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Vertex-distinguishing proper edge coloring of composition of complete graph and star

YANG Fang[1], WANG Zhi-wen[2],
CHEN Xiang-en[1], MA Chun-yan[1]

(1. *College of Mathematics and Statistics, Northwest Normal University, Lanzhou* 730070, *China*;
2. *School of Mathematics and Computer Sciences, Ningxia University, Yinchuan* 750021, *China*)

Abstract: Firstly, we gave an upper bound for the vertex-distinguishing proper edge chromatic number of composition of complete graph K_p and star S_q, which is $pq+1$ for $p \geqslant 2,\ q \geqslant 4$. Then by constructing coloring in terms of the symmetry of regular polygons and the methods of combinatorial analysis, we obtained respectively vertex-distinguishing proper edge chromatic numbers for composition of complete graph K_p and star S_q when $p=2,\ q \geqslant 4$; $p \geqslant 3,\ q=4$; p is even and $p \geqslant 4,\ q=5$; pq is odd and $p \geqslant 3,\ q \geqslant 5$.

Key words: composition; complete graph; star; vertex-distinguishing proper edge coloring; vertex-distinguishing proper edge chromatic number

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完全图和星的合成的点可区别正常边染色

杨 芳[1], 王治文[2], 陈祥恩[1], 马春燕[1]

(1. 西北师范大学 数学与统计学院, 兰州 730070;
2. 宁夏大学 数学与计算机科学学院, 银川 750021)

摘要: 首先, 给出了完全图 K_p 和星 S_q 的合成的点可区别正常边色数的一个上界: 当 $p \geqslant 2$, $q \geqslant 4$ 时, 上界是 $pq+1$. 再利用正多边形的对称性以及组合分析的方法来构造染色, 分别得到了当 $p=2,\ q \geqslant 4$; $p \geqslant 3,\ q=4$; p 是偶数且 $p \geqslant 4,\ q=5$; pq 是奇数且 $p \geqslant 3,\ q \geqslant 5$时, 完全图 K_p 和星 S_q 的合成的点可区别正常边色数.

关键词: 合成; 完全图; 星; 点可区别正常边染色; 点可区别正常边色数

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第一作者: 杨芳, 女, 硕士生, 研究方向为图论及其应用. E-mail: yangfangnwnu@126.com.

通信作者: 陈祥恩, 男, 教授, 研究方向为图论以及代数理论. E-mail: chenxe@nwnu.edu.cn.

0 Introduction and definitions

Determining chromatic numbers of various kinds of colorings is a fundamental problem of graph coloring. After the concept of vertex-distinguishing proper edge coloring of graphs was presented in [1], international scholars did many studies in [2-4].

All graphs mentioned here are simple, undirected and finite. We denote the vertex set, edge set, maximum degree, minimum degree, and edge chromatic number of a graph G by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$, and $\chi'(G)$, respectively.

A proper k-edge coloring f of a graph G is an assignment of k colors, $1, 2, \cdots, k$, to edges of G (or a mapping from $E(G)$ to $\{1, 2, \cdots, k\}$) such that no two adjacent edges receive the same color. Given such a coloring f, for any vertex $x \in V(G)$, let $S(x)$ be the set of colors assigned to the edges incident to x, i.e., $S(x) = \{f(xu)|xu \in E(G),\ u \in V(G)\}$, $S(x)$ is called the color set of vertex x. If for any two distinct vertices u and v of $V(G)$, $S(u) \neq S(v)$, then we say that f is a vertex-distinguishing proper edge coloring of graph G (in brief k-VDPEC). Let $\overline{S}(x) = \{1, 2, \cdots, k\} \setminus S(x)$. $\overline{S}(x)$ is called the complementary color set of vertex x. The minimum number of colors required for a vertex-distinguishing proper edge coloring of G, denoted by $\chi'_s(G)$, is called the vertex-distinguishing proper edge chromatic number. A graph with no more than one isolated vertex and no isolated edges is called a *vdec* graph. Obviously, a graph G has the vertex-distinguishing proper edge coloring if and only if G is a *vdec* graph.

