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# Geometric singular perturbation approach to singular singularly perturbed systems

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**Abstract:** Singularly perturbed systems for which the reduced system has a manifold of solutions are called singular singularly perturbed. Boundary value problems for such systems were examined by geometric singular perturbation approach in this paper. Assumptions were derived which ensure the existence of a locally unique solution which is near a singular orbit of the dynamics of limiting fast and slow systems.

**Key words:** geometric singular perturbation; singular singularly perturbed; exchange lemma

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## 奇异奇摄动系统的几何方法

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**摘要:** 研究了一类奇异奇摄动系统边值问题. 通过几何奇摄动理论构造了系统的奇异轨道, 并用交换引理证明了解的存在性. 最后用该方法研究了一个经典半导体模型.

**关键词:** 几何奇摄动; 奇异奇摄动; 交换引理

## 0 Introduction

We consider singularly perturbed equations of the form

$$\epsilon\frac{\mathrm{d}y}{\mathrm{d}t} = f(x, y, \epsilon), \quad \frac{\mathrm{d}x}{\mathrm{d}t} = g(x, y, \epsilon), \tag{1}$$

with fast variable $y \in \mathbf{R}^{s+u+c}$, slow variable $x \in \mathbf{R}^m$, sufficiently smooth functions $f, g$, and small parameter $0 < \epsilon \ll 1$. ($s, c, u, m$ are nonnegative integers, and $s + c + u = n$.) These equations can also be written in terms of time scale $\tau := t/\epsilon$ as

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f(x, y, \epsilon), \quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \epsilon g(x, y, \epsilon), \tag{2}$$

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when $\epsilon \neq 0$, the two systems are equivalent. Such equations possess singular orbits, which are unions of the two limiting systems obtained from (1) and (2) by setting $\epsilon = 0$.

The limiting fast system

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f(x, y, 0), \quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = 0,$$

has a manifold of equilibriums given by

$$S_0 = \{(x, y) \mid f(x, y, 0) = 0\},$$

which is called the critical manifold. We say that a compact and simply connected region of the critical manifold is normally hyperbolic if the number of eigenvalues normal to the critical manifold equals to the codimension of the critical manifold. For each point on a normally hyperbolic critical manifold there exist stable and unstable manifold with the dimension $s$ of eigenvalues with negative real parts, and $u$ of eigenvalues with positive real parts, respectively[1]. The limiting slow system describes motions along the critical manifold $S_0$ in the original time scale $t$, and is given by

$$0 = f(x, y, 0), \quad \frac{\mathrm{d}x}{\mathrm{d}t} = g(x, y, 0).$$

From the classic singular perturbation theory[2-4], one tries to find the asymptotic expansions of each part of the solution, match them to form a global solution, and use some fix point theorem in some Banach space or construct upper and lower solutions, etc., to get a true solution near the asymptotic expansions for sufficiently small $\epsilon > 0$. From the geometric singular perturbation theory or dynamical system point of view, one tries to find the singular orbit formed by solutions of limiting fast and slow systems and examine the possibility of lifting the singular orbit to a true solution for sufficiently small $\epsilon > 0$. With the development of the theory of homoclinic and heteroclinic bifurcation[5] and invariant manifold theory[1], Shadowing Lemma developed by Lin[6,7], and Exchange Lemma developed by Jones, Kopell[8-10], proved to be successfully applied to the existence of the solution and the qualitative structure of this problem.

In general, one requires the matrix $f_y(x, y, 0)$ is hyperbolic along the critical manifold $S_0$, i.e., eigenvalues of $f_y(x, y, 0) \mid_{(x,y)\in S_0}$ are uniformly separated from the imaginary axis[3,4,6-10]. This assumption ensures the critical manifold can be given as a graph of $y = u(x), x \in D \subset \mathbf{R}^m$, i.e., the reduced equation has an isolated root $y = u(x)$. In a variety of applied problems leading to singularly perturbed equations, this assumption is violated. For example, one may have $\det f_y(x, y, 0) \mid_{S_0} = 0$, and the reduced equation $f(x, y, 0) = 0$, for each fixed $x$, does not have an isolated root, but instead has a manifold of solutions. Such a case is called singular singularly perturbed (the critical cases[3]).

