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Global nonexistence for nonlinear $p(x)$-Kirchhoff systems with dynamic boundary conditions

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Abstract: This paper considered the global non-existence of solutions of nonlinear $p(x)$-Kirchhoff systems with dynamic boundary conditions, which involve nonlinear external damping terms Q and nonlinear driving forces f. Through the study of the natural energy associated to the solutions u of the systems, the nonexistence of global solutions, when the initial energy is controlled above by a critical value was proved. And the p-Kirchhoff equations involving the quasilinear homogeneous p-Laplace operator were extended to the $p(x)$-Kirchhoff equations which have been used in the last decades to model various phenomena.

Key words: $p(x)$-Kirchhoff systems; global non-existence; nonlinear source and boundary damping terms

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非线性 $p(x)$-Kirchhoff 方程在动态边界条件下的非全局存在性

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摘要: 考虑在动态边界条件下, 非线性 $p(x)$-Kirchhoff 方程组解的非全局存在性, 该方程组带有非线性外力项 Q 和非线性源项 f. 通过研究方程组解的自然能量, 证明在初始能量小于一个临界值时, 方程组解的非全局存在性. 并将带有拟线性齐次 p-拉普拉斯算子的 p-Kirchhoff 方程组推广到 $p(x)$-Kirchhoff 方程组, 该方程组近年被用来模拟很多现象.

关键词: $p(x)$-Kirchhoff 方程组; 非全局存在性; 非线性源项和外力项

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0 Introduction

In this paper we consider the following $p(x)$-Kirchhoff systems

$$\begin{cases} u_{tt} - M(g(t))\Delta_{p(x)}u + |u|^{p(x)-2}u = f(t,x,u), & \text{in } \mathbf{R}_0^+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbf{R}_0^+ \times \Gamma_0, \\ u_{tt} = -[M(g(t))|Du|^{p(x)-2}\partial_\nu u + Q(t,x,u,u_t)], & \text{on } \mathbf{R}_0^+ \times \Gamma_1, \end{cases} \tag{0.1}$$

where $p(x) > p_n$, p_n is a critical value smaller than 2. $u = (u_1, \cdots, u_N) = u(t,x)$ is the vectorial displacement, $N \geqslant 1$, $\mathbf{R}_0^+ = [0, \infty)$. Ω is a regular and bounded domain of $\mathbf{R}^n$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \phi$, $\mu_{n-1}(\Gamma_0) > 0$, where μ_{n-1} denotes the $(n-1)$-dimensional Lebesgue measure on $\partial\Omega$, while μ_n is the n-dimensional Lebesgue measure on Ω. Moreover, ν is the outward normal vector field on $\partial\Omega$. $\Delta_{p(x)}$ denotes the vectorial $p(x)$-Laplacian operator defined as $\mathrm{div}(|Du|^{p(x)-2}Du)$, and the associated $p(x)$-Dirichlet energy integral is $g(t) = \int_\Omega \frac{|Du(t,x)|^{p(x)}}{p(x)}\mathrm{d}x$. The functions f, M and Q represent a source force, a Kirchhoff dissipative term and an external damping term, respectively. We further suppose that

$$(Q(t,x,u,v),v) \geqslant 0 \text{ for all } (t,x,u,v) \in \mathbf{R}_0^+ \times \Gamma_1 \times \mathbf{R}^N \times \mathbf{R}^N,$$
$$Q \in C(\mathbf{R}_0^+ \times \Gamma_1 \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N), f \in C(\mathbf{R}_0^+ \times \Omega \times \mathbf{R}^N \to \mathbf{R}^N),$$
$$f(t,x,u) = F_u(t,x,u), F(t,x,0) = 0,$$

so $F(t,x,u) = \int_0^1 (f(t,x,\tau u),u)\mathrm{d}\tau$ is a potential for f. The Kirchhoff dissipative term M is assumed to be of the standard form

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, a, b \geqslant 0, a+b > 0, \gamma > 1 \text{ if } b > 0. \tag{0.2}$$

We choose $\mathcal{M}(\tau) = a\tau + b\tau^\gamma$, so $\mathcal{M}(\tau) = \int_0^\tau M(z)\mathrm{d}z$, where $\gamma > 1$ if $b > 0$.

The boundary conditions considered in (0.1) are usually called dynamic boundary conditions and they arise in several physical applications (for example, see [1]). Some important and interesting results about Kirchhoff equations can be found, for example, in [2]. The study of Kirchhoff type equations has already been extended to p-Kirchhoff equations[3]. In [3], the global nonexistence results are proved for scalar Kirchhoff equations, when $Eu(0) < E_1$ and all the exponents are constant, with $p(x) \equiv 2$. In particular, they considered the p-Kirchhoff system

$$\begin{cases} u_{tt} - M(\|Du(t,\cdot)\|_p^p)\Delta_p u + \mu|u|^{p-2}u = f(t,x,u), & \text{in } \mathbf{R}_0^+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbf{R}_0^+ \times \Gamma_0, \\ M(\|Du(t,\cdot)\|_p^p)|Du|^{p-2}\partial_\nu u = Q(t,x,u,u_t), & \text{on } \mathbf{R}_0^+ \times \Gamma_1, \end{cases} \tag{0.3}$$

and obtained that any local solution u of (0.3) cannot be continued in $\mathbf{R}_0^+ \times \Omega$, whenever the initial energy is controlled by a critical value.

The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. We recall that the nonhomogeneous $p(x)$-Kirchhoff operator has been used in the last decades to model various phenomena[3,5], such as the image restoration problem, the motion of electro-rheological fluids. In particular, in [6], they considered the dissipative anisotropic nonhomogeneous $p(x)$-Kirchhoff system

$$\begin{cases} u_{tt} - M(g(t))\Delta_{p(x)}u + \mu|u|^{p(x)-2}u + Q(t,x,u,u_t) = f(t,x,u), & \text{in } \mathbf{R}_0^+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbf{R}_0^+ \times \partial\Omega, \end{cases} \tag{0.4}$$

and showed the nonexistence of global solutions of (0.4), when the initial energy is controlled by a critical value.

This paper will be organized as follows. In Section 1, we will give some preliminaries on the variable exponent space. In Section 2, we will give the main theorem and its proof. In Section 3, we will show the applications of the main theorem.

1 Preliminaries

Let $h \in C_+(\overline{\Omega})$, where $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min\limits_{x\in\overline{\Omega}} h(x) > 1\}$, and define

$$h_+ = \sup_{x\in\Omega} h(x), \quad h_- = \inf_{x\in\Omega} h(x).$$

Fix $p \in C_+(\overline{\Omega})$, then the variable exponent Lebesgue space, denoted by $L^{p(\cdot)}(\Omega) = [L^{p(\cdot)}(\Omega)]^N$, consisting of all the measurable vector-valued functions $u : \Omega \to \mathbf{R}^N$ such that $\int_\Omega |u(x)|^{p(x)}\mathrm{d}x$ is finite, is endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\mathrm{d}x \leqslant 1\Big\}.$$

Since here $0 < |\Omega| < \infty$, if $q \in C_+(\overline{\Omega})$ and $p \leqslant q$ in Ω, then the embedding $L^{q(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$ is continuous (see [5, Theorem 2.8]).

Now, we define a $p(\cdot)$-modular function of the $L^{p(\cdot)}(\Omega)$ space, that is

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\mathrm{d}x.$$

If $u \in L^{p(\cdot)}(\Omega)$, since $p_+ < \infty$, then the following relations hold:

$$\|u\|_{p(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1),$$

$$\|u\|_{p(\cdot)} \geqslant 1 \Rightarrow \|u\|_{p(\cdot)}^{p_-} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p_+}, \tag{1.1}$$

$$\|u\|_{p(\cdot)} \leqslant 1 \Rightarrow \|u\|_{p(\cdot)}^{p_+} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p_-}, \tag{1.2}$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega) = [W^{1,p(\cdot)}(\Omega)]^N$, consisting of functions $u \in L^{p(\cdot)}(\Omega)$, is endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|Du\|_{p(\cdot)}.$$