Let G be a *vdec* graph and $n_d(G)$ denote the number of vertices of degree d, $\delta(G) \leqslant d \leqslant \Delta(G)$. Set

$$\pi(G) = \min\{\theta|\tbinom{\theta}{d} \geqslant n_d(G),\ \delta(G) \leqslant d \leqslant \Delta(G)\}.$$

Clearly, the following lemma is true.

Lemma 1 $\chi'_s(G) \geqslant \pi(G)$.

Burris and Schelp got the vertex-distinguishing proper edge chromatic numbers of complete graphs, complete bipartite graphs, paths and cycles in [4], and presented the following Conjecture.

Conjecture 1 If G is a *vdec* graph, then $\chi'_s(G) = \pi(G)$ or $\pi(G) + 1$.

Lemma 2[3] For any *vdec* graph of order n, then $\chi'_s(G) \leqslant n + 1$.

The composition of simple graphs G and H is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. The notation $(u, v)(u', v')$ indicates the edge between adjacent two vertices (u, v) and (u', v') in $G[H]$.

Lemma 3[5] (i) If $(u, v) \in V(G[H])$, then $d_{G[H]}(u, v) = d_G(u)|V(H)| + d_H(v)$.

(ii) $\Delta(G[H]) = \Delta(G) \cdot |V(H)| + \Delta(H)$.

Let $K_p[S_q]$ be the composition of complete graph K_p and star S_q, where K_p is a complete graph of order p, S_q is a star of order q. Then there exist edge-disjoint spanning subgraph $K(p \times q)$ and pS_q of $K_p[S_q]$, such that $K_p[S_q] = K(p \times q) \bigcup pS_q$, where $K(p \times q)$ is a complete p-partite graph with equipotent parts and q vertices in each part, pS_q is the disjoint union of p graphs which are isomorphic to S_q.

Lemma 4 If $p \geqslant 2, q \geqslant 4$, then $\pi(K_p[S_q]) = \min\{\theta|\tbinom{\theta}{pq-1} \geqslant p,\ \tbinom{\theta}{pq-q+1} \geqslant pq - p\} = pq$.

Proposition 1[6] If $p \geqslant 3, q \geqslant 2$, then $\chi'_s(K(p\times q)) = (p-1)q+2$.

Proposition 2[7] If $n \geqslant 2$, then $\chi'_s(K_{n,n}) = n+2$.

1 Main results

For convenience, let $V(K_p[S_q]) = \{(u_i, v_j)|i = 1, 2, \cdots, p,\ j = 1, 2, \cdots, q\}$.

Theorem 1 If $p \geqslant 2$, $q \geqslant 4$, then $\chi'_s(K_p[S_q]) \leqslant \chi'_s(K(p\times q)) + \chi'(pS_q)$.

Proof Firstly, we assign $\chi'_s(K(p\times q))$ colors to the edges of $K(p\times q)$ so that the resulting edge coloring is proper and vertex-distinguishing. Then we assign $\chi'(pS_q)$ new colors properly to the edges of pS_q. Combining these two colorings together gives the VDPEC of $K_p[S_q]$ using $\chi'_s(K(p\times q)) + \chi'(pS_q)$ colors. This theorem follows.

According to Proposition 1, Proposition 2 and Theorem 1, the following theorem is obvious.

Theorem 2 If $p \geqslant 2$, $q \geqslant 4$, then $\chi'_s(K_p[S_q]) \leqslant pq+1$.

Theorem 3 If $q \geqslant 4$, then $\chi'_s(K_2[S_q]) = 2q$.

Proof By Lemma 1 and Lemma 4, $\chi'_s(K_2[S_q]) \geqslant 2q$. Set

$$E(K_2[S_q]) = \{(u_1,v_j)(u_2,v_l)|j,l=1,2,\cdots,q\}\bigcup\Big(\bigcup_{i=1}^{2}\{(u_i,v_1)(u_i,v_j)|j=2,3,\cdots,q\}\Big);$$

$$E(K_{q,q}) = \{(u_1,v_j)(u_2,v_l)|j,l=1,2,\cdots,q\}.$$

By Proposition 2, we may give a $(q+2)$-VDPEC φ of $K_{q,q}$. According to the proof procedure of Proposition 2 in [7], we have

$$\begin{cases}\varphi((u_1,v_j)(u_2,v_l)) = (q+j+l)_{q+2},\\ \varphi((u_1,v_q)(u_2,v_l)) = (q-1+l)_{q+2},\end{cases}\quad j=1,2,\cdots,q-1, l=1,2,\cdots,q.$$