Singular singularly perturbed problems can be found in chemical kinetics[3], semiconductor devices[11], Poisson-Nernst-Planck system[12], etc., and gained many mathematicians' attention. In the work by Vasil'eva and Butuzov[3], initial value problem and a special structure boundary value problem of singular singularly perturbed systems were considered by the boundary function method. A general singular singularly perturbed boundary value problem was examined by Schmeiser[11] using the same method as Vasil'eva's. Within the framework of geometric singular perturbation theory, Liu[12] studied steady-state Poisson-Nernst-Planck systems.

In this paper, we will use the geometric singular perturbation theory to study a general singular singularly perturbed system. It turns out that under some quite general conditions, the singular orbit shadows a real solution for the boundary value problem. The results are equivalent to those of [11], but are accomplished with very different set of techniques that are more geometrical. Different from the method used by Liu (using the change of variables)[12], to construct the reduced system (slow flow) on the critical manifold for a general system directly, we use a well-defined projection $\pi^{S_0}$ associated with the tangent space of the critical manifold.

# 1 A dynamical system framework and the main result

Our method used in this paper is the geometric singular perturbation theory. The main idea of this approach for singularly perturbed boundary value problems are

(1) change the standard singularly perturbed boundary value problem to a "connection problem", which is convenient to use the dynamical approach, based on different time scale of the system, to derive various limiting systems for $\epsilon = 0$ and study their dynamics;

(2) use the limiting systems to construct singular orbits (zeroth order approximation) which include regular layers, boundary layers, and, sometimes, internal layers;

(3) use exchange lemma to show that there are true solutions near singular orbits for $\epsilon > 0$.

Since limiting systems have lower order than the full system, it is easier to study which makes it useful. The understanding of the limiting fast system gives the dynamics of boundary layers or internal layers. When the matrix $f_y(x, y, 0)$ is hyperbolic along the critical manifold $S_0$, the dynamics of the limiting slow system is easily found. The case, when the matrix $f_y(x, y, 0)|_{S_0}$ is nonhyperbolic, is quite different, however, as some information of the fast variables is lost.

**Hypothesis 1** *The critical manifold* $S_0$ *appears as the graph of* $y_1 = \varphi(x, y_2)$, *where* $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $x \in D_1 \subset \mathbf{R}^m$, $y_1 \in \mathbf{R}^{s+u}$, $y_2 \in D_2 \subset \mathbf{R}^c$, $\varphi \in C^r(\mathbf{R}^{m+c}, \mathbf{R}^{s+u}), r \geqslant 1$, $D_i$ *is compact and simply connected*, $i = 1, 2$.

**Hypothesis 2** *The matrix* $f_y(x, y, 0)|_{S_0} = f_y(x, \varphi(x, y_2), y_2, 0)$ *has an c-dimensional kernel, u eigenvalues with positive real parts and s eigenvalues with negative real parts.*

**Remark 1** *This assumption rules out the existence of turning points, pure imaginary eigenvalues and multiple roots.*

## 1.1 Fast dynamics and boundary layers

We consider singularly perturbed equation (3) with boundary conditions

$$B_0(y(0), x(0), \epsilon) = 0, \quad B_1(y(1), x(1), \epsilon) = 0, \tag{3}$$

where $B_i$ has values in $\mathbf{R}^{d_i}$, $d_0 + d_1 = m + n$, and assume that the function $B_i$ are of maximal rank, $i = 0, 1$. It is convenient to change this into a form that the boundary conditions can be described as manifold, and without worrying the time it takes the orbit to move from one boundary to the other. System (1), (3) become

$$\begin{aligned} &\epsilon\frac{\mathrm{d}y}{\mathrm{d}t}=f(x,y,\epsilon),\quad \frac{\mathrm{d}x}{\mathrm{d}t}=g(x,y,\epsilon),\quad \frac{\mathrm{d}\xi}{\mathrm{d}t}=1,\\ &B_L^\epsilon=\{B_0(y(0),x(0),\epsilon)=0,\xi=0\},\\ &B_R^\epsilon=\{B_1(y(1),x(1),\epsilon)=0,\xi=1\}.\end{aligned}\tag{4}$$

Thus the boundary value problem changes into a "connection problem" that requires a trajectory starts at $B_L^\epsilon$ and hits $B_R^\epsilon$ at some finite time $t$. It follows from $d_0+d_1=m+n$ that $\dim B_L^\epsilon+\dim B_R^\epsilon=m+n$.