Define $W_{\Gamma_0}^{1,p(\cdot)}(\Omega)=[W_{\Gamma_0}^{1,p(\cdot)}(\Omega)]^N$ as the Sobolev space of the functions $u\in W^{1,p(\cdot)}(\Omega)$ with $u|_{\Gamma_0}=0$. If $p_+<n$ and some conditions satisfied, then the embedding $W_{\Gamma_0}^{1,p(\cdot)}(\Omega)\to L^{p^*(\cdot)}(\Omega)$ is continuous, where p^* is the critical variable exponent related to p, defined by the relation

$$p^*(x)=\frac{np(x)}{n-p(x)}\quad\text{for all } x\in\Omega.$$

We refer to more details about Sobolev space in [7]. Hereafter, we assume that

$$p(x)\in C_+(\overline{\Omega})\text{ and } 1<p_-\leqslant p_+<n.$$

For all $h\in C(\overline{\Omega})$, with $1\leqslant h\leqslant p^*$ in Ω, we denote by $\lambda_{h(\cdot)}$ the Sobolev constant, of the continuous embedding $W_{\Gamma_0}^{1,p(\cdot)}(\Omega)\to L^{h(\cdot)}(\Omega)$, that is

$$\|u\|_{h(\cdot)}\leqslant\lambda_{h(\cdot)}\|Du\|_{p(\cdot)}. \tag{1.3}$$

For simplicity in notation, we write

$$L^{p(\cdot)}(\Omega)=[L^{p(\cdot)}(\Omega)]^N, W_{\Gamma_0}^{1,p(\cdot)}(\Omega)=[W_{\Gamma_0}^{1,p(\cdot)}(\Omega)]^N,$$

which are endowed with the norms $\|\cdot\|_{p(\cdot)}$ and $\|Du\|_{p(\cdot)}$, respectively. The usual Lebesgue space $L^2(\Omega)=[L^2(\Omega)]^N$ is equipped with the canonical norm $\|\varphi\|2=(\int_\Omega|\varphi(x)|^2\mathrm{d}x)^{\frac{1}{2}}$, while the elementary bracket pairing $\langle\varphi,\psi\rangle=\int_\Omega(\varphi(x),\psi(x))\mathrm{d}x$ is clearly well defined for all ϕ,ψ such that $(\phi,\psi)\in L^1(\Omega)$. Analogously, also $\langle\omega,\phi\rangle_{\Gamma_1}=\int_{\Gamma_1}(\omega(x),\phi(x))\mathrm{d}\mu_{n-1}$ is well defined for all ω,φ such that $(\omega,\varphi)\in L^1(\Gamma_1)$. Finally

$$K=C(\mathbf{R}_0^+\to W_{\Gamma_0}^{1,p(\cdot)}(\Omega))\cap C^1(\mathbf{R}_0^+\to L^2(\Omega))$$

denotes the main solution and test function space.

2 The main theorem

Before we state our main theorem, we first assume that for all $\varphi\in K$

(F_1) $F(t,\cdot,\varphi(t,\cdot)),(f(t,\cdot,\varphi(t,\cdot)),\varphi(t,\cdot))\in L^1(\Omega)$ for all $t\in\mathbf{R}_0^+$; $\langle f(t,\cdot,\phi(t,\cdot)),\phi(t,\cdot)\rangle\in L^1_{\mathrm{loc}}(\mathbf{R}_0^+)$.

Next, we assume the following monotonicity condition

(F_2) $$\mathcal{F}_t\geqslant 0\text{ in }\mathbf{R}_0^+\times W_{\Gamma_0}^{1,p(\cdot)}(\Omega),$$

where $\mathcal{F}_t$ is the partial derivative with respect to t of $\mathcal{F}=\mathcal{F}(t,\varphi)=\int_\Omega F(t,x,\phi(t,x))\mathrm{d}x$, with (t,φ) in $\mathbf{R}_0^+\times W_{\Gamma_0}^{1,p(\cdot)}(\Omega)$, and $\mathcal{F}\varphi(t)=\mathcal{F}(t,\varphi)$ is the potential energy of the field $\varphi\in K$, which is well defined by (F_1). Moreover, the natural total energy of the field $\varphi\in K$, associated with (0.1), is

$$\begin{cases}E\phi(t)=\dfrac{1}{2}(\|\phi_t(t,\cdot)\|_2^2+\|\phi_t(t,\cdot)\|_{2,\Gamma_1}^2)+\mathcal{A}\phi(t)-\mathcal{F}\phi(t),\\ \mathcal{A}\varphi(t)=\mathcal{M}(g\varphi(t))+\displaystyle\int_\Omega\frac{|\phi(t,x)|^{p(x)}}{p(x)}\mathrm{d}x\geqslant 0,\end{cases} \tag{2.1}$$

where $g\varphi$ is the $p(\cdot)$-Dirichlet energy integral. $E\varphi$ is well defined in K.

Finally, we consider the following condition:

(F_3) There exists a function $q \in C_+(\overline{\Omega})$ satisfying the restriction

$$\max\{2, \gamma p_+\} < q_- \leqslant p^*(x), p^*(x) = \frac{np(x)}{n-p(x)}, \tag{2.2}$$

with the property that for all $a_0 > 0$ and $\varphi \in K$ for which $\inf_{t\in\mathbf{R}_0^+} \mathcal{F}\varphi(t) \geqslant a_0$, there exist $c_1 = c_1(a_0,\varphi) > 0$ and $\varepsilon_0 = \varepsilon_0(a_0,\varphi) > 0$, such that

(i) $\qquad \mathcal{F}\varphi(t) \leqslant c_1\rho_{q(\cdot)}(\varphi(t,\cdot))$ for all $t \in \mathbf{R}_0^+$,

and for all $\varepsilon \in (0,\varepsilon_0)$ there exists $c_2 = c_2(a_0,\varphi,\varepsilon) > 0$, such that

(ii) $\qquad \langle f(t,\cdot,\phi(t,\cdot)), \phi(t,\cdot)\rangle - (q_- - \varepsilon)\mathcal{F}\phi(t) \geqslant c_2\rho_{q(\cdot)}(\phi(t,\cdot))$ for all $t \in \mathbf{R}_0^+$.

Following [8], if $u \in K$ satisfies the two following properties:

(A) Distribution Identity

$$\begin{aligned}[\langle u_t,\phi\rangle]_0^t = \int_0^t \{&\langle u_t,\phi_t\rangle - M(g\phi(t))\langle |Du|^{p(\cdot)-2}Du, D\phi\rangle - \langle |u|^{p(\cdot)-2}u,\phi\rangle \\ &+\langle f(\tau,\cdot,u),\phi\rangle - \langle Q(\tau,\cdot,u,u_t) + u_{tt},\phi\rangle_{\Gamma_1}\}\mathrm{d}\tau\end{aligned}$$

for all $t \in \mathbf{R}_0^+$ and $\phi \in K$.

(B) Energy Conservation

(i) $\mathcal{D}u(t) = \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u_t(t,\cdot)\rangle_{\Gamma_1} + \mathcal{F}_t u(t) \in L^1_{\mathrm{loc}}(\mathbf{R}_0^+)$,

(ii) $Eu(t) \leqslant Eu(0) - \int_0^t \mathcal{D}u(\tau)\mathrm{d}\tau$, for all $t \in \mathbf{R}_0^+$,

we say that u is a (weak) solution of (0.1).

Moreover, by (2.2), there exists a constant c_q such that for all $u \in K$, we have

$$\rho_{q(\cdot)}(u) \leqslant c_q\rho_{p(\cdot)}(Du).$$

In this paper, we show that any local solution u of (0.1) cannot be continued in $\mathbf{R}_0^+ \times \Omega$, whenever the initial energy is controlled above by a critical value. Now we state our main theorem as follows.

Theorem 2.1 Take $p \in (p_n, n)$. Assume (F_1)—(F_3) and the following conditions are satisfied.

- $Eu(0) < (1 - \frac{\gamma p_+}{q_-})\omega_1 = E_1$, where $\omega_1 = \inf_{t\in\mathbf{R}_0^+}\mathcal{A}u(t)$.
- There exist $T \geqslant 0$, $q_1 > 0$, m, ζ, with $1 < m < \zeta - k_0$, $0 \leqslant k_0 \leqslant p_+(1-\frac{m}{\zeta})$ and $2 \leqslant \zeta < \zeta_0$ with ζ_0 will be defined in Lemma 2.5, and non-negative functions $\delta \in L^\infty_{\mathrm{loc}}(J)$, $\psi, k \in W^{1,1}_{\mathrm{loc}}(J)$, $J = [T,\infty)$, with $k' \geqslant 0$, $\psi > 0$ in J and $\psi'(t) = o(\psi(t))$ as $t \to \infty$, such that

$$\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle_{\Gamma_1} \leqslant q_1\delta(t)^{\frac{1}{m}}\mathcal{D}u(t)^{\frac{1}{m'}}\,\|u(t,\cdot)\|_{\zeta,\Gamma_1}^{1+\frac{k_0}{m}} \tag{2.3}$$

for all $t \in J$, and

$$\delta \leqslant \Big(\frac{k}{\psi}\Big)^{m-1} \text{ in } J, \int_T^\infty \psi(t)[\max\{k(t),\psi(t)\}]^{-(1+\theta)}\mathrm{d}t = \infty, \tag{2.4}$$

for some appropriate constant $\theta \in (0,\theta_0)$, where

$$\theta_0 = \min\Big\{\frac{q_- - 2}{q_- + 2}, \frac{\overline{r}}{1-\overline{r}}\Big\},\ \overline{r} = \frac{1}{\zeta} - \Big(\frac{1-s}{q_-} + \frac{s}{p_+}\Big) \in (0,1),\ \ s = \frac{n}{p_+} - \frac{n-1}{\zeta_0} \in (0,1). \tag{2.5}$$

Then there are no solutions $u \in K$ of (0.1) in $\mathbf{R}_0^+ \times \Omega$.