The above symbol $(m)_n$ denotes the number in $\{1,2,\cdots,n\}$ which is congruent with m modulo n. Note that if m is a multiple of n, then $(m)_n = n$. For example, $(5)_5 = 5$, $(10)_5 = 5$, $(8)_5 = 3$, $(2)_5 = 2$.

Under φ, the set of two colors which are not represented at vertex (u_i, v_j) is denoted by $\mathscr{A}(u_i,v_j)$, $i=1,2, j=1,2,\cdots,q$, we have

$$\mathscr{A}(u_1,v_1) = \{q, q+1\},\qquad \mathscr{A}(u_1,v_2) = \{q+1, q+2\},$$
$$\mathscr{A}(u_2,v_1) = \{q-1, q+1\},\quad \mathscr{A}(u_2,v_3) = \{1, q+1\}.$$

Based on the coloring φ, we will color the edges of two S_q. This time we need color $q+1$ and new colors $q+3, q+4, \cdots, 2q$.

Let $(u_1,v_1)(u_1,v_2)$ and $(u_2,v_1)(u_2,v_3)$ receive color $q+1$, $(u_1,v_1)(u_1,v_j)$ receive color $q+j$, $j=3,4,\cdots,q$, $(u_2,v_1)(u_2,v_2)$ receive color $q+3$ and $(u_2,v_1)(u_2,v_j)$ receive color $q+j$, $j=4,5,\cdots,q$. The resulting edge coloring of $K_2[S_q]$ is denoted by f. Then f is proper and for this f, we have

$\overline{S}(u_1,v_1) = \{q\}$, $\overline{S}(u_1,v_2) = \{q+2\}\bigcup I$, $\overline{S}(u_1,v_j) = \mathscr{A}(u_1,v_j)\bigcup(I\backslash\{q+j\})$, $j=3,4,\cdots,q$; $\overline{S}(u_2,v_1) = \{q-1\}$, $\overline{S}(u_2,v_2) = \mathscr{A}(u_2,v_2)\bigcup(I\setminus\{q+3\})$, $\overline{S}(u_2,v_3) = \{1\}\bigcup I$, $\overline{S}(u_2,v_j) = \mathscr{A}(u_2,v_j)\bigcup(I\setminus\{q+j\})$, $j=4,5,\cdots,q$; where $I = \{q+3, q+4, \cdots, 2q\}$.

Since φ is a VDPEC, $\mathscr{A}(u_i,v_j)\neq\mathscr{A}(u_k,v_l)$, $i,k=1,2$, $1\leqslant j,l\leqslant q$, $(i,j)\neq(k,l)$.

It is easy to see that $\overline{S}(u_i,v_j)\neq\overline{S}(u_k,v_l)$, $i,k=1,2$, $1\leqslant j,l\leqslant q$, $(i,j)\neq(k,l)$. Theorem follows.

Theorem 4　If $p\geqslant 3$, then $\chi'_s(K_p[S_4])=4p$.

Proof　By Lemma 1 and Lemma 4, $\chi'_s(K_p[S_4])\geqslant 4p$. Set

$E(K_p[S_4])=\left(\bigcup\limits_{i=1}^{p-1}\bigcup\limits_{k=i+1}^{p}\{(u_i,v_j)(u_k,v_l)|j,l=1,2,3,4\}\right)\bigcup\left(\bigcup\limits_{i=1}^{p}\{(u_i,v_3)(u_i,v_j)|j=1,2,4\}\right)$.

