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Two families of spanning subgraphs of a complete graph determined by their spectra

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Abstract: A graph G is said to be *determined by its spectrum* if any graph having the same spectrum as that of G is isomorphic to G. In this paper, it was proved that $K_n - E(lP_2)$ and $K_n - E(K_{1,l})$ are determined by their spectra, respectively.
Key words: cospectral graphs; spectrum of a graph; eigenvalues
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完全图的两类生成子图是谱唯一确定的

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摘 要: 只有与G同构的图才有相同的谱, 则称图G称为谱唯一确定的. 本文证明了, $K_n - E(lP_2)$ 和 $K_n - E(K_{1,l})$ 是谱唯一确定的.
关键词: 同谱图; 图的谱; 特征值

0 Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let $d(v_i)$ denote the degree of $v_i \in V$. All graphs considered in this paper are finite undirected loopless simple graphs. Let $\boldsymbol{A}(G)$ be the (0,1)-*adjacency matrix* of G. The polynomial $P_{A(G)}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}(G))$ is called the characteristic polynomial of the graph G with respect to the adjacency matrix, where $\boldsymbol{I}$ is the identity matrix, which can be written as $P_{A(G)}(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$. Since the matrix $\boldsymbol{A}(G)$ are real and symmetric, its eigenvalues are all real numbers. Assume that $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ are the *adjacency eigenvalues* of graph G. The *adjacency spectrum* of graph G consists of the adjacency eigenvalues (together with their multiplicities). Two graphs are *cospectral* if they share the same spectrum. A graph G is said to be *determined by its spectrum* (DS for short) if for any graph H, $P_{A(H)}(\lambda) = P_{A(G)}(\lambda)$ implies that H is isomorphic to G.

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Up to now, only few graphs with very special structures have been proved to the determined by their spectra. So, "which graphs are determined by their spectrum?"[1] seems to be a difficult problem in the theory of graph spectrum. For the background and some recent surveys of the known results about this problem and related topics, we refer the reader to [2,3] and references therein.

We denote by lP_2 the disjoint union of l paths P_2, that is $lP_2 = P_2 \cup P_2 \cup \cdots \cup P_2$. In this paper, we show that $K_n - E(lP_2)(1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor)$ and $K_n - E(K_{1,l})(1 \leqslant l \leqslant n-1)$ are determined by their spectrum, respectively.

1 Main results

Before presenting proof to Theorems, we need the following Lemmas:

Lemma 1[1] Let G be a graph. For the adjacency matrix, the following can be obtained from the spectrum.

(i) The number of vertices. (ii) The number of edges.

(iii) Whether G is regular. (iv) Whether G is regular with any fixed girth.

For the adjacency matrix the following follows from the spectrum.

(v) The number of closed walk of any length. (vi) Whether G is bipartite.

Lemma 2 Let G be a graph with n vertices and $\binom{n}{2} - l$ edges, If $1 \leqslant l < n-1$, then G have only one connected component.

Proof Assume that G have r connected components, that is $G = G_{n_1} \cup G_{n_2} \cup \cdots \cup G_{n_r}$, where $|V(G_{n_l})| = n_l, l = 1, 2, \cdots, r$ and $n_1 + n_2 + \cdots + n_r = n$.

$$\begin{aligned}\frac{n(n-1)}{2} - l = |E(G)| &= |E(G_{n_1})| + |E(G_{n_2})| + \cdots + |E(G_{n_r})| \\ &\leqslant \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \cdots + \frac{n_r(n_r-1)}{2},\end{aligned}$$

namely

$$\sum_{l=1}^{r} n_l^2 + 2\sum_{1\leqslant i<j\leqslant r} n_i n_j - 2l = n^2 - 2l \leqslant \sum_{l=1}^{r} n_l^2,$$

we get

$$\sum_{1\leqslant i<j\leqslant r} n_i n_j \leqslant l, \tag{1.1}$$

since

$$n - 1 \leqslant \sum_{1\leqslant i<j\leqslant r} n_i n_j, \tag{1.2}$$

the equality hold if and only if r=2, n_l=1 and $n_2 = n-1$. By (1.1) and (1.2) we have $n-1 \leqslant l$, a contradiction.

Let $T_3(G)$ denote the number of triangles in the graph G, we have following Lemmas.

Lemma 3 If G_1 be a graph of size l, and $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_{n_1})$ is the degree sequence of G_1. Then $T_3(K_n - E(G_1)) = \binom{n}{3} - l(n-2) + \sum_{v\in V(G_1)} \binom{d(v)}{2} - T_3(G_1)$ for any $d(v) \geqslant 2$.

Proof $|E(G_1)|=l$ and every edge of K_n corresponds to $n-2$ triangles in K_n.

