

例6 求带通信号 $x(t)=\text{Sa}(t)\cos 2t$, $-\infty < t < \infty$, 通过线性相位理想低通滤波器的响应

解:

$$H(\omega) = \text{rect}\left(\frac{\omega}{2\omega_c}\right)e^{-j\omega t_d}$$

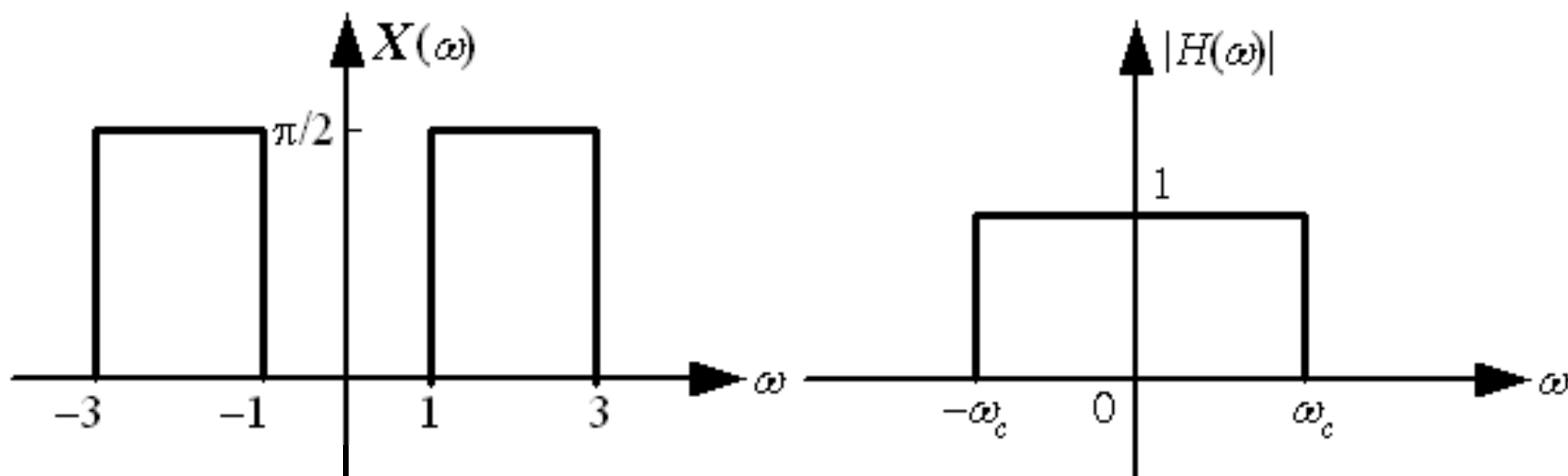
$$\therefore F[\text{Sa}(t)] = \pi \text{rect}(\omega/2)$$

$$F(\cos 2t) = \pi\delta(\omega - 2) + \pi\delta(\omega + 2)$$

$$\begin{aligned}\therefore X(\omega) &= \frac{1}{2\pi} \{F[\text{Sa}(t)] * F(\cos 2t)\} \\ &= \frac{\pi}{2} \text{rect}\left(\frac{\omega}{2}\right) * \delta(\omega - 2) + \frac{\pi}{2} \text{rect}\left(\frac{\omega}{2}\right) * \delta(\omega + 2) \\ &= \frac{\pi}{2} \text{rect}\left(\frac{\omega - 2}{2}\right) + \frac{\pi}{2} \text{rect}\left(\frac{\omega + 2}{2}\right)\end{aligned}$$

$$\therefore Y(\omega) = H(\omega)X(\omega)$$

$$= \text{rect}\left(\frac{\omega}{2\omega_c}\right) e^{-j\omega t_d} \frac{\pi}{2} \left[\text{rect}\left(\frac{\omega-2}{2}\right) + \text{rect}\left(\frac{\omega+2}{2}\right) \right]$$



1) 当 $\omega_c > 3$ 时，输入信号的所有频率分量都能通过系统，即

$$Y(\omega) = e^{-j\omega t_d} \frac{\pi}{2} \left[\text{rect}\left(\frac{\omega-2}{2}\right) + \text{rect}\left(\frac{\omega+2}{2}\right) \right]$$

$$y(t) = x(t-t_d) = \text{Sa}(t-t_d)\cos[2(t-t_d)] , \quad -\infty < t < \infty$$

2) 当 $\omega_c < 1$ 时，输入信号的所有频率分量都不能通过系统，即

$$Y(\omega) = 0$$

$$y(t) = 0 , \quad -\infty < t < \infty$$

3) 当 $1 < \omega_c < 3$ 时，只有 $1 \sim \omega_c$ 范围内的频率分量能通过系统，故

$$Y(\omega) = \frac{\pi}{2} \left\{ \text{rect} \left[\frac{\omega - \frac{\omega_c + 1}{2}}{\omega_c - 1} \right] + \text{rect} \left[\frac{\omega + \frac{\omega_c + 1}{2}}{\omega_c - 1} \right] \right\} e^{-j\omega t_d}$$

由抽样信号频谱及Fourier变换的时域和频域位移特性可得

$$y(t) = \frac{\omega_c - 1}{2} \text{Sa} \left[\frac{\omega_c - 1}{2} (t - t_d) \right] \cos \left[\frac{\omega_c + 1}{2} (t - t_d) \right]$$

