Article ID: 1000-5641(2012)01-0063-11

λ point and λ property in generalized Orlicz spaces with Luxemburg norm

SHI Zhong-rui, ZHAI Jia-yu

(*Department of Mathematics, Shanghai University, Shanghai* 200444, *China*)

Abstract: In this paper, we gave the sufficient and necessary conditions of λ point in a generalized Orlicz function space equipped with the Luxemburg norm, by methods used in classical Orlicz spaces and new methods introduced especially for generalized ones. The results indicate the difference between points in the unit balls of classical and that of generalized Orlicz spaces: all of the classical spaces are λ points, but some of the generalized spaces are not. Finally, we gave the criteria of the λ property and the uniform λ property of generalized spaces.

Key words: generalized Orlicz spaces; Luxemburg norm; λ property; λ points; uniform λ property

CLC number: O177.2 **Document code:** A

DOI: 10.3969/j.issn.1000-5641.2012.01.008

赋 Luxemburg 范数的广义 Orlicz 函数空间中的 λ 点和 λ 性质

石忠锐, 翟佳羽

(上海大学 数学系, 上海 200444)

摘要: 为了研究在更一般情形下的 Orlicz 空间的 λ 性质, 借鉴经典 Orlicz 空间中的方法并发展了广义情形下的新方法, 给出了赋 Luxemburg 范数的广义 Orlicz 空间单位球中的点是 λ 点的充分必要条件. 这些结果表明, 在某些广义 Orlicz 空间中, 并不是所有单位球中的点都是 λ 点, 这与在经典 Orlicz 空间中, 单位球中的点都是 λ 点的结果是不同的. 最后, 给出了具有 λ 性质和一致 λ 性质的赋 Luxemburg 范数广义 Orlicz 空间的充要条件.

关键词: 广义 Orlicz 空间; Luxemburg 范数; λ 性质; λ 点; 一致 λ 性质

收稿日期: 2011-03

基金项目: 国家自然科学基金(10971129); 上海市教委重点学科(J50101); 上海大学研究生创新基金(SHUCX112002)

第一作者: 石忠锐, 男, 博士生导师, 教授, 研究方向为泛函分析, 主要工作集中于空间理论. E-mail: zshi@shu.edu.cn.

第二作者: 翟佳羽, 男, 硕士研究生, 研究方向为泛函分析及空间理论. E-mail: zhaijiayumath@gmail.com

0 Introduction

Aron and Lohman[1] introduced the λ property which for infinite dimensional Banach spaces is more important than the Krein-Milman property, because for Banach spaces with the λ property we have that $\bar{\text{co}}(\text{Ext}\,B(X)) = B(X)$ although $B(X)$ need not to be compact[2]. The λ property of classical Orlicz spaces has been discussed in [3-5]. In this paper, we study the criteria for being λ point in generalized Orlicz function space equipped with the Luxemburg norm, and then the λ property and the uniform λ property of such spaces.

Let $[X, \|\cdot\|]$ be a Banach space, $S(X)$ and $B(X)$ denote the unit sphere and unit ball of X, respectively. A point $x \in S(X)$ is said to be an extreme point of $B(X)$ if x cannot be written as $x = \frac{1}{2}(y+z)$, where y and z are distinct points in $S(X)$. Denote the set of all extreme points of $B(X)$ as Ext $B(X)$. For $x \in B(X)$, we associate the number $\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y, y \in B(X), e \in \text{Ext}\ B(X)\}$, and $\lambda(x) = 0$ if Ext $B(L_{(M)}) = \emptyset$. We call x a λ point if $\lambda(x) > 0$; X to have the λ property if $\lambda(x) > 0$ for all $x \in B(X)$. We call X to have the uniform λ property if $\lambda(X) > 0$, where $\lambda(X) = \inf\{\lambda(x) : x \in B(X)\}$.

Let $\mathbf{R}$ denote the set of all real numbers. A left-continuous function $M : \mathbf{R} \to [0, +\infty]$ is called an Orlicz function if M is convex and even, $M(0) = 0$. For an Orlicz function M, set

$$\alpha = \sup\{u : M(u) = 0\}, \quad \beta = \sup\{u : M(u) < +\infty\}.$$

$u \in \mathbf{R}$ is called a strictly convex point of M, provided $M(u) < \frac{M(u+\varepsilon)+M(u-\varepsilon)}{2}$ for all $\varepsilon > 0$. For $a < b \in \mathbf{R}$, an interval (a, b) is called a structural affine interval (SAI) of M, if M is affine on (a, b) and it is not affine on either $(a-\varepsilon, b)$ or $(a, b+\varepsilon)$ for all $\varepsilon > 0$; for $a \in \mathbf{R}$, an interval $(a, +\infty)$ is called an infinite structural affine interval of M if M is affine on $(a, +\infty)$ and it is not affine on $(a-\varepsilon, +\infty)$ for all $\varepsilon > 0$. Let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be all structural affine intervals of M, and then denote $SC_M = \mathbf{R} \setminus \Big[\bigcup\limits_{i=1}^{\infty}(a_i, b_i)\Big]$.

Let (G, Σ, μ) be a non-atomic finite measurable space. For $u(t)$ a measurable function on G, its modular is defined by $\rho_M(u) = \int_G M(u(t))\mathrm{d}t$. The generalized Orlicz space $L_{(M)}$ is constructed as

$$L_{(M)} = \{u : \exists k > 0, \rho_M(ku) < \infty\},$$

equipped with the Luxemburg norm

$$\|u\|_{(M)} = \inf\{k > 0 : \rho_M(\frac{u}{k}) \leqslant 1\}.$$

For more details, please refer to [6]. In order to avoid trivial cases, we assume that there exist $u_1, u_2 > 0$ such that $M(u_1) > 0$ and $M(u_2) < \infty$.

1 Main results

For the convenience of reading, we present some auxiliary lemmas.

Lemma 1.1 For an Orlicz function M,

(1) when $M(\beta)\mu G > 1, u \in \text{Ext}B(L_{(M)}) \Leftrightarrow$

(a) $\rho_M(u) = 1$, and (b) $\mu\{t \in G : u(t) \notin SC_M\} = 0$;

(2) when $M(\beta)\mu G \leqslant 1, u \in \mathrm{Ext}B(L_{(M)}) \Leftrightarrow |u(t)| = \beta$ a.e. on G.

Proof Referring to [7], we can get the lemma.

Remark 1.1 Take $\widetilde{\mathrm{sign}}(u(t)) = \begin{cases} 1, & u(t) \geqslant 0 \\ -1, & u(t) < 0 \end{cases}$ for any $u(t)$. Then

$$u(t) = \lambda e(t) + (1-\lambda)v(t) \Leftrightarrow |u(t)| = u(t)\widetilde{\mathrm{sign}}(u(t)) = \lambda e(t)\widetilde{\mathrm{sign}}(u(t)) + (1-\lambda)v(t)\widetilde{\mathrm{sign}}(u(t)).$$

Since $e \in \mathrm{Ext}\ B(L_{(M)}) \Leftrightarrow e \cdot \varepsilon \in \mathrm{Ext}\ B(L_{(M)})$ by Lemma 1.1, where $|\varepsilon(t)| = 1$. Thus $\lambda(u) = \lambda(|u|)$ and without loss of generality, we assume $u(t) \geqslant 0$ in the following.