We start with the study of boundary layers governed by the limiting fast system of (4)

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}\tau}=f(x,y,0),\quad \frac{\mathrm{d}x}{\mathrm{d}\tau}=0,\quad \frac{\mathrm{d}\xi}{\mathrm{d}\tau}=0,\\ &B_L^0=\{B_0(y(0),x(0),0)=0,\xi=0\},\\ &B_R^0=\{B_1(y(1),x(1),0)=0,\xi=1\}.\end{aligned}\tag{5}$$

From $Hypothesis$ 1, the critical manifold $S_0=\{y_1=\varphi(x,y_2)\}$ consisting equilibrium of system (5) is a $m+c+1$-dimensional manifold of the phase space $\mathbf{R}^{m+n+1}$. From $Hypothesis$ 2, the linearization of system (5) on the critical manifold has $m+c+1$ zero eigenvalues equal to the dimension of $S_0$, and $s+u$ eigenvalues in directions normal to $S_0$. Thus, $S_0$ is normally hyperbolic, every equilibrium $p\in S_0$ has a $s$-dimensional stable manifold $W^s(p)$, and a $u$-dimensional unstable manifold $W^u(p)$.

**Hypothesis 3** *The stable manifold $W^s(S_0)$ intersects $B_L^0$ transversely, and the unstable manifold $W^u(S_0)$ intersects $B_R^0$ transversely, where $W^s(S_0)=\bigcup\limits_{p\in S_0}W^s(p)$, $W^u(S_0)=\bigcup\limits_{p\in S_0}W^u(p)$.*

Let $N_L=B_L^0\cap W^s(S_0)$, $N_R=B_R^0\cap W^u(S_0)$. The intersection has dimension

$$\begin{aligned} \dim N_L&=\dim B_L^0+\dim W^s(S_0)-m-n-1=\dim B_L^0-u,\\ \dim N_R&=\dim B_R^0+\dim W^u(S_0)-m-n-1=\dim B_R^0-s.\end{aligned}$$

### 1.2 Slow dynamics and regular layers

We now study the slow flow on the critical manifold $S_0=\{y_1=\varphi(x,y_2)\}$ for regular layers. Set $\epsilon=0$ in system (6), we get

$$y_1=\varphi(x,y_2),\quad \frac{\mathrm{d}x}{\mathrm{d}t}=g(x,y_1,y_2,0),\quad \frac{\mathrm{d}\xi}{\mathrm{d}t}=1,$$

the information on $y_2$ is lost. Because of $Hypothesis$ 1, 2, for each $p\in S_0$ the kernel of $f_y\,|_{p,\epsilon=0}$ has a unique invariant complement, so there is a well-defined projection on the kernel. Follow Fenichel's notes[1], we denote this projection by $\pi^{s_0}$, and it is associated with the tangent space of $S_0$. Fenichel defines the reduced vector field $X_R$ on $S_0$ by

$$X_R(p)=\pi^{S_0}\frac{\partial}{\partial\epsilon}X^\epsilon(p)\,|_{\epsilon=0},$$

where $p \in S_0$, $X^\epsilon$ denotes the fast vector field. The following lemma, proved by Fenichel [1], gives the reduced system in local coordinates in which $S_0$ appears as the graph of a function $y = u(x)$.

