set mapping of the auxiliary problem, i.e.,

$$S_1(\lambda) := \{x \in E(\lambda) : f(x, y, \lambda) \in S, \quad \forall y \in E(\lambda)\}.$$

Since the existence of solution for lexicographic vector equilibrium problem has been intensively studied in the literature, we focus on the stability study, assuming always that $S_1(\lambda) \neq \emptyset$ and $S(\lambda) \neq \emptyset$. Clearly, $S_1(\lambda) \subseteq S(\lambda)$.

Lemma 2.1 Suppose that the following conditions are satisfied:

(i) $E(\cdot)$ is continuous with nonempty compact values at λ_0;

(ii) $f_1(\cdot, \cdot, \cdot)$ is lower semicontinuous on $E(\Lambda) \times E(\Lambda) \times \Lambda$.

Then, $S_1(\cdot)$ is l.s.c at λ_0.

Proof For fixed $\lambda_0 \in \Lambda$. Suppose to the contrary that $S_1(\cdot)$ is not l.s.c. at λ_0. Then, there exist a net $\{\lambda_\alpha\}$ with $\lambda_\alpha \to \lambda_0$ and $x_0 \in S_1(\lambda_0)$ such that for any $x_\alpha \in S_1(\lambda_\alpha)$, $x_\alpha \not\to x_0$.

Since $E(\cdot)$ is l.s.c. at λ_0, for $x_0 \in E(\lambda_0)$, there exists $\bar{x}_\alpha \in E(\lambda_\alpha)$ such that $\bar{x}_\alpha \to x_0$. By the contradiction assumption, there must be a net $\{\bar{x}_\beta\}$ of $\{\bar{x}_\alpha\}$ such that $\bar{x}_\beta \notin S_1(\lambda_\beta)$, for all β, i.e., there exists $\bar{y}_\beta \in E(\lambda_\beta)$ such that $f(\bar{x}_\beta, \bar{y}_\beta, \lambda_\beta) \notin S$, namely,

$$f_1(\bar{x}_\beta, \bar{y}_\beta, \lambda_\beta) \leqslant 0. \tag{1}$$

Since $E(\cdot)$ is u.s.c. at λ_0 with compact values, for $\bar{y}_\beta \in E(\lambda_\beta)$, there exists $y_0 \in E(\lambda_0)$ satisfying $\bar{y}_\beta \to y_0$(taking a subnet if necessary). By the condition (ii) and Eq. (1), we have

$$f_1(x_0, y_0, \lambda_0) \leqslant 0,$$

which contradicts $x_0 \in S_1(\lambda_0)$. Thus, our result holds and the proof is complete.

Now we show that, under some suitable conditions, the solution set of the auxiliary problem is dense in the solution set to parametric lexicographic vector equilibrium problem.

Lemma 2.2 Suppose that the following conditions are satisfied:

(i) $E(\lambda_0)$ is a convex set;

(ii) for each $y \in E(\lambda_0)$, $f(\cdot, y.\lambda_0)$ is S_{lex}-concave on $E(\lambda_0)$.

Then, $S_1(\lambda_0) \subseteq S(\lambda_0) \subseteq clS_1(\lambda_0)$.

Proof It is obvious that for each $\lambda \in \Lambda$, $S_1(\lambda) \subseteq S(\lambda)$. Next we claim that $S(\lambda_0) \subseteq clS_1(\lambda_0)$. In fact, let $x_1 \in S_1(\lambda_0)$, $x \in S(\lambda_0)$ and $x_l = (1-l)x + lx_1$ with $l \in (0,1)$, we only need to prove that $x_l \in S_1(\lambda_0)$. By assumptions and the condition (ii), we have that for all $y \in E(\lambda_0)$,

$$f((1-l)x + lx_1, y, \lambda_0) \in (1-l)f(x, y, \lambda_0) + lf(x_1, y, \lambda_0) + S_{\text{lex}}$$

$$\subseteq S + S_{\text{lex}} + S_{\text{lex}} = S.$$

Namely, $x_l \in S_1(\lambda_0)$. Thus, our result holds and the proof is complete.

Now we state lower semicontinuity result of $S(\cdot)$ as follows.

Theorem 2.3 Suppose that the following conditions are satisfied:

(i) $E(\cdot)$ is continuous with nonempty compact convex values at λ_0;

(ii) $f_1(\cdot, \cdot, \cdot)$ is lower semicontinuous on $E(\Lambda) \times E(\Lambda) \times \Lambda$;

(iii) for each $y \in E(\lambda_0)$, $f(\cdot, y, \lambda_0)$ is S_{lex}-concave on $E(\lambda_0)$.

Then, $S(\cdot)$ is l.s.c at λ_0.

Proof By Lemmas 2.1 and 2.2 and Proposition 1.3, for $\lambda_0 \in \Lambda$, we have

$$S(\lambda_0) \subseteq clS_1(\lambda_0) \subseteq \liminf_{\lambda \to \lambda_0} S_1(\lambda) \subseteq \liminf_{\lambda \to \lambda_0} S(\lambda).$$

Then, by Proposition 1.3, $S(\cdot)$ is l.s.c. at λ_0. By the arbitrariness of λ_0, $S(\cdot)$ is l.s.c on Λ. This completes the proof.

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另一方面, 由推论 2.1 可知必然存在一子序列 $\{n_k\}_{k=1}^{+\infty} \subset \{n\}_{n=1}^{+\infty}$, 使得 $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$\lim_{k\to+\infty} n_k \left\{ Y_t\Big(y + z\cdot(B_{t+\frac{1}{n_k}} - B_t), t + \frac{1}{n_k}, g_1\Big) - y \right\} = g_1(t,y,z);$$

$$\lim_{k\to+\infty} n_k \left\{ Y_t\Big(y + z\cdot(B_{t+\frac{1}{n_k}} - B_t), t + \frac{1}{n_k}, g_2\Big) - y \right\} = g_2(t,y,z).$$

于是, 结合上面的几个式子可知

$$g_1(t,y,z) \geqslant g_2(t,y,z), \quad \mathrm{d}P \times \mathrm{d}t - a.e..$$

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