Remark 2.2 For all $\varphi \in K$ and $(t,x) \in \mathbf{R}_0^+ \times \Omega$, we define pointwise

$$A\varphi(t,x) = -M(g\varphi(t))\Delta_{p(x)}\varphi(t,x) + |\varphi(t,x)|^{p(x)-2}\varphi(t,x), \tag{2.6}$$

so that A is the Fréchet derivative of $\mathcal{A}$ with respect to φ. By (0.2), (2.1), we have, as $\gamma \geqslant 1$,

$$\begin{aligned}\langle A\phi(t,\cdot), \phi(t,\cdot)\rangle &= M(g\phi(t))\rho_{p(\cdot)}(D\phi(t,\cdot)) + \rho_{p(\cdot)}(\phi(t,\cdot)) \\ &\leqslant p_+\Big\{g\phi(t)M(g\phi(t)) + \int_\Omega \frac{|\phi(t,x)|^{p(x)}}{p(x)}\mathrm{d}x\Big\} \leqslant \gamma p_+\mathcal{A}\phi(t).\end{aligned} \tag{2.7}$$

Remark 2.3 If $u \in K$ is a solution of (0.1) in $\mathbf{R}_0^+ \times \Omega$, then by (2.1) there exists always $\omega_1 \geqslant 0$ such that $\mathcal{A}u(t) \geqslant \omega_1$ for all $t \in \mathbf{R}_0^+$. Moreover, by (2.1), (B)—(ii) and (F_2) we get $\mathcal{F}u(t) \geqslant \omega_1 - Eu(0) \geqslant -Eu(0)$ for all $t \in \mathbf{R}_0^+$, in other words $\mathcal{F}u$ is bounded from below in $\mathbf{R}_0^+$ along any solution $u \in K$.

Lemma 2.4 Assume (0.2), (F_1) and (F_2) hold. If $u \in K$ is a solution of (0.1) in $\mathbf{R}_0^+ \times \Omega$ then $\omega_2 = \inf_{t\in\mathbf{R}_0^+}\mathcal{F}u(t) > -\infty$. If there exists $\overline{\omega} > -1$ such that $Eu(0) < \overline{\omega}\omega_2$, then $\omega_2 > 0$. Moreover, if also $(F_3)-$(i) holds, then $\omega_1 > 0$.

Proof Let $u \in K$ be a solution of (0.1) in $\mathbf{R}_0^+ \times \Omega$. Clearly, $\mathcal{A}u$ and $\mathcal{F}u$ are bounded from below in $\mathbf{R}_0^+$ as shown in Remark 2.3. In particular $\omega_2 > -\infty$ and $\inf_{t\in\mathbf{R}_0^+}\mathcal{A}u(t) = \omega_1 \geqslant 0$. Assume that $Eu(0) < \overline{\omega}\omega_2$, with $\overline{\omega} > -1$. Then $\mathcal{F}u(t) \geqslant \omega_1 - Eu(0) > \omega_1 - \overline{\omega}\omega_2$, which gives $\omega_2 > \frac{\omega_1}{1+\overline{\omega}} \geqslant 0$ and so $\omega_2 > 0$.

Suppose that also (F_3)(i) holds. In correspondence with $a_0 = \omega_2 > 0, \varphi = u \in K$, there exist $c_1 = c_1(\omega_2, u) > 0$ and $\varepsilon_0 = \varepsilon_0(\omega_2, u) > 0$ for which (F_3)(i) is valid along u, so for all $t \in \mathbf{R}_0^+$, $\rho_{q(\cdot)}(u(t,\cdot)) \geqslant \hat{c}_1 > 0$, $\rho_{p(\cdot)}(Du) \geqslant \frac{\hat{c}_1}{c_q}$, where $\hat{c}_1 = \frac{\omega_2}{c_1}$ by embedding theorems. Hence by (0.2) and (2.1), $\mathcal{A}u(t) \geqslant a(gu(t)) + b(gu(t))^{\gamma-1}gu(t) \geqslant a_1\rho_{p(\cdot)}(Du)$, for $a_1 = \frac{a}{p_+} + b\frac{\hat{c}_1^{\gamma-1}}{c_q^{\gamma-1}p_+^\gamma} > 0$. In particular, $\omega_1 \geqslant a_1\rho_{p(\cdot)}(Du) > 0$, and the lemma is proved.

Lemma 2.5 If $p > p_n$, $\frac{2n}{n-2} < p_n = \frac{1}{2}[\sqrt{(n+1)^2+4n} + 1 - n] < 2$, then $\zeta_0 = \frac{p_+q_-(n-1+p_+)-p_+^2}{n(q_--p_+)+p_+^2} \in (\max\{2, p_+\}, \min\{p^*(x), q_-\})$.

The proof similar with the proof of Proposition 3.1 in [4], so we omit it.

Proof of theorem 2.1 Suppose as a contradiction that there exists a global solution $u \in K$ of (0.1) in $\mathbf{R}_0^+ \times \Omega$. By Lemma 2.4. and (2.2), we have $E_1 > 0$. Fix $E_2 \geqslant 0$ such that $E_2 \in (Eu(0), E_1)$ and take $\varepsilon_0 > 0$ so small that

$$\varepsilon_0\omega_1 \leqslant (q_- - \gamma p_+)\omega_1 - q_-E_2. \tag{2.8}$$

This choice is possible since $\omega_1 > 0$ and $E_2 < E_1$. Note that (2.8) forces $\varepsilon_0 \leqslant q_- - \gamma p_+$ as $E_2 \geqslant 0$. Define the function $H(t) = E_2 - Eu(0) + \int_0^t \mathcal{D}u(\tau)\mathrm{d}\tau$ for each $t \in \mathbf{R}_0^+$. Of course H is well defined and non-decreasing by (B)—(i) and (F_2), being $\mathcal{D} \geqslant 0$ and finite along u. Hence, by (B)—(ii),

$$E_2 - Eu(t) \geqslant H(t) \geqslant H_0 = E_2 - Eu(0) > 0 \text{ for } t \in \mathbf{R}_0^+, \tag{2.9}$$

where $H_0 = H(0)$. Moreover, by (2.8), (2.1), the choice of E_2, the definition of ω_2 and the inequality $\omega_2 > \frac{\gamma p_+ \omega_1}{q_-}$, it follows that for all $t \in \mathbf{R}_0^+$,

$$H(t) \leqslant E_2 - Eu(t) < E_1 + \mathcal{F}u(t) \leqslant \Big(\frac{q_-}{\gamma p_+} - 1\Big)\mathcal{F}u(t) + \mathcal{F}u(t) = \frac{q_-}{\gamma p_+}\mathcal{F}u(t). \tag{2.10}$$

Fix $\varepsilon \in (0, \varepsilon_0)$, if we put $\varphi = u$ in the Distribution Identity, we obtain by (2.1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\langle u_t, u\rangle + \langle u_t, u\rangle_{\Gamma_1}\} = {} & \|u_t(t,\cdot)\|_2^2 - \langle Au(t,\cdot), u(t,\cdot)\rangle + \langle f(t,\cdot,u), u\rangle - \langle Q(t,\cdot,u,u_t), u\rangle_{\Gamma_1} \\ & + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2 \\ = {} & c_3(\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2) + (q_- - \varepsilon)\mathcal{A}u(t) - \langle Au(t,\cdot), u(t,\cdot)\rangle \\ & + \langle f(t,\cdot,u), u\rangle - (q_- - \varepsilon)\mathcal{F}u(t) - (q_- - \varepsilon)Eu(t) - \langle Q(t,\cdot,u,u_t), u\rangle_{\Gamma_1},\end{aligned}$$

where $c_3 = 1 + \frac{q_- - \varepsilon}{2} > 0$ by the choice of ε. Using (2.7) and (F_3)(ii) with $c_2 = c_2(\omega_2, u, \varepsilon) > 0$, we obtain for all $t \in \mathbf{R}_0^+$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\langle u_t, u\rangle + \langle u_t, u\rangle_{\Gamma_1}\} \geqslant {} & c_3(\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2) + c_2\rho_{q(\cdot)}(u(t,\cdot)) - (q_- - \varepsilon)Eu(t) \\ & - \langle Q(t,\cdot,u,u_t), u\rangle_{\Gamma_1} + (q_- - \varepsilon - \gamma p_+)\mathcal{A}u(t).\end{aligned}$$

Since $\varepsilon < q_- - \gamma p_+$ by (2.8) and $Eu \leqslant E_2 - H$ by (2.9),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\langle u_t, u\rangle + \langle u_t, u\rangle_{\Gamma_1}\} \geqslant {} & c_3(\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2) + c_2\rho_{q(\cdot)}(u(t,\cdot)) \\ & + (q_- - \varepsilon - \gamma p_+)\mathcal{A}u(t) - \langle Q(t,\cdot,u,u_t), u\rangle_{\Gamma_1} + \gamma p_+ H(t) - (q_- - \varepsilon)E_2.\end{aligned}$$