§ 1.8 确定信号的相关

一、相关系数

- 两个信号波形是否相同的度量指标：均方差

以能量信号为例

$$Q = \int_{-\infty}^{\infty} |x_1(t) - x_2(t)|^2 dt$$

- 缺点：均方差无法反映两波形相似但幅度相差较大的信号的相似程度
- 为去除幅度相差的影响，对其中一个信号乘以一最佳常数 α

- 最佳常数 α 作用：使 $x_1(t)$ 和 $\alpha x_2(t)$ 的均方差最小
仍以能量信号为例

$$\begin{aligned}
 Q &= \int_{-\infty}^{\infty} |x_1(t) - \alpha x_2(t)|^2 dt \\
 &= \int_{-\infty}^{\infty} [x_1(t) - \alpha x_2(t)][x_1(t) - \alpha x_2(t)]^* dt \\
 &= \int_{-\infty}^{\infty} [x_1(t) - \alpha x_2(t)][x_1^*(t) - \alpha x_2^*(t)] dt \\
 &= \int_{-\infty}^{\infty} [x_1(t)x_1^*(t) - \alpha x_1^*(t)x_2(t) - \alpha x_1(t)x_2^*(t) + \alpha^2 x_2(t)x_2^*(t)] dt
 \end{aligned}$$

- 使均方差 Q 最小的 α 应满足：

$$\frac{\partial Q}{\partial \alpha} = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} [-x_1^*(t)x_2(t) - x_1(t)x_2^*(t) + 2\alpha x_2(t)x_2^*(t)] dt = 0$$

$$50 \quad \therefore \quad \int_{-\infty}^{\infty} [x_1^*(t)x_2(t) + x_1(t)x_2^*(t)] dt = 2\alpha \int_{-\infty}^{\infty} x_2(t)x_2^*(t) dt$$

$$\begin{aligned} \therefore \alpha &= \frac{\int_{-\infty}^{\infty} [x_1^*(t)x_2(t) + x_1(t)x_2^*(t)]dt}{2\int_{-\infty}^{\infty} x_2(t)x_2^*(t)dt} = \frac{\int_{-\infty}^{\infty} \{[x_1^*(t)x_2(t)] + [x_1^*(t)x_2(t)]^*\}dt}{2\int_{-\infty}^{\infty} |x_2(t)|^2 dt} \\ &= \frac{\int_{-\infty}^{\infty} 2\operatorname{Re}[x_1^*(t)x_2(t)]dt}{2\int_{-\infty}^{\infty} |x_2(t)|^2 dt} = \frac{\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)]dt}{\int_{-\infty}^{\infty} |x_2(t)|^2 dt} \end{aligned}$$

$$\begin{aligned} \therefore Q &= \int_{-\infty}^{\infty} [x_1(t)x_1^*(t) - \alpha x_1^*(t)x_2(t) - \alpha x_1(t)x_2^*(t) + \alpha^2 x_2(t)x_2^*(t)]dt \\ &= \int_{-\infty}^{\infty} |x_1(t)|^2 dt - \alpha \int_{-\infty}^{\infty} 2\operatorname{Re}[x_1^*(t)x_2(t)]dt + \alpha^2 \int_{-\infty}^{\infty} |x_2(t)|^2 dt \end{aligned}$$

- 将 α 代入 Q 可得到最小的均方差 Q_α

$$\begin{aligned}
Q_\alpha &= \int_{-\infty}^{\infty} |x_1(t)|^2 dt - \frac{\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)]dt}{\int_{-\infty}^{\infty} |x_2(t)|^2 dt} \int_{-\infty}^{\infty} 2 \operatorname{Re}[x_1^*(t)x_2(t)]dt \\
&\quad + \frac{\left[\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)]dt \right]^2}{\left[\int_{-\infty}^{\infty} |x_2(t)|^2 dt \right]^2} \int_{-\infty}^{\infty} |x_2(t)|^2 dt \\
&= \int_{-\infty}^{\infty} |x_1(t)|^2 dt - \frac{2 \left[\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)]dt \right]^2}{\int_{-\infty}^{\infty} |x_2(t)|^2 dt} + \frac{\left[\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)]dt \right]^2}{\int_{-\infty}^{\infty} |x_2(t)|^2 dt} \\
&= \int_{-\infty}^{\infty} |x_1(t)|^2 dt - \frac{\left[\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)]dt \right]^2}{\int_{-\infty}^{\infty} |x_2(t)|^2 dt}
\end{aligned}$$

• 两边同除以 $\int_{-\infty}^{\infty} |x_1(t)|^2 dt$

$$\frac{Q_\alpha}{\int_{-\infty}^{\infty} |x_1(t)|^2 dt} = 1 - \frac{\left[\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)] dt \right]^2}{\int_{-\infty}^{\infty} |x_1(t)|^2 dt \int_{-\infty}^{\infty} |x_2(t)|^2 dt}$$

- **定义：** 两个信号的**相关系数**

$$\rho(x_1, x_2) = \frac{\int_{-\infty}^{\infty} \operatorname{Re}[x_1^*(t)x_2(t)] dt}{\left[\int_{-\infty}^{\infty} |x_1(t)|^2 dt \int_{-\infty}^{\infty} |x_2(t)|^2 dt \right]^{1/2}}$$

$$\therefore \frac{Q_\alpha}{\int_{-\infty}^{\infty} |x_1(t)|^2 dt} = 1 - \rho^2(x_1, x_2)$$