Arrange clockwisely vertices (u_1,v_1), (u_1,v_2), (u_1,v_3), (u_1,v_4), (u_2,v_1), (u_2,v_2), (u_2,v_3), (u_2,v_4), $\cdots$, $(u_p,v_1),(u_p,v_2)$, (u_p,v_3) on the apics of a regular $(4p-1)$-gon with center point (u_p,v_4). Note that all vertices of the regular $(4p-1)$-gon and center point (u_p,v_4) together form the vertex set of $K_p[S_4]$. At the same time, the three segments of connecting (u_i,v_1) and (u_i,v_2), (u_i,v_1) and (u_i,v_4), (u_i,v_2) and (u_i,v_4) are not edges of $K_p[S_4]$, $i=1,2,\cdots,p$. Except these $3p$ segments, connecting segments between any two distinct vertices can be viewed as edges of $K_p[S_4]$. Let M_{ij} be all edges in $K_p[S_4]$ which are perpendicular to straight line connecting two vertices (u_p,v_4) and (u_i,v_j) as well as $(u_p,v_4)(u_i,v_j)$, $i=1,2,\cdots,p-1$, $j=1,2,3,4$; $i=p$, $j=3$. Let M_{pj} be all edges in $K_p[S_4]$ which are perpendicular to straight line connecting two vertices (u_p,v_4) and (u_p,v_j), $j=1,2$. Thus M_{11}, M_{12}, M_{13}, M_{14}, M_{21}, M_{22}, M_{23}, M_{24}, $\cdots$, M_{p1}, M_{p2}, M_{p3} are matching and edge-disjoint each other. Furthermore,

$$E(K_p[S_4])=M_{11}\bigcup M_{12}\bigcup M_{13}\bigcup M_{14}\bigcup M_{21}\bigcup M_{22}\bigcup M_{23}\bigcup M_{24}\bigcup\cdots\bigcup M_{p1}\bigcup M_{p2}\bigcup M_{p3}.$$

We define a proper edge coloring φ of $K_p[S_4]$ using colors $1,2,\cdots,4p-1$ as follows: assign color $4(i-1)+j$ to edges in M_{ij}, $i=1,2,\cdots,p-1$, $j=1,2,3,4$; $i=p$, $j=1,2,3$.

Based on the coloring φ, now we recolor the edge $(u_i,v_1)(u_i,v_3)$ by a new color $4p$, $i=1,2,\cdots,p-1$. The resulting edge coloring is denoted by f. Clearly f is proper.

Case 1　p is even. For this f, we have

$\overline{S}(u_i,v_1)=\{4i-2,4i+2p-3,4i+2p-2\}$, $i=1,2,\cdots,\frac{p}{2}$;

$\overline{S}(u_i,v_1)=\{4i-2p-2,4i-2p-1,4i-2\}$, $i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1$;

$\overline{S}(u_p,v_1)=\{2p-2,4p-3,4p\}$.

$\overline{S}(u_i,v_2)=\{4i-1,4i+2p-3,4p\}$, $i=1,2,\cdots,\frac{p}{2}$;

$\overline{S}(u_i,v_2)=\{4i-2p-2,4i-1,4p\}$, $i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1$;

$\overline{S}(u_p,v_2)=\{2p-2,4p-2,4p\}$.

$\overline{S}(u_i,v_3)=\{4i-2\}$, $i=1,2,\cdots,p-1$; $\overline{S}(u_p,v_3)=\{4p\}$.

$\overline{S}(u_i,v_4)=\{4i-1,4i+2p-2,4p\}$, $i=1,2,\cdots,\frac{p}{2}$;

$\overline{S}(u_i,v_4)=\{4i-2p-1,4i-1,4p\}$, $i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1$;

$\overline{S}(u_p,v_4)=\{4p-3,4p-2,4p\}$.

Note that the numbers in each set $\overline{S}(u_i,v_j)$ are arranged in ascending order, $i=1,2,\cdots,p$, $j=1,2,3,4$. We just need to prove that the complementary color sets of any two distinct vertices of same degree are different from each other.

Obviously, $\overline{S}(u_i,v_3)\neq\overline{S}(u_k,v_3)$, $1\leqslant i<k\leqslant p$.

Now we prove $\overline{S}(u_i,v_1)\neq\overline{S}(u_k,v_1)$, by contradiction, suppose $\overline{S}(u_i,v_1)=\overline{S}(u_k,v_1)$, $1\leqslant i<k\leqslant p$.