**Lemma 1** (Fenichel) *Consider a system*

$$x' = f(x, y, \epsilon), \quad y' = g(x, y, \epsilon),$$

*defined for* $(x, y)$ *in an open set* $M$ *in* $\mathbf{R}^u \times \mathbf{R}^v$, *for* $\epsilon$ *near zero. Let* $y = u(x)$ *be a function defined for* $x$ *near* $x_0$, *such that*

$$f(x, u(x), 0) = 0, \quad g(x, u(x), 0) = 0.$$

*Suppose* $(x_0, u(x_0)) \in S_R$, *so that the matrix*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} D_1 f(x_0, u(x_0), 0) & D_2 f(x_0, u(x_0), 0) \\ D_1 g(x_0, u(x_0), 0) & D_2 g(x_0, u(x_0), 0) \end{pmatrix}$$

*has rank* $v$. *Let* $\kappa = Du(x_0)$. *Then the projection has the form*

$$\pi^{S_0} = \begin{pmatrix} I + \beta(\delta - \kappa\beta)^{-1}\kappa & -\beta(\delta - \kappa\beta)^{-1} \\ \kappa + \kappa\beta(\delta - \kappa\beta)^{-1}\kappa & -\kappa\beta(\delta - \kappa\beta)^{-1} \end{pmatrix}.$$

Using the formulas of Lemma 1, we can compute the projection and the slow flow for $y_2$.

**Corollary 1** *The slow flow on the critical manifold* $S_0 = \{y_1 = \varphi(x, y_2)\}$ *for regular layers has the form*

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = g(x, y, 0), \quad \frac{\mathrm{d}\xi}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}y_1}{\mathrm{d}t} = \varphi_x g + \varphi_{y_2}\frac{\mathrm{d}y_2}{\mathrm{d}t}, \\ &\frac{\mathrm{d}y_2}{\mathrm{d}t} = f_{2y_1}\kappa\varphi_x g + (f_{2y_1}\kappa\varphi_{y_2} + I)f_{2\epsilon} - f_{2y_1}\kappa f_{1\epsilon}, \end{aligned} \tag{6}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $\kappa = (f_{1y_1} - \varphi_{y_2} f_{2y_1})^{-1}$.

**Proof** This can be calculated directly.

To identify the slow motion of the singular orbit on $S_0$, we need to examine the $\omega - limit$ (respect, the $\alpha - limit$) set of $N_L$ (respect, $N_R$). Due to the uniqueness of the flow, the map $N_L \to \omega(N_L)$ ($N_R \to \alpha(N_R)$), which takes each point of $N_L$ ($N_R$) to its limit point on $S_0$, is one-to-one, and $\omega(N_L)$, $\alpha(N_R)$ are submanifolds of $S_0$ with the same dimension as $N_L$, $N_R$, respectively. The slow orbit should be one given by (6) that connects $\omega(N_L)$ and $\alpha(N_R)$. Let $\omega(N_L) \cdot t$ be the forward flow of $\omega(N_L)$ under the slow flow (6).

**Hypothesis 4** $\omega(N_L) \cdot t$ *intersects* $\alpha(N_R)$ *transversally on* $S_0$.

The intersection has dimension

$$\dim \omega(N_L) \cdot t \cap \alpha(N_R) = \dim B_L^0 - u + 1 + \dim B_R^0 - s - (m + c + 1) = 0.$$

Thus, there exists a locally unique slow orbit on $S_0$, and two unique fast orbits on $N_L \cdot t$ and $N_R \cdot t$, respectively.

### 1.3 Main result

Based on the study of the limiting fast and slow system, we have construct a singular orbit of the boundary value problem. To show that there indeed exists a solution near the singular orbit for $\epsilon > 0$, we apply the exchange lemma to show $B_L^\epsilon \cdot t$ and $B_R^\epsilon \cdot t$ intersect transversally near the singular orbit.

**Theorem 1** *Assume Hypothesis* 1-4. *For* $\epsilon > 0$ *sufficiently small, the connection problem* (4) *has a unique solution near a singular orbit. The singular orbit is the union of two fast orbit of* (5) *and one slow orbit of* (6); *more precisely*

(1) *the fast orbit representing the boundary layer at* $t = 0$ *lies on* $B_L^0 \cdot t \cap W^s(S_0)$ *from* $B_L^0$ *to* $\omega(N_L) \subset S_0$;

(2) *the fast orbit representing the boundary layer at* $t = 1$ *lies on* $B_R^0 \cdot t \cap W^u(S_0)$ *from* $B_R^0$ *to* $\alpha(N_R) \subset S_0$;

(3) *the slow orbit on* $S_0$ *connecting the two boundary layers from* $t = 0$ *to* $t = 1$ *governed by* (6) *from* $\omega(N_L)$ *to* $\alpha(N_R)$.