Lemma 1.2[6] Let $\mathrm{Ext}\ B(X) \neq \emptyset$. If $x, y, z \in B(X)$ and $x = \alpha y + (1-\alpha)z$ for some $\alpha \in (0,1)$, then $\lambda(x) \geqslant \alpha\lambda(y)$. Consequently, $\lambda(0) = \frac{1}{2}$ and

$$\lambda(x) \geqslant \max\left\{\frac{1}{2}(1-\|x\|), \lambda\Big(\frac{x}{\|x\|}\Big)\|x\|\right\},\ (x(\neq 0) \in B(X)).$$

Remark 1.2 Since $\lambda(x) = 1$ whenever $x \in \mathrm{Ext}\ B(X)$, and by Lemma 1.2 $\lambda(x) \geqslant \frac{1}{4}\lambda\big(\frac{x}{\|x\|}\big) > 0$, we only need to discuss $x \in S(X) \setminus \mathrm{Ext}\ B(X)$ in the following.

Remark 1.3 For any u, define

$$v(t) = \begin{cases} \alpha, & 0 \leqslant u(t) < \alpha, \\ u(t), & u(t) \geqslant \alpha, \end{cases} \quad \text{and} \quad w(t) = \begin{cases} 2u(t) - \alpha, & 0 \leqslant u(t) < \alpha, \\ u(t), & u(t) \geqslant \alpha. \end{cases}$$

Then $\mu\{t : v(t) < \alpha\} = 0$, and $u = \frac{1}{2}(v+w)$ which implies $\lambda(u) \geqslant \frac{1}{2}\lambda(v)$ by Lemma 1.2. So we assume $\mu\{t : u(t) < \alpha\} = 0$ in the following.

In the following, we denote $\{(a_i, b_i)\}_i$ the set of all finite structural affine intervals of M except $(-\alpha, \alpha)$.

Lemma 1.3 If $\beta = +\infty$ and M has no infinite SAI, then $\lambda(u) > 0$ for any $u \in B(L_{(M)})$.

Proof We can prove this by similar arguments as that in classical Orlicz spaces in [6].

Lemma 1.4 If $\beta = +\infty$ and $(a, +\infty)$ is a SAI with $M(a)\mu G < 1$, then $\mathrm{Ext}\ B(L_{(M)}) = \emptyset$.

Proof For $u \in B(L_{(M)})$, if $\mu\{t : u(t) \notin SC_M\} = 0$, then $u(t) \leqslant a$ a.e. on G. Thus $\rho_M(u) \leqslant M(a)\mu G < 1$, and $u \notin \mathrm{Ext}\ B(L_{(M)})$ by Lemma 1.1(1). So $\mathrm{Ext}\ B(L_{(M)}) = \emptyset$.

Lemma 1.5 If $\beta = +\infty$ and $(a, +\infty)$ is a SAI with $M(a)\mu G \geqslant 1$. Then for any u with $\rho_M(u) = 1$, $\lambda(u) > 0 \Leftrightarrow$

(1) $[\sum\limits_i \int_{\{t:|u(t)| \in (a_i,b_i)\}} + \int_{\{t:|u(t)|>a\}}] M(u(t))\mathrm{d}t < \sum\limits_i \int_{\{t:|u(t)| \in (a_i,b_i)\}} M(b_i)\mathrm{d}t + \int_{\{t:|u(t)|>a\}} \cdot M(a)\mathrm{d}t$, or

(2) $[\sum\limits_i \int_{\{t:|u(t)| \in (a_i,b_i)\}} + \int_{\{t:|u(t)|>a\}}] M(u(t))\mathrm{d}t = \sum\limits_i \int_{\{t:|u(t)| \in (a_i,b_i)\}} M(b_i)\mathrm{d}t + \int_{\{t:|u(t)|>a\}} \cdot M(a)\mathrm{d}t$,

and there exists some $\varepsilon_0 > 0$ such that $\mu\{t : |u(t)| \in (a_i, b_i), \frac{|u(t)|-a_i}{b_i-a_i} < \varepsilon_0\} = 0, i = 1, 2, \cdots$

Proof Since $M(a)\mu G \geqslant 1$, take $E \subset G$ with $M(a)\mu E = 1$ and $u_1(t) = \begin{cases} a, & t \in E, \\ \alpha, & t \in G \setminus E. \end{cases}$ Then $\rho_M(u_1) = 1$ and $\mu\{t : u_1(t) \notin SC_M\} = 0$, which implies $u_1 \in \mathrm{Ext}\,B(L_{(M)})$ by Lemma 1.1(1), which means $\mathrm{Ext}\,B(L_{(M)}) \neq \emptyset$.

"Necessity": For u with $\rho_M(u) = 1$, if $u = \lambda v + (1-\lambda)w$ where $\lambda \in [0,1]$, $v \in \mathrm{Ext}B(L_{(M)}), w \in B(L_{(M)})$, $1 = \rho_M(u) \leqslant \lambda\rho_M(v) + (1-\lambda)\rho_M(w) = \lambda + (1-\lambda)\rho_M(w) \leqslant 1$

which follows that $\rho_M(w)=1$. Hence, $M(u(t))=\lambda M(v(t))+(1-\lambda)M(w(t))$ a.e. on G. It implies that for almost $t\in G$, $u(t),v(t),w(t)$ are either equal to each other or in the same SAI. Now we prove the necessity in two steps:

$$(1)\ \ [\sum_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}+\int_{\{t:|u(t)|>a\}}]M(u(t))\mathrm{d}t\leqslant\sum_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}M(b_i)\mathrm{d}t+\int_{\{t:|u(t)|>a\}}M(a)\mathrm{d}t.$$

Otherwise, suppose $\sum\limits_i\int_{\{t:u(t)\in(a_i,b_i)\}}[M(b_i)-M(u(t))]\mathrm{d}t<\int_{\{t:u(t)>a\}}[M(u(t))-M(a)]\mathrm{d}t$. Since $v\in\mathrm{Ext}\,B(L_{(M)})$, $\mu\{t:v(t)\notin SC_M\}=0$. A contradiction

$$\begin{aligned}1&=\rho_M(u)\\&=\int_{\{t:u(t)\in SC_M\}}M(u(t))\mathrm{d}t+\sum_i\int_{\{t:u(t)\in(a_i,b_i)\}}M(u(t))\mathrm{d}t+\int_{\{t:u(t)>a\}}M(u(t))\mathrm{d}t\\&>\int_{\{t:u(t)\in SC_M\}}M(u(t))\mathrm{d}t+\sum_i\int_{\{t:u(t)\in(a_i,b_i)\}}M(b_i)\mathrm{d}t+\int_{\{t:u(t)>a\}}M(a)\mathrm{d}t\\&\geqslant\rho_M(v)=1.\end{aligned}$$