- $\rho(x_1, x_2)$ 越大，最小的均方差 Q_α 越小
- 相关系数 $\rho(x_1, x_2)$ 是两个信号相似程度的度量

- 许瓦兹不等式:

$$\left| \int_{-\infty}^{\infty} x_1(t)x_2(t)dt \right|^2 \leq \int_{-\infty}^{\infty} |x_1(t)|^2 dt \int_{-\infty}^{\infty} |x_2(t)|^2 dt$$

when $x_1(t) = ax_2^*(t)$, a 为实常数, 等号成立

$$\Rightarrow |\rho(x_1, x_2)| \leq 1$$

- $\rho(x_1, x_2)$ 定义为两个信号的归一化相关系数
- $\rho(x_1, x_2)=1$ 表示两个信号波形完全相似
- 有时也使用不归一的相关系数:

$$r(x_1, x_2) = \int_{-\infty}^{\infty} \text{Re}[x_1^*(t)x_2(t)]dt$$

- 类似地, 归一化和不归一的相关系数的定义可推广到周期信号和功率信号的情况

- 对两个相同周期(T)的信号，**相关系数**定义为：

$$r(x_1, x_2) = \frac{1}{T} \int_{-T/2}^{T/2} \text{Re}[x_1^*(t)x_2(t)]dt$$

$$\rho(x_1, x_2) = \frac{\frac{1}{T} \int_{-T/2}^{T/2} \text{Re}[x_1^*(t)x_2(t)]dt}{\left[\frac{1}{T} \int_{-T/2}^{T/2} |x_1(t)|^2 dt \cdot \frac{1}{T} \int_{-T/2}^{T/2} |x_2(t)|^2 dt \right]^{1/2}}$$

- 对功率信号，**相关系数**定义为：

$$r(x_1, x_2) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \text{Re}[x_1^*(t)x_2(t)]dt$$

$$\rho(x_1, x_2) = \frac{\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \text{Re}[x_1^*(t)x_2(t)]dt}{\left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x_1(t)|^2 dt \cdot \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x_2(t)|^2 dt \right]^{1/2}}$$

例1 求两个信号 $x_1(t)=A+B\cos\omega_0t$ 和 $x_2(t)=C+D\cos(\omega_0t+\theta)$ 的相
关系数

解: 这是两个周期信号, 周期为: $T = 2\pi / \omega_0$

$$\begin{aligned}r(x_1, x_2) &= \frac{1}{T} \int_{-T/2}^{T/2} \operatorname{Re}[x_1^*(t)x_2(t)]dt \\&= \frac{1}{T} \int_{-T/2}^{T/2} [A + B \cos \omega_0 t][C + D \cos(\omega_0 t + \theta)]dt \\&= \frac{1}{T} \int_{-T/2}^{T/2} AC dt + \frac{1}{T} \int_{-T/2}^{T/2} AD \cos(\omega_0 t + \theta) dt \\&\quad + \frac{1}{T} \int_{-T/2}^{T/2} BC \cos \omega_0 t dt + \frac{1}{T} \int_{-T/2}^{T/2} BD \cos \omega_0 t \cos(\omega_0 t + \theta) dt \\&= AC + \frac{1}{T} \int_{-T/2}^{T/2} BD \frac{1}{2} [\cos \theta + \cos(2\omega_0 t + \theta)] dt \\&= AC + \frac{1}{2} BD \cos \theta\end{aligned}$$

$$\begin{aligned}
\frac{1}{T} \int_{-T/2}^{T/2} |x_1(t)|^2 dt &= \frac{1}{T} \int_{-T/2}^{T/2} [A + B \cos \omega_0 t]^2 dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} [A^2 + 2AB \cos \omega_0 t + B^2 \cos^2 \omega_0 t] dt \\
&= A^2 + \frac{1}{T} \int_{-T/2}^{T/2} B^2 \frac{1}{2} [1 + \cos 2\omega_0 t] dt \\
&= A^2 + \frac{1}{2} B^2
\end{aligned}$$

类似 $\Rightarrow \frac{1}{T} \int_{-T/2}^{T/2} |x_2(t)|^2 dt = C^2 + \frac{1}{2} D^2$

$$\rho(x_1, x_2) = \frac{\frac{1}{T} \int_{-T/2}^{T/2} \text{Re}[x_1^*(t)x_2(t)] dt}{\left[\frac{1}{T} \int_{-T/2}^{T/2} |x_1(t)|^2 dt \cdot \frac{1}{T} \int_{-T/2}^{T/2} |x_2(t)|^2 dt \right]^{1/2}} = \frac{AC + \frac{1}{2} BD \cos \theta}{\left[\left(A^2 + \frac{1}{2} B^2 \right) \left(C^2 + \frac{1}{2} D^2 \right) \right]^{1/2}}$$

当 $A=C=0$ 时:

$$\rho(x_1, x_2) = \frac{AC + \frac{1}{2}BD \cos \theta}{\left[\left(A^2 + \frac{1}{2}B^2 \right) \left(C^2 + \frac{1}{2}D^2 \right) \right]^{\frac{1}{2}}} = \frac{\frac{1}{2}BD \cos \theta}{\left[\left(\frac{1}{2}B^2 \right) \left(\frac{1}{2}D^2 \right) \right]^{\frac{1}{2}}} = \cos \theta$$