**Proof** All conditions for the exchange lemma[8] are satisfied, and hence, $B_L^\epsilon \cdot t$ and $B_R^\epsilon \cdot t$ intersect transversally. The intersection has dimension

$$\dim B_L^\epsilon + 1 + \dim B_R^\epsilon + 1 - (m + n + 1) = 1,$$

which is the orbit of the unique solution for the connection problem near the singular orbit.

## 2 Example

We consider the fundamental semiconductor device equations for the case of symmetric $p - n$ junction with piecewise constant doping. The singular perturbation for this problem has been examined by Vasil'eva[13], Schmeiser[11], etc. The governing equations are

$$\begin{aligned} &\epsilon\psi' = E, \quad \epsilon E' = n - p - 1, \quad \epsilon n' = nE + \frac{\epsilon J}{2}, \quad \epsilon p' = -pE - \frac{\epsilon J}{2}, \\ &\psi(0) = 0, \quad n(0) = p(0), \quad p(1) = \frac{1}{2}(-1 + \sqrt{1 + 4\delta^4}) = p_1, \quad n(1) = p_1 + 1. \end{aligned} \tag{7}$$

The variables $\psi, E, n$, and $p$ are scaled and proportional to the potential, the electric field, the electron density and the hole density in the device. $\epsilon$ and $\delta$ result from the scaling. $\epsilon$ is equal to the Debye length, and is small when the doping is large. Thus (7) is singularly perturbed in this situation. To use geometric singular perturbation theory, we will recast the singularly perturbed semiconductor systems into a connection problem:

$$\begin{aligned} &\epsilon\psi' = E, \quad \epsilon E' = n - p - 1, \\ &\epsilon n' = nE + \frac{\epsilon J}{2}, \quad \epsilon p' = -pE - \frac{\epsilon J}{2}, \quad x' = 1. \end{aligned} \tag{8}$$

System (8) will be treated as a dynamical system with the phase space $\mathbf{R}^5$, and the independent variable $t$ will be viewed as time. The boundary condition becomes

$$B_L = \{\psi = 0, n = p, x = 0\}, \quad B_R = \{p = p_1, n = p_1 + 1, x = 1\}.$$

By setting $\epsilon = 0$ in (8), we get the critical manifold

$$S_0 = \{E = 0, n = p + 1\}.$$

With the change of variable $\tau = t/\epsilon$, we derive the fast system

$$\begin{aligned} &\dot{\psi} = E, \quad \dot{E} = n - p - 1, \\ &\dot{n} = nE + \tfrac{\epsilon J}{2}, \quad \dot{p} = -pE - \tfrac{\epsilon J}{2}, \quad \dot{x} = \epsilon. \end{aligned} \tag{9}$$

Setting $\epsilon = 0$ in (9), we get the limiting fast system

$$\begin{aligned} &\dot{\psi} = E, \quad \dot{E} = n - p - 1, \\ &\dot{n} = nE, \quad \dot{p} = -pE, \quad \dot{x} = 0. \end{aligned} \tag{10}$$

The critical manifold $S_0$ consisting entirely of equilibria of system (10) is a three-dimensional manifold of the phase space $\mathbf{R}^5$. For each equilibrium, the linearization of (10) has three zero eigenvalues corresponding to the dimension of $S_0$, and two eigenvalue $\lambda_\pm = \pm\sqrt{2p+1}$ not on the imaginary axis, so $S_0$ is a normally hyperbolic manifold. Thus every equilibrium has a one-dimensional stable manifold and a one-dimensional unstable manifold.