(2) When $[\sum\limits_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}+\int_{\{t:|u(t)|>a\}}]M(u(t))\mathrm{d}t=\sum\limits_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}M(b_i)\mathrm{d}t+\int_{\{t:|u(t)|>a\}}M(a)\mathrm{d}t$, there exists some $\varepsilon_0>0$ such that $\mu\{t:|u(t)|\in(a_i,b_i),\frac{|u(t)|-a_i}{b_i-a_i}<\varepsilon_0\}=0,i=1,2,\cdots$

Otherwise, for any $\varepsilon>0$, there exist some $B_i\subset\{t:u(t)\in(a_i,b_i)\}$ with $\mu B_i>0$ satisfying $\frac{u(t)-a_i}{b_i-a_i}<\varepsilon$ for $t\in B_i$. So

$$\begin{aligned}1&=\rho_M(u)=\int_{\{t:u(t)\in SC_M\}}M(u(t))\mathrm{d}t+\sum_i\int_{\{t:u(t)\in(a_i,b_i)\}}M(u(t))\mathrm{d}t+\int_{\{t:u(t)>a\}}M(u(t))\mathrm{d}t\\&=\int_{\{t:u(t)\in SC_M\}}M(v(t))\mathrm{d}t+\sum_i\int_{\{t:u(t)\in(a_i,b_i)\}}M(b_i)\mathrm{d}t+\int_{\{t:u(t)>a\}}M(a)\mathrm{d}t,\\1&=\rho_M(v)\\&=\int_{\{t:u(t)\in SC_M\}}M(v(t))\mathrm{d}t+\sum_i\int_{\{t:v(t)=a_i\}}M(a_i)\mathrm{d}t+\sum_i\int_{\{t:v(t)=b_i\}}M(b_i)\mathrm{d}t\\&\quad+\int_{\{t:u(t)>a\}}M(a)\mathrm{d}t.\end{aligned}$$

Hence, we have that for all i,

$$\int_{\{t:u(t)\in(a_i,b_i)\}}M(b_i)\mathrm{d}t=\int_{\{t:v(t)=a_i\}}M(a_i)\mathrm{d}t+\int_{\{t:v(t)=b_i\}}M(b_i)\mathrm{d}t.$$

From $\{t:u(t)\in(a_i,b_i)\}=\{t:v(t)=a_i\}\bigcup\{t:v(t)=b_i\}$, we get $v(t)=b_i$ a.e. on $\{t:u(t)\in(a_i,b_i)\}$. From $w(t)=\frac{1}{1-\lambda}(u(t)+\lambda b_i)\in(a_i,b_i)$, we have $\lambda<\frac{u(t)-a_i}{b_i-a_i}<\varepsilon$ $(t\in B_i)$. It reaches a contradiction that $\lambda(u)=0$ by the arbitrariness of ε.

"*Sufficiency*": Since $u\notin\mathrm{Ext}\,B(L_{(M)})$ and $\rho_M(u)=1$, $\mu\{t:u(t)\notin SC_M\}>0$ by Lemma 1.1(1).

In case (1), for $\lambda \in (0,1)$, define

$$v_\lambda(t) = \begin{cases} a_i, & a_i < u(t) \leqslant \lambda a_i + (1-\lambda)b_i, i = 1,2,\cdots \\ b_i, & b_i > u(t) > \lambda a_i + (1-\lambda)b_i, i = 1,2,\cdots \\ a, & u(t) > a, \text{ for SAI } (a,+\infty), \\ u(t), & \text{others.} \end{cases}$$

$$\hat{v}_\lambda(t) = \begin{cases} a_i, & a_i < u(t) < \lambda a_i + (1-\lambda)b_i, i = 1,2,\cdots \\ b_i, & b_i > u(t) \geqslant \lambda a_i + (1-\lambda)b_i, i = 1,2,\cdots \\ a, & u(t) > a, \text{for SAI } (a,+\infty), \\ u(t), & \text{others.} \end{cases}$$

$$v_0(t) = \hat{v}_0(t) = \begin{cases} a_i, & a_i < u(t) < b_i, i = 1,2,\cdots \\ a, & u(t) > a, \text{for SAI } (a,+\infty), \\ u(t), & \text{others.} \end{cases}$$

$$v_1(t) = \hat{v}_1(t) = \begin{cases} b_i, & a_i < u(t) < b_i, i = 1,2,\cdots \\ a, & u(t) > a, \text{for SAI } (a,+\infty), \\ u(t), & \text{others.} \end{cases}$$

Then $\hat{v}_\lambda(t) \geqslant v_\lambda(t)$. From the condition(I), $\rho_M(v_1) = \rho_M(\hat{v}_1) > \rho_M(u) = 1$. By the same argument as in [6], we have that $\rho_M(\hat{v}_0) = \rho_M(v_0) < \rho_M(u) = 1$ and that $\rho_M(\hat{v}_\lambda)$ is right-continuous with respect to λ whereas $\rho_M(v_\lambda)$ is left-continuous to λ.

Set $\sigma = \sup\{\lambda : \rho_M(v_\lambda) \leqslant 1\}$ and $\hat{\sigma} = \sup\{\lambda : \rho_M(\hat{v}_\lambda) \leqslant 1\}$. Then from the left-continuity of $\rho_M(v_\lambda)$ and the right-continuity of $\rho_M(\hat{v}_\lambda)$, we see $1 > \sigma \geqslant \hat{\sigma} > 0$ and $\rho_M(v_\sigma) \leqslant 1 \leqslant \rho_M(\hat{v}_{\hat{\sigma}})$ by referring to [6].

If $\sigma = \hat{\sigma}$, from $\rho_M(v_\sigma) \leqslant 1 \leqslant \rho_M(\hat{v}_\sigma)$ and since G is a non-atomic finite measurable space, take $E_i \subset \{t : u(t) = \sigma a_i + (1-\sigma)b_i\}$ such that $\rho_M(v) = 1$, where

$$v(t) = \begin{cases} a_i, & a_i < u(t) < \sigma a_i + (1-\sigma)b_i \text{ or } t \in E_i, i = 1,2,\cdots \\ b_i, & b_i > u(t) > \sigma a_i + (1-\sigma)b_i \text{ or } t \in \{t : u(t) = \sigma a_i + (1-\sigma)b_i\} \setminus E_i, i = 1,2,\cdots \\ a, & u(t) > a, \text{ for SAI } (a,+\infty), \\ u(t), & \text{others.} \end{cases}$$

It follows that $\mu\{t : v(t) \notin SC_M\} = 0$. So $v \in \text{Ext } B(L_{(M)})$. When $\sigma \geqslant \frac{1}{2}$, set $w = \frac{1}{\sigma}[u-(1-\sigma)v]$ (i.e.$(1-\sigma)v + \sigma w = u$), which implies that $u(t), v(t), w(t)$ are either equal to each other or in the same SAI by referring to [6]. And then

$$1 = \rho_M(u) = (1-\sigma)\rho_M(v) + \sigma\rho_M(w) = \rho_M(u) = 1 - \sigma + \sigma\rho_M(w),$$

which implies $\rho_M(w) = 1$. Therefore $\lambda(u) \geqslant 1 - \sigma > 0$ by Lemma 1.2. When $\sigma < \frac{1}{2}$, set $w = \frac{1}{1-\sigma}(u - \sigma v)$. We can deduce $\lambda(u) \geqslant \sigma > 0$ by replacing σ above by $1-\sigma$.