Now set $C_2 = \frac{(q_- - \varepsilon - \gamma p_+)\varepsilon a_1}{q_-} > 0$, so

$$\begin{aligned}& (q_- - \varepsilon - \gamma p_+)\mathcal{A}u(t) - (q_- - \varepsilon)E_2 \geqslant (q_- - \varepsilon - \gamma p_+)\Big(1 - \frac{q_- - \varepsilon}{q_-}\Big)\mathcal{A}u(t) \\ & + (q_- - \varepsilon - \gamma p_+)\frac{q_- - \varepsilon}{q_-}\omega_1 - (q_- - \varepsilon)E_2 \geqslant C_2\rho_{p(\cdot)}(Du(t,\cdot)),\end{aligned}$$

by $\mathcal{A}u(t) \geqslant a_1\rho_{p(\cdot)}(Du(t,\cdot))$ in Lemma 2.4 and the fact $\frac{(q_- - \varepsilon)[(q_- - \varepsilon - \gamma p_+)\omega_1 - q_- E_2]}{q_-} \geqslant 0$ thanks to (2.8). Consequently, putting $c_2 = \min\{c_2, C_2\} > 0$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\langle u_t, u\rangle + \langle u_t, u\rangle_{\Gamma_1}\} \geqslant {} & c_3(\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2) + c_2(\rho_{q(\cdot)}u(t,\cdot) \\ & + \rho_{p(\cdot)}Du(t,\cdot)) - \langle Q(t,\cdot,u,u_t), u\rangle_{\Gamma_1} + \gamma p_+ H(t).\end{aligned} \tag{2.11}$$

Since $\zeta < \zeta_0$ there exists $S_0 > 0$ such that $\|u(t,\cdot)\|_{\zeta,\Gamma_1} \leqslant S_0\|u(t,\cdot)\|_{\zeta_0,\Gamma_1}$. On the other hand, by the choice of s in (2.5), as $\zeta_0 > p(\cdot)$ by Lemma 2.4, we have

$$\|u(t,\cdot)\|_{\zeta,\Gamma_1} \leqslant S\|u(t,\cdot)\|_{q(\cdot)}^{1-s}\|Du(t,\cdot)\|_{p(\cdot)}^{s}, \tag{2.12}$$

where S is an appropriate constant. The proof see [4]. Furthermore, $\frac{n}{p_+}-\frac{n-1}{\zeta}<s<\frac{(\frac{q_-}{\zeta}-1)}{(\frac{q_-}{p_+}-1)}$ as $\zeta<\zeta_0$. Let $\alpha_1,\ \alpha_2$ and $\beta_1,\ \beta_2$ denote the numbers

$$\frac{1}{\alpha_1}=\frac{1}{m}-\frac{s}{p_+}\Big(1+\frac{k_0}{m}\Big),\ \beta_1=(1-s)\Big(1+\frac{k_0}{m}\Big)-q_-\Big\{\frac{1}{m}-\frac{s}{p_+}\Big(1+\frac{k_0}{m}\Big)\Big\},$$
$$\frac{1}{\alpha_2}=\frac{1}{\zeta}-\frac{s}{p_+},\ \beta_2=1-s-q_-\Big(\frac{1}{\zeta}-\frac{s}{p_+}\Big).$$

We can prove that $1<\alpha_1\leqslant\alpha_2$ and $\beta_1\leqslant\beta_2<0$. Hence, using (2.3) and (2.12), we get for all $t\in J$,

$$\begin{aligned}\langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1}&\leqslant q_1(\delta(t)^{\frac{1}{m-1}}\mathcal{D}u(t))^{\frac{1}{m'}}(S\left\|u(t,\cdot)\right\|_{q(\cdot)}^{1-s}\left\|Du(t,\cdot)\right\|_{p(\cdot)}^{s})^{1+\frac{k_0}{m}}\\&=q_2(\delta(t)^{\frac{1}{m-1}}\mathcal{D}u(t))^{\frac{1}{m'}}\left\|u(t,\cdot)\right\|_{q(\cdot)}^{(1-s)(1+\frac{k_0}{m})}\left\|Du(t,\cdot)\right\|_{p(\cdot)}^{s(1+\frac{k_0}{m})}\\&=q_2(\delta(t)^{\frac{1}{m-1}}\mathcal{D}u(t))^{\frac{1}{m'}}\left\|u(t,\cdot)\right\|_{q(\cdot)}^{\frac{q_-}{\alpha_1}}\left\|Du(t,\cdot)\right\|_{p(\cdot)}^{s(1+\frac{k_0}{m})}\left\|u(t,\cdot)\right\|_{q(\cdot)}^{\beta_1},\end{aligned}$$

where $q_2=q_1S^{1+\frac{k_0}{m}}$. Let $l\in(0,1)$. Applying Young's inequality, then it gives

$$\begin{aligned}\langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1}&\leqslant q_2\left(\frac{2\delta(t)}{l}\right)^{\frac{1}{m-1}}\mathcal{D}u(t))+\frac{1}{2}l\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_-}+\frac{1}{2}l\left\|Du(t,\cdot)\right\|_{p(\cdot)}^{p_+}]\left\|u(t,\cdot)\right\|_{q(\cdot)}^{\beta_1}\\&\leqslant q_2[l^{-\frac{m'}{m}}\delta(t)^{\frac{1}{m-1}}\mathcal{D}u(t)+l(\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_-}+\left\|Du(t,\cdot)\right\|_{p(\cdot)}^{p_+})]\left\|u(t,\cdot)\right\|_{q(\cdot)}^{\beta_2},\end{aligned}\tag{2.13}$$

where $\widetilde{q}_2=q_2 2^{\frac{1}{m-1}}\max\{1,(\widehat{c}_1)^{\beta_1-\beta_2}\}>0$. By direct calculation, we have $\overline{r}=-\frac{\beta_2}{q_-}\in(0,1)$.

Moreover, by (F_3)(i), if $\left\|u(t,\cdot)\right\|_{q(\cdot)}\geqslant1$ then $\mathcal{F}u(t)\leqslant c_1\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_-}$ by (1.1). On the other hand, if $\left\|u(t,\cdot)\right\|_{q(\cdot)}\leqslant1$ then $\omega_2\leqslant c_1(t,\cdot)q(\cdot)^{q_-}$ by (F_3)(i), the definition of ω_2 and (1.2). Hence $\left\|u(t,\cdot)\right\|_{q(\cdot)}\geqslant(\omega_2/c_1)^{1/q_-}>0$, so that $\mathcal{F}u(t)\leqslant c_1\rho_{q(\cdot)}(u(t,\cdot))\leqslant c_1(\frac{c_1}{\omega_2})^{\frac{q_+}{q_-}}\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_+}$ by (F_3)(i). In conclusion, if u is the solution of (0.1), then we have for all $t\in\mathbf{R}_0^+$, $\mathcal{F}u(t)\leqslant c_1'\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_+}$ with $c'_1=\max\{c_1,c_1(\frac{c_1}{\omega_2})^{\frac{q_+}{q_-}}\}$. Since $\|u\|_{q(\cdot)}$ is finite, we get $\mathcal{F}u(t)\leqslant c_1'\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_+}\leqslant\bar{c}_1\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_-}$, for some $\overline{c}_1>0$. Then by (2.10), we have

$$\left\|u(t,\cdot)\right\|_{q(\cdot)}^{\beta_2}=\left\|u(t,\cdot)\right\|_{q(\cdot)}^{-\bar{r}q_-}\leqslant\bar{c}_1^{\bar{r}}\mathcal{F}u(t)^{-\bar{r}}\leqslant\left(\frac{\bar{c}_1q_-}{\gamma p_+}\right)^{\bar{r}}[H(t)]^{-\bar{r}}.$$

Therefore, for all $t\in J$,

$$\langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1}\leqslant c_4[l^{-\frac{m'}{m}}\delta(t)^{\frac{1}{m-1}}\mathcal{D}u(t)+l(\left\|u(t,\cdot)\right\|_{q(\cdot)}^{q_-}+\left\|Du(t,\cdot)\right\|_{p(\cdot)}^{p_+})][H(t)]^{-\bar{r}}$$

where $c_4=\widetilde{q}_2(\frac{\overline{c}_1\,q_-}{\gamma p_+})^{\overline{r}}$. Put

$$r_0=\min\Big\{\overline{r},\frac{1}{2}-\frac{1}{q_-}\Big\}.\tag{2.14}$$

Note that θ_0 in (2.5) can be expressed as $\theta_0 = \frac{r_0}{1-r_0}$, and take from now on $r = \frac{\theta}{1+\theta}$, so that $r \in (0, r_0)$. Consequently, we get

$$\begin{aligned}\langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1} \leqslant c_4\{lH_0^{-\bar{r}}(\|u(t,\cdot)\|_{q(\cdot)}^{q_-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p_+}) \\ + l^{-\frac{m'}{m}} H_0^{r-\bar{r}}\delta(t)^{\frac{1}{m-1}}[H(t)]^{-r}\mathcal{D}u(t)\},\end{aligned} \tag{2.15}$$

where in the last step we have used that $0 < r < r_0 < \overline{r}$ by (2.14) and $H > H_0$ by (2.9).