两个频率相同的正弦信号间的相关系数是这两个信号之间相位差的余弦函数

进一步, 当相位差为: $\theta = \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, \dots$

$$\rho(x_1, x_2) = 0 \quad \text{两个信号不相关}$$

不相关通常称为正交

二、相关函数

- **相关系数**不足以表示两个信号的相似程度

例1中 $x_1(t) = B \cos \omega_0 t$, $x_2(t) = D \cos(\omega_0 t + \theta) = D \cos \omega_0(t + \theta / \omega_0)$
 $\Rightarrow \rho(x_1, x_2) = \cos \theta$

相关系数与其中一个波形的时间移动有关

- $x_1(t)$ 和 $x_2(t)$ 的**互相关函数(cross-correlation)**定义为:

对能量信号

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)dt$$

对功率信号

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t)x_2^*(t-\tau)dt$$

对周期信号

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^*(t - \tau) dt$$

- 当 $x_2(t) = x_1(t) = x(t)$, 类似地将 $x(t)$ 的**自相关函数(auto-correlation)**定义为:

对能量信号

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t - \tau) dt$$

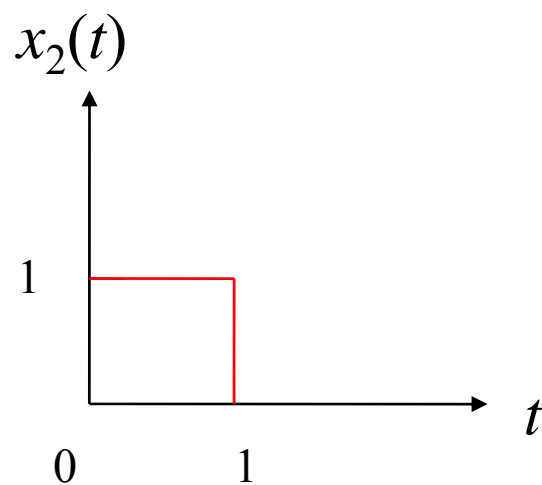
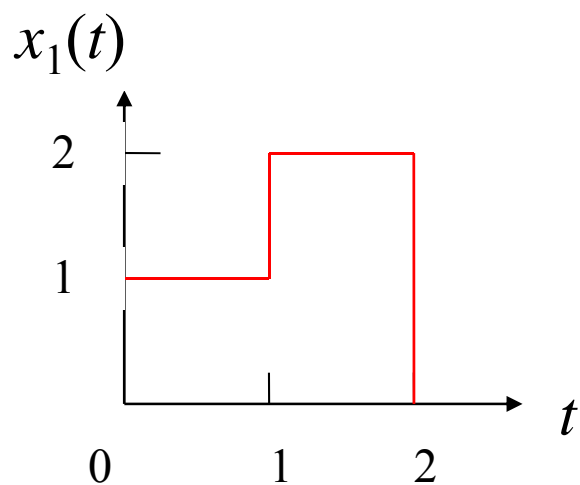
对功率信号

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t - \tau) dt$$

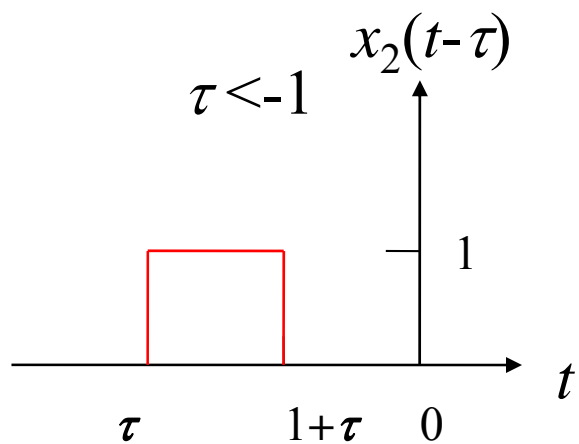
对周期信号

$$R_x(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t - \tau) dt$$

例2 求图示两个信号 $x_1(t)$ 和 $x_2(t)$ 的互相关函数

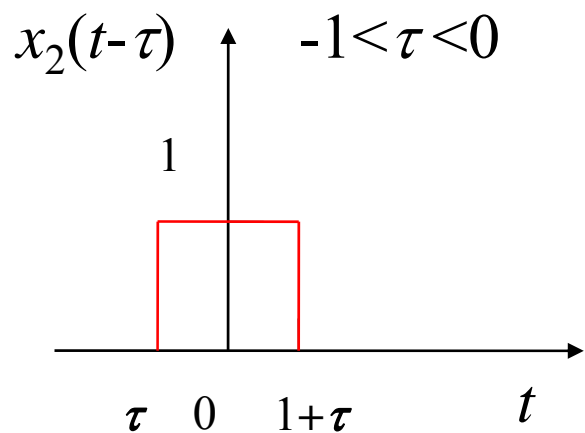


解:
$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$



当 $\tau < -1$ 时,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt = 0$$



当 $-1 < \tau < 0$ 时 ,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

$$= \int_0^{1+\tau} (1 \times 1)dt = 1 + \tau$$

当 $0 < \tau < 1$ 时 ,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

$$= \int_{\tau}^1 (1 \times 1)dt + \int_1^{1+\tau} (2 \times 1)dt = 1 - \tau + 2\tau = 1 + \tau$$