If $\sigma > \hat{\sigma}$, for any $\lambda \in (\hat{\sigma}, \sigma)$, from the definition of $\hat{\sigma}$ and σ, we get $\rho_M(v_\lambda) \leqslant 1 < \rho_M(\hat{v}_\lambda)$. Replacing the σ by λ, by the same arguments to the case of $\sigma = \hat{\sigma}$, we have $\lambda(u) > 0$.

In case (2), take $v = v_1$ as above. Then $\rho_M(v) = 1$ and $\mu\{t : v(t) \notin SC_M\} = 0$. Therefore $v \in \text{Ext } B(L_{(M)})$. Take $w = \frac{1}{1-\lambda}(u - \lambda v)$, $\lambda \in (0, \min\{\varepsilon_0, 1\})$. We see that when

$u(t) \in (a_i, b_i)$, $v(t) = b_i$, $w(t) = \frac{1}{1-\lambda}[u(t) - \lambda b_i] < b_i$, and $\frac{u(t)-w(t)}{b_i-w(t)} = \lambda < \varepsilon_0 \leqslant \frac{u(t)-a_i}{b_i-a_i}$ which implies $w(t) \geqslant a_i$. So $u(t), v(t), w(t)$ are either equal to each other or in the same SAI. Thus $1 = \rho_M(u) = \lambda\rho_M(v) + (1-\lambda)\rho_M(w) = \lambda + (1-\lambda)\rho_M(w)$, and then $\rho_M(w) = 1$. Consequently, $\lambda(u) \geqslant \lambda > 0$ by Lemma 1.2.

Lemma 1.6 If $\beta < +\infty$, $M(\beta)\mu G > 1$, then for any u with $\rho_M(u) = 1$, $\lambda(u) > 0$.

Proof For $u \notin \operatorname{Ext} B(L_{(M)})$ with $\rho_M(u) = 1$, $\mu\{t : u(t) \notin SC_M\} > 0$ by Lemma 1.1(1). Moreover, $|u(t)| \leqslant \beta$ a.e. on G. For $\lambda \in (0,1)$, set

$$v_\lambda(t) = \begin{cases} a_i, & a_i < u(t) \leqslant \lambda a_i + (1-\lambda)b_i, i = 1, 2, \cdots \\ b_i, & b_i > u(t) > \lambda a_i + (1-\lambda)b_i, i = 1, 2, \cdots \\ u(t), & \text{others.} \end{cases}$$

$$\hat{v}_\lambda(t) = \begin{cases} a_i, & a_i < u(t) < \lambda a_i + (1-\lambda)b_i, i = 1, 2, \cdots \\ b_i, & b_i > u(t) \geqslant \lambda a_i + (1-\lambda)b_i, i = 1, 2, \cdots \\ u(t), & \text{others.} \end{cases}$$

$$v_0(t) = \hat{v}_0(t) = \begin{cases} a_i, & a_i < u(t) < b_i, i = 1, 2, \cdots \\ u(t), & \text{others.} \end{cases}$$

$$v_1(t) = \hat{v}_1(t) = \begin{cases} b_i, & a_i < u(t) < b_i, i = 1, 2, \cdots \\ u(t), & \text{others.} \end{cases}$$

Repeating the arguments in (I) of the sufficiency's proof of Lemma 1.5, we have $\lambda(u) > 0$.

Lemma 1.7 If $\beta < +\infty$, $M(\beta) = +\infty$, $\alpha > 0$, then for any $u \in B(L_{(M)})$, $\lambda(u) > 0$.

Proof When $\rho_M(u) = 1$, we have $\lambda(u) > 0$ by Lemma 1.6. When $\rho_M(u) < 1$, from $M(\beta) = +\infty$, we see $|u(t)| < \beta$ a.e. on G. Take $\varepsilon \in (0, \beta - \alpha)$ such that $M(\beta-\varepsilon)\mu\{t : u(t) \leqslant \beta - \varepsilon\} > 1$. Denote

$$f(h) = \int_{\{t:u(t)\geqslant h\}} M(u(t))\mathrm{d}t, \quad \hat{f}(h) = \int_{\{t:u(t)>h\}} M(u(t))\mathrm{d}t,$$

$$g(h) = \int_{\{t:u(t)<h\}} M(\beta-\varepsilon)\mathrm{d}t, \quad \hat{g}(h) = \int_{\{t:u(t)\leqslant h\}} M(\beta-\varepsilon)\mathrm{d}t.$$

Referring to [6], we get that $f(h)$ and $g(h)$ are left-continuous on $(0, +\infty)$, whereas $\hat{f}(h)$ and $\hat{g}(h)$ are right-continuous on $[0, +\infty)$. For $h \geqslant \beta - \varepsilon$, $\hat{g}(h) > 1$ and $f(h) + g(h) > 1$; for $h < \alpha$, $f(h) + g(h) = \hat{f}(h) + \hat{g}(h) = \rho_M(u) < 1$; for $h \leqslant \beta - \varepsilon$, $f(h) + g(h) \leqslant \hat{f}(h) + \hat{g}(h)$.

Set $H = \sup\{h : f(h) + g(h) \leqslant 1\}$ and $\widehat{H} = \sup\{h : \hat{f}(h) + \hat{g}(h) \leqslant 1\}$. Then from the left-continuity of f, g and the right-continuity of $\hat{f}, \hat{g}$, we see $\alpha < \widehat{H} \leqslant H < \beta - \varepsilon$ and $f(H) + g(H) \leqslant 1 \leqslant \hat{f}(\widehat{H}) + \hat{g}(\widehat{H})$.