Case 1 G_1 contains no C_3. For a $P_3=uvw$ in G_1, since the edges uv and vw correspond to same one triangle, denote it uvw, hence uv and vw correspond to 2(n-2)-1 triangles in $K_n-E(G_1)$, so the l edges in $E(G_1)$ correspond to $l(n-2)-\sum_{v\in V(G_1)}\binom{d(v)}{2}$ triangles in $K_n-E(G_1)$, where $\sum_{v\in V(G_1)}\binom{d(v)}{2}$ is the number of P_3 in G_1.

Case 2 G_1 contains C_3. Similarly, for a $C_3=uvw$, the edges uv, vw and wu correspond to same one triangle, denote it uvw. Since for the triangle uvw, we counted 3 times in $l(n-2)$ and $\sum_{v\in V(G_1)}\binom{d(v)}{2}$, respectively. So the l edges in $E(G_1)$ correspond to $l(n-2)-\sum_{v\in V(G_1)}\binom{d(v)}{2}+T_3(G_1)$ triangles in $K_n-E(G_1)$. Thus the number of triangles in $K_n-E(G_1)$ is $\binom{n}{3}-l(n-2)+\sum_{v\in V(G_1)}\binom{d(v)}{2}-T_3(G_1)$.

Theorem 1 The graph $K_n-E(lP_2)(n\geqslant 3, 1\leqslant l\leqslant\lfloor\frac{n}{2}\rfloor)$ is determined by its spectrum.

Proof Suppose a graph G is cospectral with $K_n-E(lP_2)$ respect to the adjacency spectrum. By Lemma 1, G is a graph with n vertices and $\binom{n}{2}-l$ edges. Since $n\geqslant 3$, hence $l\leqslant\lfloor\frac{n}{2}\rfloor\leqslant\frac{n}{2}<n-1$, by Lemma 2, G have only one connected component. So G must isomorphic to a graph which is obtained from K_n by deleting l edges, write the graph consist of the l edges is G_1 and $E(G_1)=\{e_1,e_2,\cdots,e_l\}$. Assume that there exist at least two edges $e_i,e_j\in E(G_1)$ such that them are jointed but no triangle in G_1, let u be the common vertex of e_i and e_j, then $d(u)\geqslant 2$, by Lemma 3, $T_3(G)=\binom{n}{3}-l(n-2)+\sum_{v\in V(G_1)}\binom{d(v)}{2}\geqslant\binom{n}{3}-l(n-2)+1>\binom{n}{3}-l(n-2)=T_3(K_n-E(lP_2))$. Assume that there exist at least one triangle in G_1, then $T_3(G)\geqslant\binom{n}{3}-l(n-2)+3\binom{2}{2}-1>\binom{n}{3}-l(n-2)=T_3(K_n-E(lP_2))$. This is a contradiction with (v) of Lemma 1. Thus the edges in E_{G_1} is pairwise disjoint, that is $H\cong G$.

The disjoint union of k disjoint paths $P_{n_1}\cup P_{n_2}\cup\cdots\cup P_{n_k}$ is determined by its spectrum[4], by Theorem 1 we can get the following corollary.

Corollary 1 $\overline{lP_2}$ is determined by its spectrum.

Theorem 2 The graph $K_n-E((l-2)P_2\cup P_3)(n\geqslant 6, 2\leqslant l\leqslant\lfloor\frac{n}{2}\rfloor)$ is determined by its spectrum.

Proof Suppose a graph G is cospectral with $K_n-E((l-2)P_2\cup P_3)$ respect to the adjacency spectrum. Similar to the proof of Theorem 1, G isomorphic to a graph which is obtained from K_n by deleting l edges, write the graph consist of the l edges is G_1, that is $G=K_n-E(G_1)$. By Lemma 3, $T_3(G)=\binom{n}{3}-l(n-2)+\sum_{v\in V(G_1)}\binom{d(v)}{2}-T_3(G_1)$ and $T_3(K_n-E((l-2)P_2\cup P_3))=\binom{n}{3}-l(n-2)+1$. By (v) of Lemma 1, we have $T_3(G)=T_3(K_n-E((l-2)P_2\cup P_3))$, that is

$$\sum_{v\in V(G_1)}\binom{d(v)}{2}-T_3(G_1)=1. \tag{1.3}$$

Assume that there exist at least one triangle in G_1, then $\sum_{v\in V(G_1)}\binom{d(v)}{2}-T_3(G_1)\geqslant 3-1=2\neq 1$, a contradiction, so contains no triangle in G_1. By (1.3) we have $\sum_{v\in V(G_1)}\binom{d(v)}{2}=1$, so there exist one vertex $v\in V(G_1)$ such that $d(v)$=2 and $d(u)$=1 for other vertices $u\in V(G_1)-\{v\}$, thus $G_1\cong(l-2)P_2\cup P_3$ and $G\cong K_n-E((l-2)P_2\cup P_3)$.