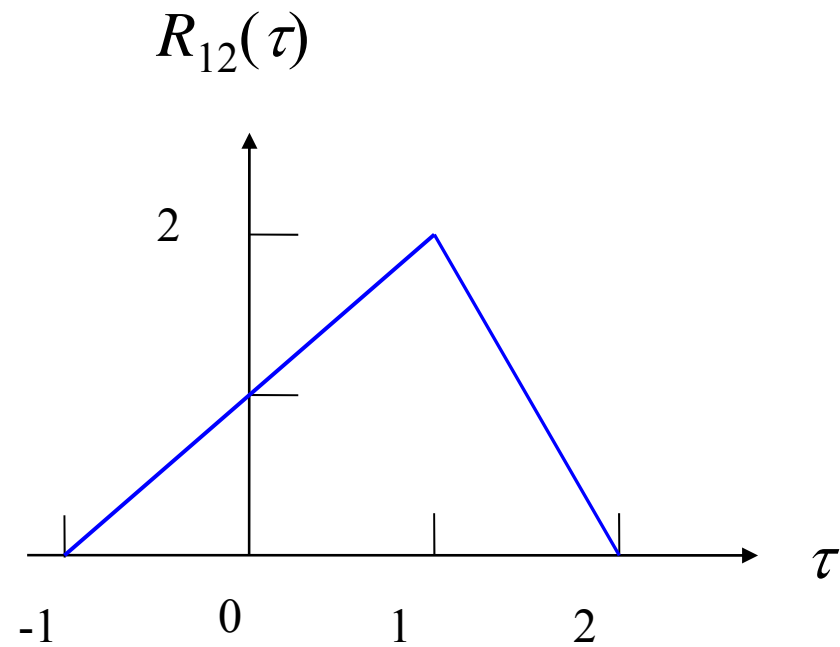
当 $1 < \tau < 2$ 时 ,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

$$= \int_{\tau}^2 (2 \times 1)dt = 2(2 - \tau) = 4 - 2\tau$$

当 $\tau > 2$ 时 , $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt = 0$

$$\therefore R_{12}(\tau) = \begin{cases} 1 + \tau & -1 < \tau < 1 \\ 4 - 2\tau & 1 < \tau < 2 \\ 0 & \text{else} \end{cases}$$



三、相关函数的性质

1、与频谱的关系

- **对能量信号，若**

$$F[x_1(t)] = X_1(\omega), F[x_2(t)] = X_2(\omega), F[R_{12}(\tau)] = E_{12}(\omega)$$

$$\Rightarrow E_{12}(\omega) = X_1(\omega)X_2^*(\omega)$$

证：

$$\begin{aligned} R_{12}(\tau) &= \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)dt = \int_{-\infty}^{\infty} x_1(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega)e^{j\omega(t-\tau)}d\omega \right]^* dt \\ &= \int_{-\infty}^{\infty} x_1(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2^*(\omega)e^{-j\omega(t-\tau)}d\omega dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2^*(\omega) \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt e^{j\omega\tau}d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2^*(\omega)X_1(\omega)e^{j\omega\tau}d\omega \end{aligned}$$

$$\therefore E_{12}(\omega) = F[R_{12}(\tau)] = X_1(\omega)X_2^*(\omega)$$

$E_{12}(\omega)$ 具有能量谱的量纲

称 $E_{12}(\omega)$ 为 $x_1(t)$ 和 $x_2(t)$ 的互能量谱，简称互谱

推论： $F[R_x(\tau)] = X(\omega)X^*(\omega) = E_x(\omega)$

维纳-辛钦定理：

能量信号自相关函数的傅里叶变换为信号的能量谱

- **对功率信号，类似地有：**

功率信号 $x_1(t)$ 和 $x_2(t)$ 的互相关函数的傅里叶变换 $P_{12}(\omega)$ 称为功率信号 $x_1(t)$ 和 $x_2(t)$ 的互功率谱，也简称互谱

$$F[R_{12}(\tau)] = P_{12}(\omega)$$

维纳-辛钦定理：

功率信号自相关函数的傅里叶变换为信号的功率谱

$$F[R_x(\tau)] = P_x(\omega)$$

2、相关函数的最大值

• 对能量信号

根据许瓦兹不等式：

$$\left| \int_{-\infty}^{\infty} x_1(t)x_2(t)dt \right|^2 \leq \int_{-\infty}^{\infty} |x_1(t)|^2 dt \int_{-\infty}^{\infty} |x_2(t)|^2 dt$$

when $x_1(t) = ax_2^*(t)$, a 为实常数, 等号成立

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt \int_{-\infty}^{\infty} |x^*(t-\tau)|^2 dt \right]^{1/2} = [EE]^{1/2} = E$$

$$R_x(0) = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \left[\int_{-\infty}^{\infty} |x(t)|^2 dt \int_{-\infty}^{\infty} |x^*(t)|^2 dt \right]^{1/2} = [EE]^{1/2} = E$$

$$\therefore R_x(\tau) \leq R_x(0) = E_x$$

能量信号自相关函数在 $\tau=0$ 时有最大值，最大值为信号的能量

互相关函数在 $\tau=0$ 时不一定有最大值：

$$\begin{aligned} R_{12}(0) &= \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)e^{j\omega\tau} \Big|_{\tau=0} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{12}(\omega)d\omega \end{aligned}$$