If $H = \widehat{H}$, from $f(H) + g(H) \leqslant 1 \leqslant \hat{f}(\widehat{H}) + \hat{g}(\widehat{H})$, take $E \subset \{t : u(t) = H\}$ such that $\rho_M(v) = 1$, where

$$v(t) = \begin{cases} \beta - \varepsilon, & \alpha \leqslant u(t) < H, \text{ or } t \in E, \\ u(t), & u(t) > H \text{ or } t \in \{t : u(t) = H\} \setminus E. \end{cases}$$

Take $\lambda \in (\frac{\alpha}{2\beta}, \frac{\alpha}{\beta-\varepsilon})$ which implies $\lambda(\beta-\varepsilon) < \alpha$, and set $w(t) = \frac{1}{1-\lambda}(u(t) - \lambda v(t))$ (i.e. $u = \lambda v + (1-\lambda) w$). When $v(t) = \beta-\varepsilon, \alpha \leqslant u(t) \leqslant H < \beta-\varepsilon$ and $0 \leqslant w(t) = \beta-\varepsilon+\frac{u(t)-(\beta-\varepsilon)}{1-\lambda} < \beta-\varepsilon$. It follows $\rho_M(w) \leqslant \rho_M(v) = 1$. Therefore, $\lambda(v) > \frac{\alpha}{2\beta}$ by Lemma 1.6, and $\lambda(u) \geqslant \lambda \cdot \lambda(v) > \frac{\alpha}{2\beta}\lambda(v) > 0$ by Lemma 1.2.

If $H > \widehat{H}$, for any $h \in (\widehat{H}, H)$, $f(h) + g(h) \leqslant 1 < \hat{f}(h) + \hat{g}(h)$. Replacing the H above by h, we obtain $\lambda(u) > \frac{\alpha}{2\beta}\lambda(v) > 0$.

Lemma 1.8 If $\beta < +\infty$, $M(\beta) = +\infty$, $\alpha = 0$, then for any $u \in B(L_{(M)})$, $\lambda(u) > 0$.

Proof For $u \in S(L(M))$, when $\rho_M(u) = 1$, we have $\lambda(u) > 0$ by Lemma 1.6. When $\rho_M(u) < 1$, firstly, $\mu\{t : u(t) \geqslant \frac{1}{2}\beta\} > 0$. Otherwise, $u(t) < \frac{1}{2}\beta$ a.e. on G, and then $\rho_M(\frac{3u}{2}) \leqslant M(\frac{3\beta}{4})\mu G < +\infty$. Combining $\rho_M(u) < 1$, a contradiction with that $u \in S(L_{(M)})$. From $M(\beta) = +\infty$, take $\varepsilon > 0$ with $M(\beta-\varepsilon)\mu\{t : \frac{1}{2}\beta \leqslant u(t) \leqslant \beta-\varepsilon\} > 1$. It follows $M(\beta-\varepsilon)\mu\{t : \frac{1}{2}(\beta-\varepsilon) \leqslant u(t) \leqslant \beta-\varepsilon\} > 1$. Denote

$$f(h) = \int_{\{t:u(t)\geqslant h\}} M(u(t))\mathrm{d}t,\ \hat{f}(h) = \int_{\{t:u(t)>h\}} M(u(t))\mathrm{d}t,$$

$$g(h) = \begin{cases} \int_{\{t:u(t)<h\}} M(2u(t))\mathrm{d}t, & h \leqslant \frac{1}{2}(\beta-\varepsilon), \\ \int_{\{t:u(t)<\frac{1}{2}(\beta-\varepsilon)\}} M(2u(t))\mathrm{d}t + \int_{\{t:\frac{1}{2}(\beta-\varepsilon)\leqslant u(t)<h\}} M(\beta-\varepsilon)\mathrm{d}t, & h > \frac{1}{2}(\beta-\varepsilon), \end{cases}$$

$$\hat{g}(h) = \begin{cases} \int_{\{t:u(t)\leqslant h\}} M(2u(t))\mathrm{d}t, & h \leqslant \frac{1}{2}(\beta-\varepsilon), \\ \int_{\{t:u(t)<\frac{1}{2}(\beta-\varepsilon)\}} M(2u(t))\mathrm{d}t + \int_{\{t:\frac{1}{2}(\beta-\varepsilon)\leqslant u(t)\leqslant h\}} M(\beta-\varepsilon)\mathrm{d}t, & h > \frac{1}{2}(\beta-\varepsilon). \end{cases}$$

Referring to [6], we can obtain that f and g are left-continuous on $(0, +\infty)$, whereas $\hat{f}$ and $\hat{g}$ are right-continuous on $[0, +\infty)$; and that $f(0) + g(0) = \hat{f}(0) + \hat{g}(0) = \rho_M(u) < 1$, $\hat{g}(h) > 1$ and $f(h) + g(h) > 1$ for $h \geqslant \beta-\varepsilon$. For $h \leqslant \beta-\varepsilon$, $f(h) + g(h) \leqslant \hat{f}(h) + \hat{g}(h)$.

Set $H = \sup\{h : f(h) + g(h) \leqslant 1\}$ and $\widehat{H} = \sup\{h : \hat{f}(h) + \hat{g}(h) \leqslant 1\}$. Then $0 < \widehat{H} \leqslant H < \beta-\varepsilon$ and $f(H) + g(H) \leqslant 1 \leqslant \hat{f}(\widehat{H}) + \hat{g}(\widehat{H})$.

If $H = \widehat{H} \leqslant \frac{1}{2}(\beta-\varepsilon)$, from $f(H) + g(H) \leqslant 1 \leqslant \hat{f}(\widehat{H}) + \hat{g}(\widehat{H})$, take $E \subset \{t : u(t) = H\}$ such that $\rho_M(v) = 1$, where

$$v(t) = \begin{cases} 2u(t), & u(t) < H,\ \text{or}\ t \in E, \\ u(t), & u(t) > H,\ \text{or}\ t \in \{t : u(t) = H\} \setminus E. \end{cases}$$

Define

$$w(t) = \begin{cases} 0, & u(t) < H,\ \text{or}\ t \in E, \\ u(t), & u(t) > H,\ \text{or}\ t \in \{t : u(t) = H\} \setminus E. \end{cases}$$

then $u = \frac{1}{2}v + \frac{1}{2}w$ and $\rho_M(w) \leqslant \rho_M(v) = 1$. Thus, $\lambda(v) > 0$ by Lemma 1.6 and $\lambda(u) \geqslant \frac{1}{2}\lambda(v) > 0$ by Lemma 1.2.

If $H = \widehat{H} > \frac{1}{2}(\beta-\varepsilon)$, from $f(H) + g(H) \leqslant 1 \leqslant \hat{f}(\widehat{H}) + \hat{g}(\widehat{H})$, take $E \subset \{t : u(t) = H\}$, such that $\rho_M(v) = 1$, where

$$v(t) = \begin{cases} 2u(t), & u(t) < \frac{1}{2}(\beta-\varepsilon), \\ \beta-\varepsilon, & \frac{1}{2}(\beta-\varepsilon) \leqslant u(t) < H,\ \text{or}\ t \in E, \\ u(t), & u(t) > H\ \text{or}\ t \in \{t : u(t) = H\} \setminus E. \end{cases}$$

Take $w(t)=2u(t)-v(t)$ (i.e. $u=\frac{1}{2}v+\frac{1}{2}w$). Then $\rho_M(w)\leqslant\rho_M(v)=1$. It follows $\lambda(v)>0$ by Lemma 1.6, and $\lambda(u)\geqslant\frac{1}{2}\lambda(v)>0$ by Lemma 1.2.