For $1\leqslant i\leqslant\frac{p}{2}$, $\frac{p+2}{2}\leqslant k\leqslant p-1$, from $4i+2p-3=4k-2p-1$ we know that $k-i=p-\frac{1}{2}$;

for $1\leqslant i\leqslant\frac{p}{2}$, $k=p$, from $4i+2p-2=4p$ we know that $i=\frac{p+1}{2}$;

for $\frac{p+2}{2}\leqslant i\leqslant p-1$, $k=p$, from $4i-2=4p$ we know that $i=p+\frac{1}{2}$.

These are contradictions. Thus $\overline{S}(u_i,v_1)\neq\overline{S}(u_k,v_1)$, $1\leqslant i<k\leqslant p$.

Similarly, we can show that $\overline{S}(u_i,v_2)\neq\overline{S}(u_k,v_2)$, $\overline{S}(u_i,v_4)\neq\overline{S}(u_k,v_4)$, $1\leqslant i<k\leqslant p$.

We will prove $\overline{S}(u_p,v_1)\neq\overline{S}(u_i,v_2)$, $\overline{S}(u_p,v_1)\neq\overline{S}(u_i,v_4)$. By contradiction, suppose $\overline{S}(u_p,v_1)=\overline{S}(u_i,v_2)$, $\overline{S}(u_p,v_1)=\overline{S}(u_i,v_4)$, $i=1,2,\cdots,p-1$.

For $1\leqslant i\leqslant\frac{p}{2}$, from $2p-2=4i-1$ we know that $i=\frac{p}{2}-\frac{1}{4}$; for $\frac{p+2}{2}\leqslant i\leqslant p-1$, from $4p-3=4i-1$ we know that $i=p-\frac{1}{2}$.

These are contradictions. Thus $\overline{S}(u_p,v_1)\neq\overline{S}(u_i,v_2)$, $\overline{S}(u_p,v_1)\neq\overline{S}(u_i,v_4)$, $i=1,2,\cdots,p$. The color $4p$ belongs to $\overline{S}(u_i,v_2)$ and $\overline{S}(u_i,v_4)$, but does not belong to $\overline{S}(u_k,v_1)$, $i=1,2,\cdots,p$, $k=1,2,\cdots,p-1$. Thus, $\overline{S}(u_k,v_1)\neq\overline{S}(u_i,v_2)$, $\overline{S}(u_k,v_1)\neq\overline{S}(u_i,v_4)$, $i,k=1,2,\cdots,p$.

We will prove $\overline{S}(u_i,v_2)\neq\overline{S}(u_k,v_4)$, by contradiction, suppose $\overline{S}(u_i,v_2)=\overline{S}(u_k,v_4)$, $i,k=1,2,\cdots,p$.

For $1\leqslant i\leqslant\frac{p}{2}$, $1\leqslant k\leqslant\frac{p}{2}$, from $4i+2p-3=4k+2p-2$ we know that $i-k=\frac{1}{4}$;

for $1\leqslant i\leqslant\frac{p}{2}$, $\frac{p+2}{2}\leqslant k\leqslant p-1$, from $4i+2p-3=4k-1$ we know that $k-i=\frac{p-1}{2}$;

for $1\leqslant i\leqslant\frac{p}{2}$, $k=p$, from $4i-1=4p-3$ we know that $i=p-\frac{1}{2}$;

for $\frac{p+2}{2}\leqslant i\leqslant p-1$, $1\leqslant k\leqslant\frac{p}{2}$, from $4i-2p-2=4k-1$ we know that $i-k=\frac{p}{2}+\frac{1}{4}$;

for $\frac{p+2}{2}\leqslant i\leqslant p-1$, $\frac{p+2}{2}\leqslant k\leqslant p-1$, from $4i-2p-2=4k-2p-1$ we know that $i-k=\frac{1}{4}$;

for $\frac{p+2}{2}\leqslant i\leqslant p-1$, $k=p$, from $4i-1=4p-2$ we know that $i=p-\frac{1}{4}$;

for $i=p$, $1\leqslant k\leqslant\frac{p}{2}$, from $2p-2=4k-1$ we know that $k=\frac{p}{2}-\frac{1}{4}$;

for $i=p$, $\frac{p+2}{2}\leqslant k\leqslant p-1$, from $2p-2=4k-2p-1$ we know that $k=p-\frac{1}{4}$.