称 $R_{12}(0)$ 为信号 $x_1(t)$ 和 $x_2(t)$ 的交叉能量

• 对功率信号，类似地有：

$$R_x(\tau) \leq R_x(0) = P_x$$

功率信号自相关函数在 $\tau=0$ 时有最大值，最大值为信号的功率

互相关函数在 $\tau=0$ 时不一定有最大值：

$$R_{12}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{12}(\omega) d\omega$$

称 $R_{12}(0)$ 为信号 $x_1(t)$ 和 $x_2(t)$ 的交叉功率

3、共轭对称性

信号自相关函数具有共轭对称性

$$R_x(\tau) = R_x^*(-\tau)$$

以能量信号为例来证明，结论对功率信号也成立

证： $R_x^*(-\tau) = \left[\int_{-\infty}^{\infty} x(t)x^*(t+\tau)dt \right]^* = \int_{-\infty}^{\infty} x^*(t)x(t+\tau)dt$

令： $t = t' - \tau \rightarrow dt = dt'$

$$\therefore R_x^*(-\tau) = \int_{-\infty}^{\infty} x^*(t' - \tau)x(t')dt' = \int_{-\infty}^{\infty} x(t)x^*(t - \tau)dt = R_x(\tau)$$

当 $x(t)$ 为实信号时， $R_x(\tau)$ 是实函数

$$R(\tau) = R^*(\tau) \Rightarrow R(\tau) = R(-\tau)$$

实信号的自相关函数是 τ 的偶函数

对于互相关函数，有：

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

仍以能量信号为例来证明，结论对功率信号也成立

证： $R_{21}^*(-\tau) = \left[\int_{-\infty}^{\infty} x_2(t)x_1^*(t+\tau)dt \right]^* = \int_{-\infty}^{\infty} x_2^*(t)x_1(t+\tau)dt$

令： $t = t' - \tau \rightarrow dt = dt'$

$$\therefore R_{21}^*(-\tau) = \int_{-\infty}^{\infty} x_2^*(t' - \tau)x_1(t')dt' = \int_{-\infty}^{\infty} x_1(t)x_2^*(t - \tau)dt = R_{12}(\tau)$$

4、相关与卷积的关系

- **以能量信号为例来说明**

卷积：

$$x(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau, \quad X(\omega) = X_1(\omega)X_2(\omega)$$

相关：

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)dt, \quad E_{12}(\omega) = X_1(\omega)X_2^*(\omega)$$

相同点：时间延迟、相乘、积分

不同点：卷积信号反转；相关信号共轭

- **互相关可化为卷积来计算(卷积存在快速算法)**

先将第二个信号时间域上反转，记为第三个信号：

$$x_3(t) = x_2(-t)$$

再计算第一个信号与第三个信号的共轭信号的卷积：

$$\begin{aligned}x_1(\tau) * x_3^*(\tau) &= \int_{-\infty}^{\infty} x_1(t)x_3^*(\tau - t)dt = \int_{-\infty}^{\infty} x_1(t)x_3^*[-(t - \tau)]dt \\ &= \int_{-\infty}^{\infty} x_1(t)x_2^*(t - \tau)dt = R_{12}(\tau)\end{aligned}$$

计算结果就是第一个信号与第二个信号的互相关函数

四、线性系统与相关函数的关系

• 对能量信号

确定性信号通过线性系统：

$$\left. \begin{aligned}y(t) &= x(t) * h(t) \\ Y(\omega) &= X(\omega)H(\omega)\end{aligned} \right\} \Rightarrow$$

$$\begin{aligned}F[R_y(\tau)] &= E_y(\omega) = Y(\omega)Y^*(\omega) = [X(\omega)H(\omega)][X(\omega)H(\omega)]^* \\ &= X(\omega)X^*(\omega)H(\omega)H^*(\omega) = E_x(\omega) |H(\omega)|^2\end{aligned}$$

$$\because F[R_h(\tau)] = H(\omega)H^*(\omega) = |H(\omega)|^2$$

$$\therefore R_y(\tau) = R_x(\tau) * R_h(\tau)$$

类似地有：

$$\begin{aligned} F[R_{yx}(\tau)] &= E_{yx}(\omega) = Y(\omega)X^*(\omega) = [X(\omega)H(\omega)]X^*(\omega) \\ &= X(\omega)X^*(\omega)H(\omega) = E_x(\omega)H(\omega) \end{aligned}$$