Corollary 2 $\overline{(l-2)P_2\cup P_3}$ is determined by its spectrum.

Theorem 3 The graph $K_n - E(K_{1,l})(1 \leqslant l \leqslant n-1)$ is determined by its spectrum.

Proof If $l = n-1$, then $K_n - E(K_{1,l})$ must isomorphic to the disjoint union of K_{n-1} and K_1, so graph $K_n - E(K_{1,l})$ is determined by its adjacency spectrum. Next, we will assume that $1 \leqslant l < n-1$.

Similar to the proof of Theorem 2, suppose a graph G is cospectral with $K_n - E(K_{1,l})$ respect to the adjacency spectrum, then G is a graph with n vertices and $\binom{n}{2} - l$ edges. By Lemma 2, G have only one connected component. So G must isomorphic to a graph which is obtained from K_n by deleting l edges, write the set of l edges is $E_l = \{e_1, e_2, \cdots, e_l\}$. We denote by $\mathcal{G}_1$ the set of all graphs consist of the l edges in E_l.

Case 1 l=1. By Theorem 1, the graph $G = K_n - E(P_2)$ is determined by its spectrum.

Case 2 l=2. Then $G = K_n - E(2P_2)$ or $G \cong K_n - E(K_{1,2})$. By Lemma 3, $T_3(K_n - E(2P_2)) = \binom{n}{3} - l(n-2)$ and $T_3(K_n - E(K_{1,2})) = \binom{n}{3} - l(n-2) + \binom{2}{2}$, $T_3(K_n - E(2P_2)) \neq T_3(K_n - E(K_{1,2}))$, this is a contradiction with (v) of Lemma 1. so $G \cong K_n - E(K_{1,2})$.

Case 3 l=3. Then $G = K_n - E(P_4)$ or $G = K_n - E(C_3)$ or $G = K_n - E(3P_2)$ or $G = K_n - E(P_2 \cup P_3)$ or $G \cong K_n - E(K_{1,3})$.

Sub-case 3.1 If $G = K_n - E(P_4)$, then by Lemma 3, $T_3(K_n - E(P_4)) = \binom{n}{3} - l(n-2) + 2\binom{2}{2}$ and $T_3(K_n - E(K_{1,3})) = \binom{n}{3} - l(n-2) + \binom{3}{2}$, $T_3(K_n - E(P_4)) \neq T_3(K_n - E(K_{1,3}))$. this is a contradiction with (v) of Lemma 1.

Sub-case 3.2 Similar to Subcase 3.1, if $G = K_n - E(C_3)$, then by Lemma 3, $T_3(K_n - E(C_3)) = \binom{n}{3} - l(n-2) + 3\binom{2}{2} - 1$ and $T_3(K_n - E(K_{1,3})) = \binom{n}{3} - l(n-2) + \binom{3}{2}$, $T_3(K_n - E(C_3)) \neq T_3(K_n - E(K_{1,3}))$. this is a contradiction with (v) of Lemma 1.

Sub-case 3.3 If $G = K_n - E(3P_2)$, or $G = K_n - E(P_2 \cup P_3)$, then by Theorem 1 and 2, the graphs $K_n - E(3P_2)$ and $K_n - E(P_2 \cup P_3)$ is determined by its spectrum, respectively. Thus $G \cong K_n - E(K_{1,3})$.

Next, we will assume that $4 \leqslant l < n-1$. For a star $K_{1,l} \in \mathcal{G}_1$, all edges in $K_{1,l}$ is joint with each other, hence $K_{1,l}$ contain the most P_3, the number of P_3 in $K_{1,l}$ is $\sum_{v \in V(K_{1,l})} \binom{d(v)}{2} = \binom{l}{2}$. For any graph $G_1 \in \mathcal{G}_1 - \{K_{1,l}\}$, since $l \geqslant 4$, hence there exist at least two edges in G_1 is disjoint, so the number of P_3 in G_1 less than the number of P_3 in $K_{1,l}$, that is $\sum_{v \in V(G_1)} \binom{d(v)}{2} < \binom{l}{2}$ $(G_1 \in \mathcal{G}_1 - \{K_{1,l}\})$. By Lemma 3, $T_3(K_n - E(G_1)) = \binom{n}{3} - l(n-2) + \sum_{v \in V(G_1)} \binom{d(v)}{2} - T_3(G_1) < \binom{n}{3} - l(n-2) + \binom{l}{2} = T_3(K_n - E(K_{1,l}))$, By (v) of Lemma 1 the graph $G_1 \in \mathcal{G}_1 - \{K_{1,l}\}$ is not cospectral with $G_1 \in \mathcal{G}_1 - \{K_{1,l}\}$ respect to the adjacency spectrum. This completes the proof.

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