If $H>\widehat{H}$, for any $h\in(\widehat{H},H)$, $f(h)+g(h)\leqslant 1<\hat{f}(h)+\hat{g}(h)$. Replacing the H by h and repeating the same arguments, we have $\lambda(u)\geqslant\frac{1}{2}\lambda(v)>0$.

Lemma 1.9 If $\beta<+\infty$, $M(\beta)<+\infty$, $M(\beta)\mu G>1$, $\alpha>0$, then for any $u\in B(L_{(M)})$, $\lambda(u)>0$.

Proof When $\rho_M(u)=1$, $\lambda(u)>0$ by Lemma 1.6. When $\rho_M(u)<1$, noticing $M(\beta)<+\infty$ and $M(\beta)\mu G>1$, replace the $\beta-\varepsilon$ in the proof of Lemma 1.7 by the β here and repeat the argument of Lemma 1.7. Then $\lambda(u)>\frac{\alpha}{2\beta}\lambda(v)>0$.

Lemma 1.10 If $\beta<+\infty$, $M(\beta)<+\infty$, $M(\beta)\mu G>1$, $\alpha=0$, then for any $u\in B(L_{(M)})$, $\lambda(u)>0$.

Proof When $\rho_M(u)=1$, we have $\lambda(u)>0$ by Lemma 1.6. When $\rho_M(u)<1$, from $M(\beta)<+\infty$, we have $\mu\{t:u(t)\leqslant\beta\}=\mu G$. We discuss in the two cases as follows:

(1) In the case of $M(\beta)\mu\{t:0<u(t)\leqslant\beta\}>1$, there exists one $\lambda\in(0,1)$, such that $M(\beta)\{t:\lambda\beta\leqslant u(t)\leqslant\beta\}>1$. Replacing the $\beta-\varepsilon$ in the proof of Lemma 1.8 by the β here, we can prove $\lambda(u)>0$ by the same arguments as in Lemma 1.8.

(2) In the case of $M(\beta)\mu\{t:0<u(t)\leqslant\beta\}\leqslant 1$, combining $M(\beta)\mu\{t:0\leqslant u(t)\leqslant\beta\}>1$, we deduce $\mu\{t:u(t)=0\}>0$. Pick $\delta\in(0,\beta)$ with $M(\delta)<\frac{1-\rho_M(u)}{\mu\{t:u(t)=0\}}$, and set

$$v(t)=\begin{cases}\delta, & u(t)=0,\\ u(t), & u(t)>0,\end{cases}\quad\text{and } w(t)=\begin{cases}-\delta, & u(t)=0,\\ u(t), & u(t)>0.\end{cases}$$

So, $u=\frac{1}{2}v+\frac{1}{2}w$, $\rho(v)=\rho(w)=M(\delta)\mu\{t:u(t)=0\}+\rho(u)<1$, and $M(\beta)\mu\{t:0<v(t)\leqslant\beta\}=M(\beta)\mu\{t:u(t)\leqslant\beta\}>1$. Hence by (I), $\lambda(v)>0$; further $\lambda(u)\geqslant\frac{1}{2}\lambda(v)>0$ by Lemma 1.2.

Lemma 1.11 If $M(\beta)\mu G\leqslant 1$, then $L_{(M)}$ has the uniform λ property.

Proof For any $u\in B(L_{(M)})$, then $\rho_M(u)\leqslant 1$, $u(t)\leqslant\beta$ a.e. on G. Set $v(t)=\beta$, $w(t)=2u(t)-\beta$. Then $v\in\operatorname{Ext}B(L_{(M)})$ by lemma 1.1(2) and $|w(t)|\leqslant\beta$. From $u=\frac{1}{2}v+\frac{1}{2}w$, and $\rho_M(w)\leqslant M(\beta)\mu G\leqslant 1$ which implies $\|w\|_{(M)}\leqslant 1$, we see $\lambda(u)\geqslant\frac{1}{2}\lambda(v)=\frac{1}{2}$ by Lemma 1.2. This implies $\lambda(L_{(M)})=\inf\{\lambda(u):u\in B(L_{(M)})\}\geqslant\frac{1}{2}$, so $L_{(M)}$ has the uniform λ property.

Theorem 1.1 If M is an Orlicz function, then for any $u\in B(L_{(M)})$,

(1) if $\beta<+\infty$, or $\beta=+\infty$ and M has no infinite SAI, then $\lambda(u)>0$;

(2) if $\beta=+\infty$, $(a,+\infty)$ is a SAI with $M(a)\mu G<1$, then $\operatorname{Ext}B(L_{(M)})=\emptyset$;

(3) if $\beta=+\infty$, $(a,+\infty)$ is a SAI with $M(a)\mu G\geqslant 1$, then $\lambda(u)>0\Leftrightarrow$.

(a) $\rho_M(u)<1$; or

(b) $\rho_M(u)=1$, and $[\sum\limits_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}+\int_{\{t:|u(t)|>a\}}]M(u(t))\mathrm{d}t<\sum\limits_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}\cdot M(b_i)\mathrm{d}t+\int_{\{t:|u(t)|>a\}}M(a)\mathrm{d}t$; or

(c) $\rho_M(u)=1$, and $[\sum\limits_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}+\int_{\{t:|u(t)|>a\}}]M(u(t))\mathrm{d}t=\sum\limits_i\int_{\{t:|u(t)|\in(a_i,b_i)\}}\cdot M(b_i)\mathrm{d}t+\int_{\{t:|u(t)|>a\}}M(a)\mathrm{d}t$, and there exists some $\varepsilon_0>0$ such that $\mu\{t:|u(t)|\in(a_i,b_i),\frac{|u(t)|-a_i}{b_i-a_i}<\varepsilon_0\}=0, i=1,2,\cdots$

Proof (1) It is followed by the conclusions of Lemmas 1.3, 1.7, 1.8, 1.9, 1.10, 1.11;

(2) It is followed by Lemma 1.4;

(3) *"Sufficiency"*: For this M, $M \in \Delta_2$, thus $\|u\|_{(M)} = 1 \Leftrightarrow \rho_M(u) = 1$.

In the case of (a), $\rho_M(u) < 1$ implies $\|u\|_{(M)} < 1$. Then $\lambda(u) > 0$ by Lemma 1.2.

In the case of (b) or (c), since $\rho_M(u) = 1$, $\lambda(u) > 0$ by Lemma 1.5.

"Necessity": If $\rho_M(u) < 1$, (a) is true; if not, i.e. $\rho_M(u) = 1$, it is followed by the necessity of Lemma 1.5 that either (b) or (c) is ture.

Theorem 1.2 $L_{(M)}$ has the λ property if and only if

(1) $\beta < +\infty$, or

(2) $\beta = +\infty$ and M has no infinite SAI.