Define the auxiliary function

$$Z(t) = \lambda k(t)[H(t)]^{1-r} + \psi(t)\{\langle u_t(t,\cdot), u(t,\cdot)\rangle + \langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1}\}$$

for all $t \in J$, and $\lambda > 0$ to be fixed later. Clearly $Z \in W^{1,1}_{\mathrm{loc}}(J)$, so a.e. in J. On the one hand

$$\begin{aligned}Z'(t) = \lambda k(1-r)H^{-r}H' + \lambda k' H^{1-r} + \psi'\{\langle u_t(t,\cdot), u(t,\cdot)\rangle + \langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1}\} \\ + \psi\frac{\mathrm{d}}{\mathrm{d}t}\{\langle u_t(t,\cdot), u(t,\cdot)\rangle + \langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1}\}.\end{aligned} \tag{2.16}$$

Since Cauchy's and Young's inequalities, and the definition of K, we get

$$|\langle u_t(t,\cdot), u(t,\cdot)\rangle| \leqslant \|u_t(t,\cdot)\|_2\|u(t,\cdot)\|_2 \leqslant \|u_t(t,\cdot)\|_2^2 + \|u(t,\cdot)\|_2^2.$$

Consider now the relation $z^{\xi} \leqslant z + 1 \leqslant (1 + \frac{1}{\eta})(z + \eta)$, which holds for all $z \geqslant 0$, $\xi \in [0, 1]$, $\eta > 0$, and take $z = \|u(t,\cdot)\|_2^{q_-}$, $\xi = \frac{2}{q_-} < 1$, since $q_- > 2$ by (2.2), and $\eta = H_0$. We obtain that $\|u(t,\cdot)\|_2^2 \leqslant (1 + \frac{1}{H_0})(\|u(t,\cdot)\|_2^{q_-} + H_0)$. Since the embedding $L^{q(\cdot)}(\Omega) \to L^2(\Omega)$ is continuous by (2.2), there exists a positive constant B, independent of u, such that $\|u(t,\cdot)\|_2 \leqslant B\|u(t,\cdot)\|_{q(\cdot)}$. So we have

$$\|u(t,\cdot)\|_2^2 \leqslant c_5\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p_+} + H(t)\}$$

where $c_5 = (1 + \frac{1}{H_0})\max\{1, B^{q_-}\} > 0$, as $H > H_0$ in J by (2.9). Analogously, using again Cauchy's and Young's inequalities, we get

$$|\langle u_t(t,\cdot), u(t,\cdot)\rangle|_{\Gamma_1} \leqslant \|u_t(t,\cdot)\|_{2,\Gamma_1}\|u(t,\cdot)\|_{2,\Gamma_1} \leqslant \|u_t(t,\cdot)\|_{2,\Gamma_1}^2 + \|u(t,\cdot)\|_{2,\Gamma_1}^2$$

Fix $\alpha = \frac{1}{1-r}$. By the choice (2.14) of r and r_0, $\alpha \in (1, 2)$. Put $\nu = \frac{2}{\alpha}$ so that $\nu > 1$. Take $z = \|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'}$, $\xi = \frac{2}{\alpha\nu'}$ and $\eta = H_0$. We get $\|u(t,\cdot)\|_{2,\Gamma_1}^2 \leqslant (1 + \frac{1}{H_0})(H_0 + \|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'})$. Using the proof in [4], we have

$$\begin{aligned}&\|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'} \leqslant c_6\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p_+}\}, \\ &\|u(t,\cdot)\|_{2,\Gamma_1}^2 \leqslant c_{5'}\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p_+} + H(t)\},\end{aligned} \tag{2.17}$$

with $c_6 > 0, c'_5 > 0$. Combining these facts with (1.6), and inserting them into (2.16), we have

$$\begin{aligned}Z' \geqslant{}& \lambda k(1-r)H^{-r}H' + \lambda k' H^{1-r} - \psi'\{\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2 \\ &+ (c_5 + c_{5'})(\|u(t,\cdot)\|_{q(\cdot)}^{q_-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p_+} + H(t))\} + \psi\{c_3(\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2) \\ &+ c_2(\rho_{q(\cdot)}(u(t,\cdot)) + \rho_{p(\cdot)}(Du(t,\cdot))) - \langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1} + \gamma p_+ H(t)\}.\end{aligned}$$

If $t \in \mathbf{R}_0^+$ and $\rho_{q(\cdot)}(u(t,\cdot)) \geqslant 1$, then $\rho_{q(\cdot)}(u(t,\cdot)) \geqslant \|u(t,\cdot)\|_{q(\cdot)}^{q-}$, by (1.1). On the other hand, $\rho_{q(\cdot)}(u(t,\cdot)) \leqslant 1$, then $\omega_2 \leqslant c_1\|u(t,\cdot)\|_{q(\cdot)}^{q-}$ by (F$_3$)(i), by ω_2 and (0.2). Hence $\|u(t,\cdot)\|_{q(\cdot)} \geqslant (\omega_2/c_1)^{1/q-} > 0$, so that $\rho_{q(\cdot)}(u(t,\cdot)) \geqslant \frac{c_1}{\omega_2}\|u(t,\cdot)\|_{q(\cdot)}^{q-}$. Hence for all $t \in \mathbf{R}_0^+$, we get $\rho_{q(\cdot)}(u(t,\cdot)) \geqslant \min\{1, \frac{c_1}{\omega_2}\}\|u(t,\cdot)\|_{q(\cdot)}^{q-} = c_0\|u(t,\cdot)\|_{q(\cdot)}^{q-}$ with $c_0 = \min\{1, \frac{c_1}{\omega_2}\}$.

Likewise, if $t \in \mathbf{R}_0^+$ and $\rho_{p(\cdot)}(Du(t,\cdot)) \geqslant 1$, then $\rho_{p(\cdot)}(Du(t,\cdot)) \geqslant 1$, by (1.1). Otherwise $\rho_{p(\cdot)}(Du(t,\cdot)) \leqslant 1 \Leftrightarrow \|Du(t,\cdot)\|_{p(\cdot)} \leqslant 1$, which gives to

$$\|Du(t,\cdot)\|_{p(\cdot)}^{p-} \geqslant \rho_{p(\cdot)}(Du(t,\cdot)) \geqslant \frac{\hat{c}_1}{c_q} > 0$$

by $\rho_{p(\cdot)}(Du(t,\cdot)) \geqslant \frac{\hat{c}_1}{c_q}$ in Lemma 2.4. We get $\|Du(t,\cdot)\|_{p(\cdot)} \geqslant (\frac{\hat{c}_1}{c_q})^{\frac{1}{p-}} > 0$, so that $\rho_{p(\cdot)}(Du(t,\cdot)) \geqslant \frac{c_q}{\hat{c}_1}\|Du(t,\cdot)\|_{p(\cdot)}^{p-}$. Hence for all $t \in \mathbf{R}_0^+$, we get

$$\rho_{p(\cdot)}(Du(t,\cdot)) \geqslant \min\Big\{1, \frac{c_q}{\hat{c}_1}\Big\}\|Du(t,\cdot)\|_{p(\cdot)}^{p-} \geqslant C_0\|Du(t,\cdot)\|_{p(\cdot)}^{p+}$$

for some $C_0 > 0$, since $\|u(t,\cdot)\|_{p(\cdot)}$ is finite. Putting $c_0 = \min\{c_0, C_0\}$. By (2.4) and (2.14) and the fact that $\lambda k' H^{1-r} \geqslant 0$, it follows that a.e. in J

$$\begin{aligned}
Z' \geqslant\; & k\{\lambda(1-r) - c_4 l^{-\frac{m'}{m}} H_0^{r-\bar{r}}\}H^{-r}H' + \psi\Big\{\gamma p_+ - (c_5 + c_{5'})\Big|\frac{\psi'}{\psi}\Big|\Big\}H \\
& + \psi\Big\{c_3 - \Big|\frac{\psi'}{\psi}\Big|\Big\}(\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2) \\
& + \psi\Big\{c_2 c_0 - c_4 l H_0^{-\bar{r}} - (c_5 + c_{5'})\Big|\frac{\psi'}{\psi}\Big|\Big\}(\|u(t,\cdot)\|_{q(\cdot)}^{q-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p+}).
\end{aligned}$$

Now, since $\psi'(t) = o(\psi(t))$ as $t \to \infty$, there exists $T_1 \in J$ such that

$$2\Big|\frac{\psi'}{\psi}\Big| \leqslant \min\Big\{c_3, \frac{\gamma p_+}{c_5 + c_5}, \frac{c_2 c_0}{c_5 + c_5}\Big\}$$

for all $t \in J_1 = [T_1, \infty)$. Moreover, we take $l > 0$ so small such that $4c_4 l \leqslant c_2 c_0 H_0^{\overline{r}}$ and $\lambda > 0$ so large that $\lambda \geqslant \max\{\frac{c_4 H_0^{r-\overline{r}}}{l^{\frac{m'}{m}}(1-r)}, 1\}$ and $Z(T_1) > 0$. Therefore, for a.e. $t \in J_1$,

$$Z'(t) \geqslant c\psi(t)\{H(t) + \|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2 + \|u(t,\cdot)\|_{q(\cdot)}^{q-} + \|Du(t,\cdot)\|_{p(\cdot)}^{p+}\} \tag{2.18}$$

where $2c = \min\{\gamma p_+, c_3, \frac{c_2 c_0}{2}\}$. Since $k(T_1), H(T_1) > 0$, in particular $Z(t) > Z(T_1) > 0$ for all $t \in J_1$.