System (10) possesses a complete set of integrals

$$H_1 = \frac{E^2}{2} - n - p + \ln n, \quad H_2 = \psi - \ln n, \quad H_3 = np, \quad H_4 = x,$$

thus the stable and unstable manifold can be characterized in detail. The stable and unstable manifold $W^s(S_0)$ and $W^u(S_0)$:

$$\begin{aligned} &\frac{E^2}{2} - n - p + \ln n = -2n^* + 1 + \ln n^*, \quad \psi - \ln n = \psi^* - \ln n^*, \\ &np = n^*(n^* - 1), \quad x = x^*, \quad (n^* = p^* + 1), \end{aligned}$$

where $(\psi^*, 0, p^* + 1, p^*, x^*) \in S_0$.

The stable manifold $W^s(S_0)$ intersects $B_L$ transversally at point

$$\begin{aligned} &\psi = 0, \quad E = \sqrt{4\sqrt{p^*(p^*+1)} - 4p^* - 2 + 2\ln\sqrt{\frac{p^*+1}{p^*}}}, \\ &p = \sqrt{p^*(p^*+1)}, \quad n = \sqrt{p^*(p^*+1)}, \quad x = 0. \end{aligned}$$

The unstable manifold $W^u(S_0)$ intersects $B_R$ transversally at point

$$E = 0, \quad \psi = \psi^*, \quad p = p_1, \quad n = p_1 + 1, \quad x = 1.$$

Let $N_L = B_L \cap W^s(S_0)$, $N_R = B_R \cap W^u(S_0)$ then

$$\begin{aligned} &\omega(N_L) = \left(\ln\sqrt{\frac{p^*+1}{p^*}}, 0, p^* + 1, p^*, 0\right), \\ &\alpha(N_R) = (\psi^*, 0, p_1 + 1, p_1, 1). \end{aligned}$$

We now examine the slow flow in the vicinity of the critical manifold $S_0$:

$$\dot{z} = \pi^{\epsilon} D_{\epsilon} h(z, 0),$$

where $z = (\psi, p, x, E, n)^{\mathrm{T}}$ and

$$\pi^{\epsilon} = \begin{pmatrix} 1 & \dfrac{1}{2p+1} & 0 & 0 & -\dfrac{1}{2p+1} \\ 0 & \dfrac{p+1}{2p+1} & 0 & 0 & \dfrac{p}{2p+1} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{p+1}{2p+1} & 0 & 0 & \dfrac{p}{2p+1} \end{pmatrix}.$$

Thus we get the reduced system (slow flow)

$$\begin{aligned} &\psi' = -\frac{J}{2p+1}, \quad p' = -\frac{J}{4p+2}, \\ &x' = 1, \quad E' = 0, \quad n' = -\frac{J}{4p+2}. \end{aligned} \tag{11}$$

The slow orbit should be one given by (11) that connects $\omega(N_L)$ and $\alpha(N_R)$:

$$x = t, \quad E = 0, \quad n = p + 1, \quad p = \frac{-1 + \sqrt{2J - 2Jt + 4p_1^2 + 4p_1 + 1}}{2},$$

$$\psi = \sqrt{2J + 1 - 2Jt + 4p_1^2 + 4p_1} - \sqrt{2J + 1 + 4p_1^2 + 4p_1} + \ln \frac{\sqrt{1 + 2J + 4p_1^2 + 4p_1} + 1}{\sqrt{2J + 4p_1^2 + 4p_1}},$$

and

$$p^* = \frac{-1 + \sqrt{1 + 2J + 4p_1^2 + 4p_1}}{2},$$

$$\psi^* = 2p_1 + 1 - \sqrt{2J + 1 + 4p_1^2 + 4p_1} + \ln \frac{\sqrt{1 + 2J + 4p_1^2 + 4p_1} + 1}{\sqrt{2J + 4p_1^2 + 4p_1}}.$$

Based on the study of the limiting behavior of fast and slow system, we construct a singular orbit of the boundary value problem. All conditions for Theorem 1 are satisfied, and hence, there is a orbit of the unique solution for the connection problem near the singular orbit.

**Remark 2** *The results are equivalent to those of* [11], *but are accomplished with very different set of techniques that are more geometrical.*

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