These are contradictions. Thus $\overline{S}(u_i,v_2)\neq\overline{S}(u_k,v_4)$, $i,k=1,2,\cdots,p$.

In summary, the above coloring is a $4p$-VDPEC coloring of $K_p[S_4]$.

Case 2 p is odd. For this f, we have

$$\overline{S}(u_i,v_1)=\{4i-2,4i+2p-3,4i+2p-2\},\ \ i=1,2,\cdots,\frac{p-1}{2};$$

$$\overline{S}(u_{\frac{p+1}{2}},v_1)=\{1,2p,4p-1\};$$

$$\overline{S}(u_i,v_1)=\{4i-2p-2,4i-2p-1,4i-2\},\ \ i=\frac{p+3}{2},\frac{p+5}{2},\cdots,p-1;$$

$$\overline{S}(u_p,v_1)=\{2p-2,4p-3,4p\}.$$

$$\overline{S}(u_i,v_2)=\{4i-1,4i+2p-3,4p\},\ \ i=1,2,\cdots,\frac{p+1}{2};$$

$$\overline{S}(u_i,v_2)=\{4i-2p-2,4i-1,4p\},\ \ i=\frac{p+3}{2},\frac{p+5}{2},\cdots,p-1;$$

$$\overline{S}(u_p,v_2)=\{2p-2,4p-2,4p\}.$$

$$\overline{S}(u_i,v_3)=\{4i-2\},\ \ i=1,2,\cdots,p-1;\overline{S}(u_p,v_3)=\{4p\}.$$

$$\overline{S}(u_i,v_4)=\{4i-1,4i+2p-2,4p\},\ \ i=1,2,\cdots,\frac{p-1}{2};$$

$$\overline{S}(u_i,v_4)=\{4i-2p-1,4i-1,4p\},\ \ i=\frac{p+1}{2},\frac{p+3}{2},\cdots,p-1;$$
$$\overline{S}(u_p,v_4)=\{4p-3,4p-2,4p\}.$$

Note that the numbers in each set $\overline{S}(u_i,v_j)$ are arranged in ascending order, $i=1,2,\cdots,p,\ j=1,2,3,4$. We just need to prove that the complementary color sets of any two distinct vertices of same degree are different from each other.

The proof of Case 2 is similar to that of Case 1.

Theorem 5 If $p(\geqslant 4)$ is even, then $\chi'_s(K_p[S_5])=5p$.

Proof By Lemma 1 and Lemma 4, $\chi'_s(K_p[S_5])\geqslant 5p$. Set

$E(K_p[S_5])=(\bigcup\limits_{i=1}^{p-1}\bigcup\limits_{k=i+1}^{p}\{(u_i,v_j)(u_k,v_l)|j,l=1,2,3,4,5\})\bigcup(\bigcup\limits_{i=1}^{p}\{(u_i,v_3)(u_i,v_j)|j=1,2,4,5\})$.

We define a proper edge coloring φ of $K_p[S_5]$ using colors $1,2,\cdots,5p-1$ in the same way as that of Theorem 4 as follows: assign color $5(i-1)+j$ to edges in M_{ij}, $i=1,2,\cdots,p-1$, $j=1,2,3,4,5$; $i=p, j=1,2,3,4$.

Based on the coloring φ, now we recolor the edge $(u_i,v_3)(u_i,v_5)$ by a new color $5p$, $i=1,2,\cdots,p-1$. The resulting edge coloring is denoted by f. Clearly f is proper and for this f, we have