On the other hand, from the definition of Z, we obtain

$$\begin{aligned}
Z(t) &\leqslant \lambda k(t)H(t)^{\frac{1}{\alpha}} + \psi(t)\{|\langle u_t(t,\cdot), u(t,\cdot)\rangle| + |\langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1}|\} \\
&\leqslant \lambda k(t)H(t)^{\frac{1}{\alpha}} + \psi(t)\{\|u_t(t,\cdot)\|_2\|u(t,\cdot)\|_2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}\|u(t,\cdot)\|_{2,\Gamma_1}\}.
\end{aligned} \tag{2.19}$$

Using once more the relation $z^\xi \leqslant z + 1 \leqslant (1 + \frac{1}{\eta})(z + \eta)$, with $z = \|u(t,\cdot)\|_2^{q-}$, $\xi = \frac{\alpha\nu'}{q_-}$ and $\eta = H_0$. Since $\nu = \frac{2}{\alpha}$, $\alpha = \frac{1}{1-r}$, we get $\frac{1}{\alpha\nu'} = \frac{\nu-1}{\alpha\nu} = \frac{1}{\alpha} - \frac{1}{2} = \frac{1}{2} - r > \frac{1}{q_-}$, then $\xi < 1$. It follows by (2.9) that

$$\|u(t,\cdot)\|_2^{\alpha\nu'} \leqslant \Big(1 + \frac{1}{H_0}\Big)(H_0 + \|u(t,\cdot)\|_2^{q-}) \leqslant c_5(H(t) + \|u(t,\cdot)\|_{q(\cdot)}^{q-}) \tag{2.20}$$

where c_5 defined as before. Hence using (2.17) and (2.20), from (2.19), we get

$$
\begin{aligned}
Z^{\alpha} &\leqslant 4^{\alpha-1}[\max\{\lambda k(t),\psi(t)\}]^{\alpha}\{H(t)+\|u_t(t,\cdot)\|_2^{\alpha\nu}+\|u(t,\cdot)\|_2^{\alpha\nu'}+\|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu}+\|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'}\}\\
&\leqslant B[\max\{\lambda k(t),\psi(t)\}]^{\alpha}\{H(t)+\|u_t(t,\cdot)\|_2^{2}+\|u_t(t,\cdot)\|_{2,\Gamma_1}^{2}+\|u(t,\cdot)\|_{q(\cdot)}^{q_-}+\|Du(t,\cdot)\|_{p(\cdot)}^{p_+}\},
\end{aligned}
$$

where $B=4^{\alpha-1}(c_5+\max\{1,c_6\})$ for a.e. $t\in J_1$.

Combing this with (2.16) and $\lambda\geqslant 1$, we obtain a.e. in J,

$$Z^{-\alpha}Z'\geqslant \frac{c\psi(t)}{B\lambda^{\alpha}[\max\{k,\psi\}]^{\alpha}}.$$

Finally, since $\alpha=1+\theta$, as $r=\frac{\theta}{1+\theta}$, we see that Z cannot be global by (2.4). We finish the proof.

3 Applications

In this section we provide some concrete examples of functions f and Q, and give useful applications to the main Theorem 2.1. Assume that

$$f(t,x,u)=g(t,x)\,|u|^{\sigma(x)-2}\,u+c(x)\,|u|^{q(x)-2}\,u, \tag{3.1}$$

where $\sigma,q\in C_+(\overline{\Omega})$, $c\in L^{\infty}(\Omega)$ is a non-negative function, $g\in C(\mathbf{R}_0^+\times\Omega)$ is differentiable with respect to t and $g_+\in C(\mathbf{R}_0^+\times\Omega)$. Moreover, assume

$$
\begin{cases}
\sigma_+\leqslant q_-, \max\{2,\gamma p_+\}<q_-\leqslant q\leqslant p^* \ \text{ in }\ \Omega, c=\|c\|_{\infty}>0;\\
0\leqslant -g(t,x), g_t(t,x)\leqslant h(x)\ \text{ in }\ \mathbf{R}_0^+\times\Omega,\ \text{ for some }\ h\in L^1(\Omega);\\
g(t,\cdot)\in L^{\eta(\cdot)}(\Omega)\ \text{ in }\ \mathbf{R}_0^+\times\Omega,\ \text{ where }\ \eta(\cdot)=\begin{cases} q(x)/[q(x)-\sigma(x)], & if\ \ \sigma_+<q_-,\\ \infty, & if\ \ \sigma_+=q_-.\end{cases}
\end{cases}
\tag{3.2}
$$

The next lemma says that the function f given in (3.1)—(3.2) satisfies the principal structural assumptions (F_1)—(F_3).

Lemma 3.1[4,Lemma 4.1] Assume that the external force f is of the type given in (3.1) and (3.2). Then (F_1)—(F_2) and (F_3)(i) hold. Furthermore, if in addition

$$\sigma_+<q_-\ \text{ and }\ \bar{c}=\text{ess}\ \inf_{\overline{\Omega}} c(x)>0, \tag{3.3}$$

then (F_3)(ii) is verified, and in particular

$$\langle f(t,\cdot,\phi(t,\cdot)),\phi(t,\cdot)\rangle\geqslant q_-\mathcal{F}\phi(t), \tag{3.4}$$

for all $\phi\in K$ and $t\in\mathbf{R}_0^+$.

In the same manner as for f, we provide a concrete function Q which represents the typical nonlinear boundary damping for (0.1). This is done with the following:

Lemma 3.2[4,Lemma4.2] Assume that the continuous damping function Q given in the Introduction verifies also the following pointwise condition:

(Q) There exist constants $t_Q \geqslant 0, m, \zeta$ satisfy $1 < m < \zeta - k_0$, $0 \leqslant k_0 \leqslant p_+(1-\frac{m}{\zeta})$ and $2 \leqslant \zeta < \zeta_0$, ζ_0 is defined in Lemma 2.5, and non-negative function $d \in C(\mathbf{R}_0^+ \to L^{\zeta/\zeta-k_0-m}(\Gamma_1))$ such that

$$|Q(t,x,u,v)| \leqslant (d(t,x)\,|u|^{k_0})^{1/m}(Q(t,x,u,v),v)^{1/m'}, \tag{3.5}$$

where $(t,x,u,v) \in [t_Q,\infty) \times \Gamma_1 \times \mathbf{R}^N \times \mathbf{R}^N$. Then (2.3) is satisfied along any solution u of the problem (0.1), with $T \geqslant t_Q$, $\delta(t) = \|d(t,\cdot)\|_{\zeta/\zeta-k_0-m,\Gamma_1}$, provided that (F_2) holds.

Lemma 3.3[4,Lemma4.3] Assume (3.1)—(3.2). If $u \in K$ is a solution of (0.1) in $\mathbf{R}_0^+ \times (\Omega)$, then for all $t \in \mathbf{R}_0^+$,

$$Eu(t) \geqslant \frac{s}{(\Lambda p_+)^\gamma} \min\{v(t)^{p_-}, v(t)^{p_+}\}^\gamma - \frac{c}{q_-} \max\{v(t)^{p_-}, v(t)^{p_+}\},$$

where $v(t) = \|u(t,\cdot)\|_{q(\cdot)}$,

$$\Lambda = \max\{\lambda_{q(\cdot)}^{p_+}, \lambda_{q(\cdot)}^{p_-}, (s\gamma/cp_+^{\gamma-1})^{1/\gamma}\}, \tag{3.6}$$

and $\lambda_{q(\cdot)}$ is the constant introduced in (1.3).

From Lemma 3.3 we obtain

$$Eu(t) \geqslant \varphi(v(t)) \ \text{ for all } \ t \in \mathbf{R}_0^+, \tag{3.7}$$

where $\varphi : \mathbf{R}_0^+ \to \mathbf{R}$ is defined by $\varphi(v) = \varphi_1(v)$ if $v \in [0,1]$, while $\varphi(v) = \varphi_2(v)$ if $v \geqslant 1$, with $\varphi_1(v) = \frac{s}{(\Lambda p_+)^\gamma} v^{\gamma p_+} - \frac{c}{q_-} v^{q_-}$, $\varphi_2(v) = \frac{s}{(\Lambda p_+)^\gamma} v^{\gamma p_-} - \frac{c}{q_-} v^{q_+}$.

It is easy to see that φ attains its maximum at

$$v_1 = a_1^{1/(q_- - \gamma p_+)}, \ \text{ where } \ a_1 = \frac{s\gamma p_+}{c(\Lambda p_+)^\gamma}. \tag{3.8}$$

The choice of Λ in (3.6) guarantees that $v_1 \in (0,1]$. Clearly φ_2 takes its maximum at $v_2 = a_2^{1/(q_+ - \gamma p_-)}$, where $a_2 = p_- q_- a_1 / p_+ q_+ \leqslant a_1 \leqslant 1$. Hence φ is strictly decreasing for $v \geqslant v_1$, with $\varphi(v) \to -\infty$ as $v \to \infty$. Finally,

$$\varphi(v_1) = \Big(1 - \frac{\gamma p_+}{q_-}\Big)\omega_0 = E_0 > 0, \ \text{ where } \ \omega_0 = \frac{s v_1^{\gamma p_+}}{(\Lambda p_+)^\gamma} > 0. \tag{3.9}$$

Put $\Sigma = \big\{(v,E) \in R^2 : v > v_1, E < E_0\big\}$.