$$R_{yx}(\tau) = R_x(\tau) * h(\tau)$$

• 对功率信号，同样有：

$$F[R_y(\tau)] = P_y(\omega) = P_x(\omega) |H(\omega)|^2$$

$$R_y(\tau) = R_x(\tau) * R_h(\tau)$$

$$F[R_{yx}(\tau)] = P_{yx}(\omega) = P_x(\omega)H(\omega)$$

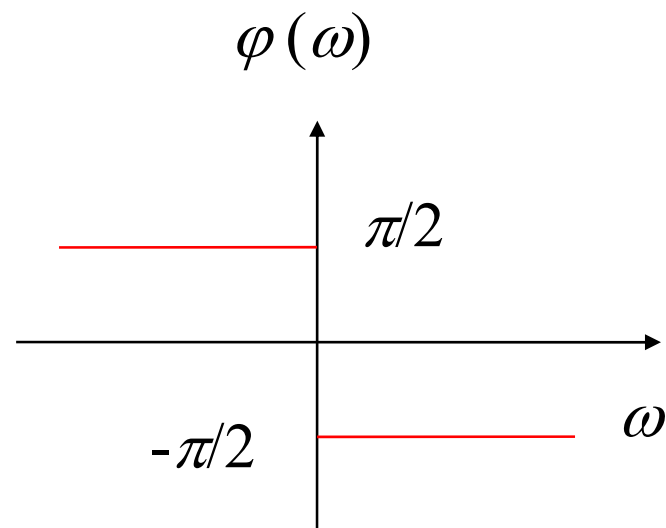
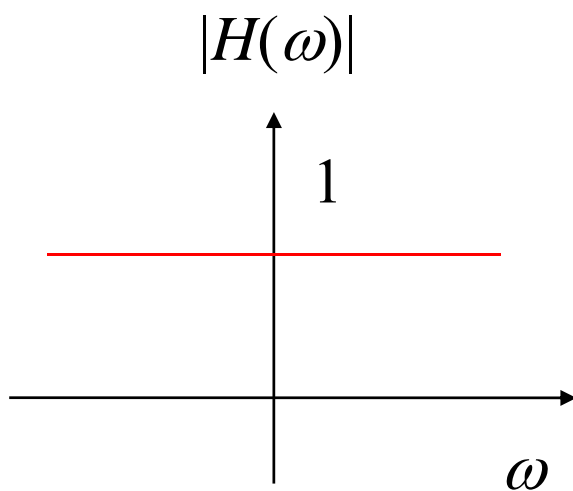
$$R_{yx}(\tau) = R_x(\tau) * h(\tau)$$

§ 1.9 Hilbert变换

一、Hilbert变换的基本概念

- **Hilbert变换**(希尔伯特变换/H变换): 移相网络

$$H(\omega) = \begin{cases} e^{-j\frac{\pi}{2}} & \omega > 0 \\ e^{j\frac{\pi}{2}} & \omega < 0 \end{cases}$$



$$H(\omega) = \begin{cases} e^{-j\frac{\pi}{2}} & \omega > 0 \\ e^{j\frac{\pi}{2}} & \omega < 0 \end{cases} = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases} = -j\text{Sgn}(\omega)$$

$$\therefore \text{F}[\text{Sgn}(t)] = \frac{2}{j\omega} \quad \text{对称性: } \text{F}[x(t)] = X(\omega) \Rightarrow \text{F}[X(t)] = 2\pi x(-\omega)$$

$$\therefore \text{F}\left[\frac{2}{jt}\right] = 2\pi \text{Sgn}(-\omega) = -2\pi \text{Sgn}(\omega)$$

$$H(\omega) = -j\text{Sgn}(\omega) \Rightarrow h(t) = \frac{1}{\pi t}$$

• $x(t)$ 的Hilbert变换: $\hat{x}(t)$



时域： $\hat{x}(t) = x(t) * h(t) = x(t) * \frac{1}{\pi t}$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} x(\tau) \frac{1}{t - \tau} d\tau = \frac{1}{\pi} \int_{-\infty}^{\infty} x(t - \tau) \frac{1}{\tau} d\tau$$

频域： $F[\hat{x}(t)] = X(\omega)H(\omega) = -jSgn(\omega)X(\omega)$

• **Hilbert反变换：**

$$H_1(\omega) = \frac{1}{H(\omega)} = \frac{1}{-jSgn(\omega)} = \frac{j}{-j^2 Sgn(\omega)} = jSgn(\omega)$$

$$\therefore h_1(t) = -\frac{1}{\pi t}$$

$$x(t) = \hat{x}(t) * h_1(t) = \hat{x}(t) * \left[-\frac{1}{\pi t} \right] = -\hat{x}(t) * \frac{1}{\pi t}$$

例1 求信号 $x(t)=\cos\omega_0t$ 的H变换

解： 从时间域求解：

$$\begin{aligned}\hat{x}(t) &= x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos[\omega_0(t-\tau)] \frac{1}{\tau} d\tau \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \cos\omega_0t \cos\omega_0\tau \frac{1}{\tau} d\tau + \frac{1}{\pi} \int_{-\infty}^{\infty} \sin\omega_0t \sin\omega_0\tau \frac{1}{\tau} d\tau \\ &= \frac{\cos\omega_0t}{\pi} \int_{-\infty}^{\infty} \frac{\cos\omega_0\tau}{\tau} d\tau + \frac{\sin\omega_0t}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega_0\tau}{\tau} d\tau\end{aligned}$$

$\because \cos\omega_0\tau$ 为 τ 的偶函数; $\sin\omega_0\tau$ 、 τ 为 τ 的奇函数

$$\begin{aligned}\hat{x}(t) &= \frac{2\sin\omega_0t}{\pi} \int_0^{\infty} \frac{\sin\omega_0\tau}{\tau} d\tau = \frac{2\sin\omega_0t}{\pi} \int_0^{\infty} \frac{\sin\omega_0\tau}{\omega_0\tau} d(\omega_0\tau) \\ &= \frac{2\sin\omega_0t}{\pi} \int_0^{\infty} Sa(x) dx = \frac{2\sin\omega_0t}{\pi} Si(\infty) = \frac{2\sin\omega_0t}{\pi} \frac{\pi}{2} = \sin\omega_0t\end{aligned}$$