Proof By Theorem 1.1(1), we see the sufficiency. On the other hand, suppose that both (1) and (2) are not true, $\beta = +\infty$ and $(a, +\infty)$ is a SAI for some $a > 0$. We consider in the two cases as follows.

In the case of $M(a)\mu G < 1$, $\operatorname{Ext} B(L_{(M)}) = \emptyset$ by Theorem 1.1(2). Thus $L_{(M)}$ does not have the λ property.

In the case of $M(a)\mu G \geqslant 1$, set $E \subset G$ with $M(a+1)\mu E = 1$, and $u(t) = \begin{cases} a+1, & t \in E, \\ 0, & t \in G \setminus E. \end{cases}$
Then $\rho_M(u) = 1$. From $a > \alpha$, we have $\Big[\sum\limits_i \int_{\{t:|u(t)| \in (a_i, b_i)\}} + \int_{\{t:|u(t)|>a\}}\Big] M(u(t))\mathrm{d}t = M(a+1)\mu E > M(a)\mu E = \sum\limits_i \int_{\{t:|u(t)| \in (a_i, b_i)\}} M(b_i)\mathrm{d}t + \int_{\{t:|u(t)|>a\}} M(a)\mathrm{d}t$.

Therefore $\lambda(u) = 0$ by Theorem 1.1(3). So we reach a contradiction that $L_{(M)}$ has the λ property.

If Orlicz function M satisfies, as in [6], that $\alpha = 0$, $\beta = +\infty$, $\lim\limits_{u \to 0} \frac{M(u)}{u} = 0$, and $\lim\limits_{u \to +\infty} \frac{M(u)}{u} = +\infty$ which implies that M has no infinite SAI. By Theorem 1.2, we get the following corollary which has been proved in [6].

Corollary 1.1 $L_{(M)}$ always has the λ property.

Theorem 1.3 $L_{(M)}$ has the uniform λ property if and only if

(1) $M(\beta)\mu G \leqslant 1$, or (2) $M(\beta)\mu G > 1$ and M is strictly convex on (α, β).

Proof *"Necessity"*: Suppose that $M(\beta)\mu G > 1$ and that M is not strictly convex on (α, β). Then there exists some SAI $(a, b) \subset (\alpha, \beta)$. For any $\varepsilon \in (0, 1)$, set $c = \varepsilon a + (1 - \varepsilon)b$, then take $s \in SC_M$ and $E \subset G$ with $\mu E > 0$ such that $M(c)\mu E + M(s)\mu(G \setminus E) = 1$. In fact, by Theorem 1.2, $\beta < +\infty$, or $\beta = +\infty$ and M has no infinite SAI. We pick s and E in the following cases:

(a) If $M(c)\mu G < 1$.

(a-1) If $\beta = +\infty$ and M has no infinite SAI, we take $s \in SC_M$ such that $M(s)\mu G > 1$.

(a-2) If $\beta < +\infty$ and $M(\beta) = +\infty$, (a, β) is not a SAI for any $a < \beta$, from $M^{-1}(\frac{1}{\mu G}) < \beta$, we take $s \in SC_M$ such that $M(s)\mu G > 1$.

(a-3) If $\beta < +\infty$ and $M(\beta) < +\infty$, $\beta \in SC_M$, from $M(\beta)\mu G > 1$, set $s = \beta$. Then $M(s)\mu G > 1$.

Thus $M(s) > \frac{1}{\mu G} > M(c)$. Further, take $E \subset G$ with $\mu E = \frac{M(s)\mu G - 1}{M(s) - M(c)}$. Thus $0 < \mu E < \mu G$ and $M(c)\mu E + M(s)\mu(G \setminus E) = 1$.

(b) If $M(c)\mu G \geqslant 1$, set $s = \alpha$. Then $s \in SC_M$ and $M(s) = 0 < \frac{1}{\mu G} \leqslant M(c)$. Take $E \subset G$ such that $\mu E = \frac{1}{M(c)}$. Hence $0 < \mu E \leqslant \mu G$ and $M(c)\mu E + M(s)\mu(G \setminus E) = 1$.

Set $u = c\chi_E + s\chi_{G\setminus E}$. If $u = \lambda v + (1-\lambda)w$ with $\lambda \in (0,1), v \in \text{Ext}B(L_{(M)})$ and $w \in B(L_{(M)})$, $1 = M(c)\mu E + M(s)\mu(G \setminus E) = \rho_M(u) \leqslant \lambda\rho_M(v) + (1-\lambda)\rho_M(w) \leqslant 1$ which implies $\rho_M(w) = 1$. Therefore, $M(u(t)) = \lambda M(v(t)) + (1-\lambda)M(w(t))$ a.e. on G which follows that $u(t) = v(t) = w(t) = s$ for almost $t \in G \setminus E$, whereas $u(t), v(t), w(t)$ are all in (a,b) for almost $t \in E$. By Lemma 1.1(1), $v(t) = a$ or b a.e. on E. And we have $\mu\{t \in E : v(t) = a\} > 0$. Indeed, if $v(t) = b$ a.e. on E, $\rho_M(v) = M(b)\mu E + M(s)\mu(G\setminus E) > M(c)\mu E + M(s)\mu(G\setminus E) = 1$ which is a contradiction with that $v \in B(L_{(M)})$. For $t \in \{t \in E : v(t) = a\} \subset E$, $\varepsilon a + (1-\varepsilon)b = c = u(t) = \lambda a + (1-\lambda)v(t) \leqslant \lambda a + (1-\lambda)b$ which implies $\lambda \leqslant \varepsilon$. Thus $\lambda(u) \leqslant \varepsilon$, moreover $\lambda(L_{(M)}) = 0$ by the arbitrariness of ε, a contradiction.

"Sufficiency": When $M(\beta)\mu G \leqslant 1$, $L_{(M)}$ has the uniform λ property by Lemma 1.11; when $M(\beta)\mu G > 1$ and M is strictly convex on (α,β), we will discuss in the following five cases. Firstly, by Remark 1.3, without loss of generality, we assume $\mu\{t : u(t) < \alpha\} = 0$. On the other hand, $\lambda(u) \geqslant \max\{\frac{1}{2}(1 - \|u\|_{(M)}), \lambda(\frac{u}{\|u\|_{(M)}})\|u\|_{(M)}\} \geqslant \frac{1}{4}\lambda(\frac{u}{\|u\|_{(M)}})$ for $u \in B(L_{(M)}) \setminus \{0\}$ by Lemma 1.2.

(a) If $\beta = +\infty$, for $u \in S(L_{(M)}) \setminus \text{Ext}B(L_{(M)})$, from the strict convexity of M on $(\alpha, +\infty)$, we get $\rho_M(u) < 1$. Using the same argument as in the proof of classical Orlicz spaces[6], we can get $\lambda(u) = 1$. Thus $\lambda(L_{(M)}) \geqslant \frac{1}{4} > 0$.