$$\overline{S}(u_i,v_1)=\Big\{5i-2,5i+\frac{5}{2}p-4,5i+\frac{5}{2}p-3,5p\Big\},\ \ i=1,2,\cdots,\frac{p}{2};$$
$$\overline{S}(u_i,v_1)=\Big\{5i-\frac{5}{2}p-3,5i-\frac{5}{2}p-2,5i-2,5p\Big\},\ \ i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1;$$
$$\overline{S}(u_p,v_1)=\Big\{\frac{5}{2}p-3,\frac{5}{2}p-2,5p-4,5p\Big\}.$$
$$\overline{S}(u_i,v_2)=\Big\{5i-2,5i+\frac{5}{2}p-4,5i+\frac{5}{2}p-2,5p\Big\},\ \ i=1,2,\cdots,\frac{p}{2};$$
$$\overline{S}(u_i,v_2)=\Big\{5i-\frac{5}{2}p-3,5i-\frac{5}{2}p-1,5i-2,5p\Big\},\ \ i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1;$$
$$\overline{S}(u_p,v_2)=\Big\{\frac{5}{2}p-3,5p-3,5p-2,5p\Big\}.$$
$$\overline{S}(u_i,v_3)=\{5i-1\},\ \ i=1,2,\cdots,p-1;\ \ \overline{S}(u_p,v_3)=\{5p\}.$$
$$\overline{S}(u_i,v_4)=\Big\{5i-2,5i+\frac{5}{2}p-3,5i+\frac{5}{2}p-1,5p\Big\},\ \ i=1,2,\cdots,\frac{p}{2};$$
$$\overline{S}(u_i,v_4)=\Big\{5i-\frac{5}{2}p-2,5i-\frac{5}{2}p,5i-2,5p\Big\},\ \ i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1;$$
$$\overline{S}(u_p,v_4)=\Big\{\frac{5}{2}p-2,5p-2,5p-1,5p\Big\}.$$
$$\overline{S}(u_i,v_5)=\Big\{5i-2,5i-1,5i+\frac{5}{2}p-2,5i+\frac{5}{2}p-1\Big\},\ \ i=1,2,\cdots,\frac{p}{2};$$
$$\overline{S}(u_i,v_5)=\Big\{5i-\frac{5}{2}p-1,5i-\frac{5}{2}p,5i-2,5i-1\Big\},\ \ i=\frac{p+2}{2},\frac{p+4}{2},\cdots,p-1;$$
$$\overline{S}(u_p,v_5)=\{5p-4,5p-3,5p-1,5p\}.$$

Note that the numbers in each set $\overline{S}(u_i,v_j)$ are arranged in ascending order, $i=1,2,\cdots,p,\ j=1,2,3,4,5$. We just need to prove that the complementary color sets of any two distinct vertices of same degree are different from each other.

The proof is similar to Case 1 of Theorem 4.

Theorem 6 If pq $(p \geqslant 3, q \geqslant 5)$ is odd, then $\chi'_s(K_p[S_q]) = pq$.

Proof By Lemma 1 and Lemma 4, $\chi'_s(K_p[S_q]) \geqslant pq$. Set

$E(K_p[S_q]) = \Big(\bigcup_{i=1}^{p-1}\bigcup_{k=i+1}^{p}\{(u_i,v_j)(u_k,v_l)|\, j,l=1,2,\cdots,q\}\Big) \bigcup \Big(\bigcup_{i=1}^{p}\{(u_i,v_{\frac{q+1}{2}})(u_i,v_j)|\, j = 1,2,\cdots,q, \text{ and } j \neq \frac{q+1}{2}\}\Big)$.

Arrange clockwisely vertices $(u_1,v_1),(u_1,v_2),\cdots,(u_1,v_q),(u_2,v_1),(u_2,v_2),\cdots,(u_2,v_q),\cdots,(u_p,v_1),(u_p,v_2),\cdots,(u_p,v_q)$ on the apics of a regular pq-gon with center point O. Note that all vertices of the regular pq-gon form the vertex set of $K_p[S_q]$. At the same time, the segment of connecting (u_i,v_j) and (u_i,v_l) is not edge of $K_p[S_q]$, $i=1,2,\cdots,p,\ j=1,2,\cdots,q-1$, $l=j+1,j+2,\cdots,q$ and $j,l\neq\frac{q+1}{2}$. Except these $\frac{q^2-3q+2}{2}p$ segments, connecting segments between any two distinct vertices can be viewed as edges of $K_p[S_q]$. Let M_{ij} be all edges in $K_p[S_q]$ which are perpendicular to straight line connecting O and (u_i,v_j), $i=1,2,\cdots,p,\ j=1,2,\cdots,q$. Thus $M_{11},M_{12},\cdots,M_{1q},M_{21},M_{22},\cdots,M_{2q},\cdots,M_{p1},M_{p2},\cdots,M_{pq}$ are matching and edge-disjoint each other. Furthermore,

$$E(K_p[S_q]) = M_{11}\bigcup M_{12}\bigcup\cdots\bigcup M_{1q}\bigcup M_{21}\bigcup\cdots\bigcup M_{2q}\bigcup\cdots\bigcup M_{p1}\bigcup\cdots\bigcup M_{pq}.$$