Theorem 3.4 Assume(3.1), (3.2) and (Q). If u is a solution of (0.1) in $\mathbf{R}_0^+ \times \Omega$, then $\omega_2 = \inf_{t \in \mathbf{R}_0^+} \mathcal{F}u(t) > -\infty$. If, moreover, $Eu(0) < E_1$, with E_1 given in Theorem 2.1, then $\omega_2 > 0$ and $(v(t), Eu(t)) \in \overline{\Sigma}$ for all $t \in \mathbf{R}_0^+$, where

$$\overline{\Sigma} = \big\{(v,E) \in R^2 : v > v_1, E < E_1\big\}, \tag{3.10}$$

and v_1 is defined in (3.8). Consequently, if in addition (3.3) holds, then there are no solutions $u \in K$ of the problem (0.1) in $\mathbf{R}_0^+ \times \Omega$, with $Eu(0) < E_1$, for which there exist positive functions ψ, k verifying (2.4), (2.5) as in Theorem 2.1.

Proof Clearly Lemmas 3.1 and 3.2 are available, so that assumptions (F_1), (F_2), (F_3)(i) and (Q) of Theorem 2.1 are satisfied along any solution u of (0.1). The fact that ω_2 is finite and positive are an immediate consequence of Theorem 2.1. By (F_2), (Q) and (B)(ii) clearly $Eu(t) \leqslant Eu(0) < E_1$ for all $t \in \mathbf{R}_0^+$. Suppose now that there exists $t_1 \in \mathbf{R}_0^+$ such that $v(t_1) \leqslant v_1$. Then, by (1.2) we have $\omega_2 \leqslant \mathcal{F}u(t_1) \leqslant cv(t_1)^{q_-}/q_-$. On the other hand, $\mathcal{A}u(t_1) \geqslant sv(t_1)^{\gamma p_+}/(\Lambda p_+)^\gamma$. Now, by (2.1), (F_2), (Q) and (B)—(ii), it follows that

$$\Big(1-\frac{\gamma p_+}{q_-}\Big)\mathcal{A}u(t_1) \geqslant \Big(1-\frac{\gamma p_+}{q_-}\Big)\omega_1 = E_1 > Eu(0) \geqslant \mathcal{A}u(t_1) - \mathcal{F}u(t_1) \geqslant \mathcal{A}u(t_1) - \frac{c}{q_-}v(t_1)^{q_-}.$$

That is $v(t_1) > [s\gamma p_+/c(\Lambda p_+)^\gamma]^{1/(q_- - \gamma p_+)} = v_1$ by (3.8). This is an obvious contradiction. Therefore $v(t_1) > v_1$ and $(v(t), Eu(t)) \in \overline{\Sigma}$ for all $t \in \mathbf{R}_0^+$, as required.

The last part of the theorem is again a direct consequence of Theorem 2.1.

Theorem 3.5　Assume (3.1), (3.2) and (Q). Let $u \in K$ be a solution of (0.1) in $\mathbf{R}_0^+ \times \Omega$, such that $Eu(0) < E_0$, with E_0 given in (3.9). Then $v_1 \notin \overline{v(\mathbf{R}_0^+)}$ and $\omega_1 = \inf_{t\in\mathbf{R}_0^+} \mathcal{A}u(t) \neq \omega_0$, where v_1 and ω_0 are defined in (3.8) and (3.9), respectively. Moreover, $\omega_1 > \omega_0$ if and only if $v(\mathbf{R}_0^+) \subset (v_1, \infty)$.

Proof　Let $u \in K$ be a solution of (0.1) in $\mathbf{R}_0^+ \times \Omega$, with $Eu(0) < E_0$. Proceed by contradiction and suppose that $v_1 \in \overline{v(\mathbf{R}_0^+)}$. It follows that there exists a sequence $(t_j)_j$ in $\mathbf{R}_0^+$ such that $v(t_j) \to v_1$ as $j \to \infty$. By (3.7) we have $E_0 > Eu(0) \geqslant Eu(t_j) \geqslant \varphi(v(t_j))$, which provides $E_0 > E_0$ by the continuity of $\varphi \circ v$, then we prove that $v_1 \notin \overline{v(\mathbf{R}_0^+)}$.

We show that $\omega_1 \neq \omega_0$. Otherwise, $\mathcal{A}u(t) \geqslant \omega_0$ for all $t \in \mathbf{R}_0^+$. Therefore, by (2.1) and (3.9), we have

$$\Big(1-\frac{\gamma p_+}{q_-}\Big)\mathcal{A}u(t) \geqslant \Big(1-\frac{\gamma p_+}{q_-}\Big)\omega_1 = E_1 > Eu(0) \geqslant \mathcal{A}u(t) - \mathcal{F}u(t) \geqslant \mathcal{A}u(t) - \frac{c}{q_-}v(t)^{q_-}.$$

Hence, if $t \in \mathbf{R}_0^+$ and $v(t) \leqslant 1$, then $\frac{c}{q_-}v(t)^{q_-} \geqslant \frac{\gamma p_+}{q_-}\mathcal{A}u(t) \geqslant \frac{\gamma p_+}{q_-}\frac{sv(t)^{\gamma p_+}}{(\Lambda p_+)^\gamma}$, that is $v(t) > v_1$. On the other hand, if $v(t) > 1$, then automatically $v(t) > v_1$, being $v_1 \leqslant 1$. Hence, $v(t) > v_1$ for each $t \in \mathbf{R}_0^+$. Consequently, the first part of the theorem yields $\overline{v(\mathbf{R}_0^+)} \subset (v_1, \infty)$. On the other hand, there exists a sequence $(t_j)_j$ such that $\mathcal{A}u(t_j) \to \omega_1 = \omega_0$ as $j \to \infty$, so $\limsup_{j\to\infty} v(t_j) \leqslant \lim_{j\to\infty}[\frac{(\Lambda p_+)^\gamma \mathcal{A}u(t_j)}{s}]^{\frac{1}{\gamma p_+}} = v_1$ by (3.9), which contradicts the fact that $\overline{v(\mathbf{R}_0^+)} \subset (v_1, \infty)$. Hence $\omega_1 \neq \omega_0$.

If $\omega_1 > \omega_0$, then $Eu(0) < E_1$ and $v(t) > v_1$ for all $t \in \mathbf{R}_0^+$ by Theorem 3.4, so $v(\mathbf{R}_0^+) \subset (v_1, \infty)$, since $v_1 \notin \overline{v(\mathbf{R}_0^+)}$.

On the other hand, if $\overline{v(\mathbf{R}_0^+)} \subset (v_1, \infty)$, then $v(t) > v_1$ and $\mathcal{A}u(t) > \frac{sv(t)^{\gamma p_+}}{(\Lambda p_+)^\gamma} = \omega_0$ for all $t \in \mathbf{R}_0^+$. Hence $\omega_1 > \omega_0$, since the case $\omega_1 = \omega_0$ cannot occur by the argument above.

In the next corollary we present an application of both Theorems 2.1 and 3.4. In particular, we provide sufficient conditions under which assumptions (2.4), (2.5) of Theorem 2.1 are satisfied. Let $Q = Q(t, x, u, v)$ be a continuous damping function as in the Section 0 and assume also that there exists $t^* \gg 1$ such that for all $(t, x, u, v) \in [t^*, \infty) \times \Gamma_1 \times \mathbf{R}^N \times \mathbf{R}^N$,

$$Q(t, x, u, v) = d(t, x)\,|u|^k\,|v|^{m-2}\,v, \tag{3.11}$$

where m, ζ, k_0, d satisfy condition (Q), with $d(t, x) \geqslant 0$ in $\mathbf{R}_0^+ \times \Gamma_1$. Put $\delta(t) = \|d(t,\cdot)\|_{\zeta/(\zeta-k_0-m),\Gamma_1}$ for all $t \in \mathbf{R}_0^+$. Hence

$$|Q(t, x, u, v)| \leqslant [d(t, x)\,|u|^k]^{1/m}[(Q(t, x, u, v), v)]^{1/m'},$$

for all $(t,x,u,v)\in[t^*,\infty)\times\Gamma_1\times\mathbf{R}^N\times\mathbf{R}^N$, so that (Q) holds with $t_Q=t^*$.

Corollary 3.6 Assume (3.1)—(3.3), (3.11) and that $\delta(t)\leqslant\delta_1(1+t)^l$ for each $t\in[t^*,\infty)$, for some appropriate numbers $\delta_1\geqslant 1$ and $l\leqslant m-1$. Then there are no solutions $u\in K$ of (0.1) in $\mathbf{R}_0^+\times\Omega$, with $Eu(0)<E_1$.