(b) If $\beta < +\infty$, $M(\beta) = +\infty$, $\alpha > 0$, for $u \in S(L_{(M)}) \setminus \text{Ext}B(L_{(M)})$, since the strict convexity of M on (α,β), $\rho_M(u) < 1$. By the proof of Lemma 1.7, we can get $\lambda(u) \geqslant \frac{\alpha}{2\beta}\lambda(v)$ and $\rho_M(v) = 1$. Since $v(t) \geqslant u(t) \geqslant \alpha$ and the strict convexity of M on (α,β), $v \in \text{Ext}B(L_{(M)})$. Then $\lambda(v) = 1$ and $\lambda(u) \geqslant \frac{\alpha}{2\beta}$. Thus $\lambda(L_{(M)}) \geqslant \frac{1}{4} \cdot \frac{\alpha}{2\beta} = \frac{\alpha}{8\beta} > 0$.

(c) If $\beta < +\infty$, $M(\beta) = +\infty$, $\alpha = 0$, for $u \in S(L_{(M)}) \setminus \text{Ext}B(L_{(M)})$, since the strict convexity of M on (α,β), $\rho_M(u) < 1$. By the same argument as in the proof of Lemma 1.8, we also obtain $\lambda(u) \geqslant \frac{1}{2}\lambda(v)$ and $\rho_M(v) = 1$. From the strict convexity of M on (α,β), we see $v \in \text{Ext}B(L_{(M)})$. Thus $\lambda(v) = 1$ and $\lambda(u) \geqslant \frac{1}{2}$. Therefore $\lambda(L_{(M)}) \geqslant \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} > 0$.

(d) If $\beta < +\infty$, $M(\beta) < +\infty$, $\alpha > 0$, for $u \in S(L_{(M)}) \setminus \text{Ext}B(L_{(M)})$, since the strict convexity of M on (α,β), $\rho_M(u) < 1$. By the proof of Lemma 1.9, we have $\lambda(u) \geqslant \frac{\alpha}{2\beta}\lambda(v)$ and $\rho_M(v) = 1$. From $v(t) \geqslant u(t) \geqslant \alpha$ and the strict convexity of M on (α,β), we see $v \in \text{Ext}B(L_{(M)})$. Then $\lambda(v) = 1$ and $\lambda(u) \geqslant \frac{\alpha}{2\beta}$. So $\lambda(L_{(M)}) \geqslant \frac{1}{4} \cdot \frac{\alpha}{2\beta} = \frac{\alpha}{8\beta} > 0$.

(e) If $\beta < +\infty$, $M(\beta) < +\infty$, $\alpha = 0$, for $u \in S(L_{(M)}) \setminus \text{Ext}B(L_{(M)})$, since the strict convexity of M on (α,β), $\rho_M(u) < 1$. If $\mu\{t : u(t) = 0\} = 0$, for $h \in [0,\beta]$, we define two decreasing functions as

$$f(h) = \int_{\{t:u(t)\geqslant h\}} [M(\beta) - M(u(t))]\text{d}t + \rho_M(u),\ \hat{f}(h) = \int_{\{t:u(t)>h\}} [M(\beta) - M(u(t))]\text{d}t + \rho_M(u).$$

Then f is left-continuous on $(0,\beta]$, whereas $\hat{f}$ is right-continuous on $[0,\beta)$; and $f(0) = \hat{f}(0) = M(\beta)\mu G - \int_{\{t:u(t)\geqslant 0\}} M(u(t))\text{d}t + \rho_M(u) > 1$, $f(\beta) = \hat{f}(\beta) = \rho_M(u) < 1$, and $f(h) \geqslant \hat{f}(h)$.

Set $H = \sup\{h : f(h) \geqslant 1\}$, $\widehat{H} = \sup\{h : \hat{f}(h) \geqslant 1\}$. Then from the left-continuity of f and the right-continuity of $\hat{f}$, we have $0 < \widehat{H} \leqslant H < \beta$ and $\hat{f}(\widehat{H}) \leqslant 1 \leqslant f(H)$.

If $H = \widehat{H}$, from $\hat{f}(\widehat{H}) \leqslant 1 \leqslant f(H)$, take $E \subset \{t : u(t) = H\}$ such that $\rho_M(v) = 1$, where

$$v(t) = \begin{cases} u(t), & 0 < u(t) < H, \text{ or } t \in E, \\ \beta, & H < u(t) \leqslant \beta, \text{ or } t \in \{t : u(t) = H\} \setminus E. \end{cases}$$

From the strict convexity of M on (α, β), we see $v \in \mathrm{Ext}B(L_{(M)})$. Thus $\lambda(v) = 1$. Denote $w(t) = 2u(t) - v(t)$. Then $u = \frac{1}{2}v + \frac{1}{2}w$, and when $v(t) = \beta$, $-\beta < w(t) = 2u(t) - \beta \leqslant \beta$. Thus $\rho_M(w) \leqslant \rho_M(v) = 1$, which by Lemma 1.2, implies $\lambda(u) \geqslant \frac{1}{2}\lambda(v) = \frac{1}{2}$.

If $H > \widehat{H}$, for any $h \in (\widehat{H}, H)$, $\hat{f}(\hat{h}) \leqslant 1 \leqslant f(h)$. Replacing the H by h and repeating the same arguments as in the proof of the case $H = \widehat{H}$ above, we get $\lambda(u) \geqslant \frac{1}{2}$.

If $\mu\{t : u(t) = 0\} > 0$, using the same method as in the case (II) of the proof of Lemma 1.10, we see $\lambda(u) \geqslant \frac{1}{4}$. Therefore $\lambda(L_{(M)}) \geqslant \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} > 0$.

If M is defined as in [6], so $\alpha = 0$ and $\beta = +\infty$, we reach the result obtained in [6].

Corollary 1.2 $L_{(M)}$ has the uniform λ property iff M is strictly convex.

[References]

[1] ARON R M, LOHMAN R H. A geometric function determined by extreme points of the unit ball of a normed space [J]. Pacific Journal of Mathematics, 1987, 127: 209-231.

[2] FORALEWSKI P, HUDZIK H, PLUCIENNIK R. Orlicz spaces without extreme points [J]. J Math Anal Appl, 2010, 361: 506-519.

[3] GRANERO A S. λ-property in Orlicz spaces [J]. Bulletin of Polish Academy of Sciences, Mathematics, 1989, 37: 421-431.

[4] CHEN S T, SUN H Y, Wu C X. λ-property of Orlicz spaces [J]. Bulletin of Polish Academy of Sciences, Mathematics, 1991, 39: 63-69.

[5] CHEN S T, SUN H Y. λ-property of Orlicz sequence spaces [J]. Annales Polonici Mathematici, 1994, 59: 239-249.

[6] CHEN S T. Geometry of Orlicz spaces [M]. Warszawa: Dissertationes Mathematicae, 356, 1996.

[7] LIU X B, WANG T F, YU F F. Extreme points and strongly extreme points of Musielak-Orlicz sequences spaces [J]. Acta Mathematica Sinica, English Series, 2005, 21: 267-278.