We define a proper edge coloring f of $K_p[S_q]$ using colors $1,2,\cdots,pq$ as follows: assign color $q(i-1)+j$ to edge in M_{ij}, $i=1,2,\cdots,p,\ j=1,2,\cdots,q$.

The color $\frac{(2i-1)q+1}{2}$ belongs to $\overline{S}(u_i,v_j)$, but $\frac{(2i-1)q+1}{2}$ does not belong to $\overline{S}(u_k,v_l)$, thus

$$\overline{S}(u_i,v_j)\neq\overline{S}(u_k,v_l),\quad 1\leqslant i<k\leqslant p,\quad j,l=1,2,\cdots,q.$$

For $i=1,2,\cdots,p$, let

$I=\{1,2,\cdots,pq\}$, $I_i=\{(i-1)q+1,(i-1)q+2,\cdots,(i-1)q+q\}$,

$A_{ij}=\{(i-1)q+\frac{j+2}{2},(i-1)q+\frac{j+4}{2},\cdots,(i-1)q+\frac{j+q-1}{2}\}$, $j=2,4,\cdots,q-1$,

$B_{ij}=\{(i-1)q+\frac{j+1}{2},(i-1)q+\frac{j+3}{2},\cdots,(i-1)q+\frac{j+q}{2}\}$, $j=1,3,\cdots,q$.

Of course, $A_{ij}\subseteq I_i$, $B_{ij}\subseteq I_i$. Moreover, A_{ij} and B_{ij} have $\frac{q-1}{2}$ and $\frac{q+1}{2}$ consecutive natural numbers, respectively.

If $i=1,2,\cdots,p$, $q\equiv 1(\mathrm{mod}\,4)$, then

$$\overline{S}(u_i,v_j)=A_{ij}\bigcup C_{ij},\quad C_{ij}\subseteq I\setminus I_i,\ \ j=2,4,\cdots,q-1;$$

$$\overline{S}(u_i,v_j)=\Big(B_{ij}\setminus\Big\{(i-1)q+\frac{j+\frac{q+1}{2}}{2}\Big\}\Big)\bigcup C_{ij},\ \ C_{ij}\subseteq I\setminus I_i, j=1,3,\cdots,q,\ \ j\neq\frac{q+1}{2};$$

$$\overline{S}(u_i,v_{\frac{q+1}{2}})=\Big\{(i-1)q+\frac{q+1}{2}\Big\}.$$

If $i=1,2,\cdots,p$, $q\equiv 3(\mathrm{mod}\,4)$, then

$$\overline{S}(u_i,v_j)=B_{ij}\bigcup D_{ij},\quad D_{ij}\subseteq I\setminus I_i,\quad j=1,3,\cdots,q;$$

$$\overline{S}(u_i,v_j)=\Big(A_{ij}\setminus\Big\{(i-1)q+\frac{j+\frac{q+1}{2}}{2}\Big\}\Big)\bigcup D_{ij},\ D_{ij}\subseteq I\setminus I_i,\ j=2,4,\cdots,q-1, j\neq\frac{q+1}{2};$$

$$\overline{S}(u_i,v_{\frac{q+1}{2}})=\Big\{(i-1)q+\frac{q+1}{2}\Big\}.$$

Thus for each $i \in \{1,2,\cdots,p\}$ and each odd number $q(\geqslant 5)$, we have that $\overline{S}(u_i,v_j) \neq \overline{S}(u_i,v_l)$, $1 \leqslant j < l \leqslant q$.

In conclusion, the above coloring is a pq-VDPEC coloring of $K_p[S_q]$.

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