This corollary is similar with Corollary 4.1 in [6], so we omit the proof here. From now on in this section we assume the assumptions (3.1), (3.2), (3.3), and (3.11), with $\delta(t)\leqslant\delta_1(1+t)^l$ for each $t\in[t^*,\infty)$ and some $\delta_1\geqslant 1$ with $l\leqslant m-1$.

Corollary 3.7 Problem (0.1) does not possess solutions $u\in K$ in $\mathbf{R}_0^+\times\Omega$, with

$$\|u(0,\cdot)\|_{q(\cdot)}>v_1,\quad Eu(0)<E_0,\tag{3.12}$$

where E_0 is defined in (3.9).

Proof Assume as a contradiction that $u\in K$ is a solution of (0.1) in $\mathbf{R}_0^+\times\Omega$, verifying (3.12). By Theorem 3.5 then $\omega_1>\omega_0$. Hence $Eu(0)<E_0<E_1$, and the contradiction follows at once by an application of Corollary 3.6.

Proposition 3.8 If $u\in K$ is a solution of (0.1) in $\mathbf{R}_0^+\times\Omega$, with $Eu(0)<E_0$, where E_0 is defined in (3.9), then

$$\omega_1\leqslant\omega_0.\tag{3.13}$$

Proof Otherwise $\omega_1>\omega_0$, so $Eu(0)<E_1$, and u could not be global by Corollary 3.6.

In the rest of this section we assume also:

(D) There exists $t_*>0$ such that either

(i) $g_t(t,x)\geqslant g_0(t)>0$ for each $(t,x)\in[0,t_*]\times\Omega$, or

(ii) $\phi\in K$ and $\langle Q(t,\cdot,\phi,\phi_t),\phi_t\rangle_{\Gamma_1}=0$ in $[0,t_*]$ implies either $\phi(t,\cdot)=0$ or $\phi_t(t,\cdot)=0$ for all $t\in[0,t_*]$.

Theorem 3.9 Problem (0.1) does not possess solutions $u\in K$ in $\mathbf{R}_0^+\times\Omega$, with

$$\|u(0,\cdot)\|_{q(\cdot)}>v_1,\quad Eu(0)=E_0.\tag{3.14}$$

Proof Assume by contradiction that $u\in K$ is a global solution of (0.1) in $\mathbf{R}_0^+\times\Omega$, verifying (3.14). By Proposition 3.8 we have $\omega_1\leqslant\omega_0$. We first claim that $\omega_1<\omega_0$ cannot occur. As a matter of fact, if $\omega_1<\omega_0$ there would exist t_0 such that $\mathcal{A}u(t_0)<\omega_0$, and this is possible only if $v(t_0)<v_1$; indeed if $v(t_0)\geqslant v_1$ we would immediately have $\mathcal{A}u(t_0)\geqslant\omega_0$. Hence $t_0>0$ by (3.14) and by the continuity of v there exists $s\in(0,t_0)$ such that $v(s)=v_1$. Thus

$$E_0=Eu(0)\geqslant Eu(s)\geqslant\omega_0-\frac{c}{q_-}v_1^{q_-}=E_0$$

by (3.7). In other words, $Eu(s)=E_0$ and $\int_0^s\mathcal{D}u(\tau)\mathrm{d}\tau=0$ by (B)(ii). Consequently $\mathcal{D}u\equiv 0$ in $[0,s]$ and so, by (F_2) and (3.11), we obtain $\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u_t(t,\cdot)\rangle_{\Gamma_1}=0$ and $\mathcal{F}_tu(t)=0$ for all $t\in[0,s]$.

Now, if (D)(i) holds, then

$$0=\mathcal{F}_tu(t)=\int_\Omega g_t(t,x)\frac{|u(t,x)|^{\sigma(x)}}{\sigma(x)}\mathrm{d}x\geqslant\frac{g_0(t)}{\sigma_+}\rho_{\sigma(\cdot)}(u(t,\cdot))\geqslant 0$$

for each $t \in [0, s_0]$, where $s_0 = \min\{t_*, s\}$. Therefore $\rho_{\sigma(\cdot)}(u(t,\cdot)) = 0$ and in turn $u = 0$ in $[0, s_0]\times\Omega$ by (1.1) and (1.2). But this occurrence is impossible, since $\|u(0,\cdot)\|_{q(\cdot)} = v(0) > v_1 > 0$ by (3.14), so we reach a contradiction.

However, if (D)(ii) holds, since $\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u_t(t,\cdot)\rangle_{\Gamma_1} = 0$ for all $t \in [0, s_0]$, we get that either $u(t,\cdot) = 0$ or $u_t(t,\cdot) = 0$ for all $t \in [0, s_0]$, where as above $s_0 = \min\{t_*, s\}$. Again, as already shown, the first case $u(t,\cdot) = 0$ cannot occur since $v(0) > v_1$. In the latter, u is clearly constant with respect to t in $[0, s_0]$, and so $u(t,x) = u(0,x)$ for each $t \in [0, s_0]$. Taking $\phi(t,x) = u(0,x)$ in the Distribution Identity (A), then for each $t \in [0, s_0]$ we have $t\,\langle Au(0,\cdot),u(0,\cdot)\rangle = \int_0^t \langle f(\tau,\cdot,u(0,\cdot)),u(0,\cdot)\rangle \mathrm{d}\tau$, since $\langle Q(t,\cdot,u(0,\cdot),0),u(0,\cdot)\rangle_{\Gamma_1} = 0$, as $\mathcal{D}u = 0$ in $[0, s_0]$. Therefore $\langle Au(0,\cdot),u(0,\cdot)\rangle = \langle f(t,\cdot,u(0,\cdot)),u(0,\cdot)\rangle$ for each $t \in [0, s_0]$, and so $\langle Au(0,\cdot),u(0,\cdot)\rangle = \langle f(0,\cdot,u(0,\cdot)),u(0,\cdot)\rangle$. Now $\gamma p_+\mathcal{A}u(0) \geqslant q_-\mathcal{F}u(0)$ by (2.7) and (F_3). On the other hand, $E_0 = Eu(0) = \mathcal{A}u(0) - \mathcal{F}u(0)$, since $u_t(t,0) = 0$. By (3.9) we have $\mathcal{A}u(0) > \omega_0 > 0$, and so

$$E_0 \geqslant \Big(1 - \frac{\gamma p_+}{q_-}\Big)\mathcal{A}u(0) \geqslant \Big(1 - \frac{\gamma p_+}{q_-}\Big)\omega_0 = E_0,$$

by (3.9). This contradiction shows the claim.

Hence $\omega_1 = \omega_0$. In particular $\mathcal{A}u(t) \geqslant \omega_0$ for all $t \in R_0^+$ and we assert that equality cannot occur at a finite time. Indeed, if there is a τ such that $\mathcal{A}u(\tau) = \omega_0$, then $v(\tau) \leqslant v_1$. On the other hand, as shown in the proof of Theorem 3.5, we get $v(\tau) > v_1$. This contradiction shows that it remains to consider only the case $\omega_1 = \omega_0$, $\mathcal{A}u(t) > \omega_0$ and $v(t) > v_1$ for all $t \in \mathbf{R}_0^+$. A continuity argument shows at once that $\liminf\limits_{t\to\infty} \mathcal{A}u(t) = \omega_0$, $\liminf\limits_{t\to\infty} v(t) = v_1$.

Indeed, since $\inf_{t\in\mathbf{R}_0^+} \mathcal{A}u(t) = \omega_1 = \omega_0$ there exists a $(t_k)_k \in \mathbf{R}_0^+$ such that $\lim\limits_{k\to\infty} \mathcal{A}u(t_k) = \omega_0$ and $(t_k)_k$ cannot be bounded because $\mathcal{A}u$ reaches its infimum at infinity. Hence $\liminf\limits_{t\to\infty} \mathcal{A}u(t) \leqslant \omega_0$ and this forces $\inf\limits_{t\to\infty} \mathcal{A}u(t) = \omega_0$, as $\omega_0 = \inf_{t\in\mathbf{R}_0^+} \mathcal{A}u(t)$. Put now $v_1' = \liminf\limits_{t\to\infty} v(t)$. Since $\mathcal{A}u(t) \geqslant \frac{s}{(\Lambda p_+)^\gamma} v(t)^{\gamma p_+}$ for all $t \in \mathbf{R}_0^+$, then $\omega_0 \geqslant \frac{s}{(\Lambda p_+)^\gamma}(v_1')^{\gamma p_+}$, which gives $v_1' \leqslant v_1$ by (3.9). On the other hand $v_1' \geqslant v_1$, as $v(t) \geqslant v_1$ for all $t \in \mathbf{R}_0^+$ and in turn $v_1' = v_1$, as required.

Now, by (2.1) and (B)(ii) we have $\omega_0 - \mathcal{F}u(t) < Eu(t) \leqslant E_0$, so $\limsup\limits_{t\to\infty} Eu(t) = E_0$. Hence $\int_0^\infty \mathcal{D}u(\tau)\mathrm{d}\tau = 0$ by monotonicity. In particular $\mathcal{D}u \equiv 0$ in $\mathbf{R}_0^+$, which is again impossible by (D), using the argument already produced. This completes the proof.

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