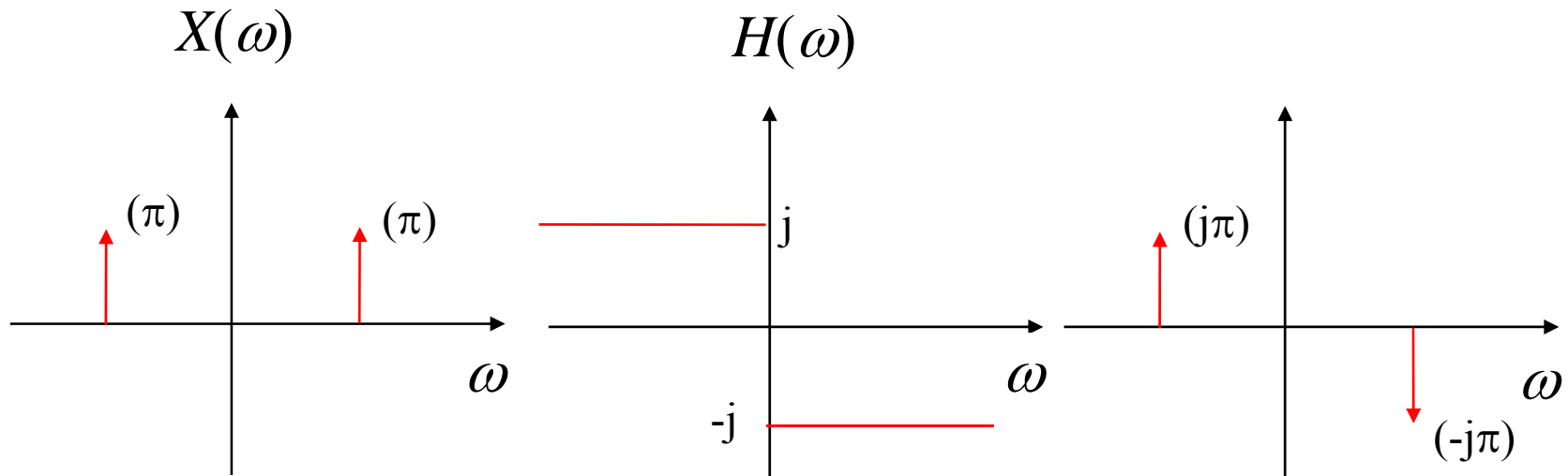
从频率域求解：

$$x(t) = \cos \omega_0 t \xleftrightarrow{F} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$H(\omega) = -j\text{Sgn}(\omega)$$

$$\therefore \hat{x}(t) \xleftrightarrow{F} X(\omega)H(\omega) = -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$$

$$\Rightarrow \hat{x}(t) = \sin \omega_0 t$$



- 常见的Hilbert变换:

$$\cos \omega_0 t \xleftrightarrow{H} \sin \omega_0 t \quad \sin \omega_0 t \xleftrightarrow{H} -\cos \omega_0 t$$

$$\delta(t) \xleftrightarrow{H} \frac{1}{\pi t} \quad \frac{1}{\pi t} \xleftrightarrow{H} -\delta(t)$$

- Hilbert变换 $\Leftrightarrow -\pi/2$ 移相器

$$\cos \omega_0 t \xleftrightarrow{H} \cos\left(\omega_0 t - \frac{\pi}{2}\right) = \sin \omega_0 t$$

$$\sin \omega_0 t \xleftrightarrow{H} \sin\left(\omega_0 t - \frac{\pi}{2}\right) = -\cos \omega_0 t$$

$$e^{j\omega_0 t} \xleftrightarrow{H} e^{j\left(\omega_0 t - \frac{\pi}{2}\right)} = e^{-j\frac{\pi}{2}} e^{j\omega_0 t} = -je^{j\omega_0 t} = \sin \omega_0 t - j \cos \omega_0 t$$

二、Hilbert变换的性质

1、H变换只改变信号的相位谱，振幅谱不变

$$H(\omega) = \begin{cases} e^{-j\frac{\pi}{2}} & \omega > 0 \\ e^{j\frac{\pi}{2}} & \omega < 0 \end{cases}$$

- 对能量信号 → 能量谱不变 → 能量不变
- 对功率信号 → 功率谱不变 → 功率不变

2、实信号的H变换仍为实信号

$$\begin{aligned} \hat{x}(t) &= x(t) * h(t) = x(t) * \frac{1}{\pi t} \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} x(\tau) \frac{1}{t-\tau} d\tau = \frac{1}{\pi} \int_{-\infty}^{\infty} x(t-\tau) \frac{1}{\tau} d\tau \end{aligned}$$

3、实信号 $x(t)$ 与其H变换 $\hat{x}(t)$ 是正交的:

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$$

证: 对于实信号:

$$\begin{aligned} R_{12}(\tau) &= \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)dt = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{12}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)e^{j\omega\tau}d\omega \end{aligned}$$

令: $\tau = 0, x_1(t) = x(t), x_2(t) = \hat{x}(t)$

$$\begin{aligned} \therefore R_{12}(0) &= \int_{-\infty}^{\infty} x_1(t)x_2(t)dt = \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\{F[\hat{x}(t)]\}^*d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\{X(\omega)[-j\text{Sgn}(\omega)]\}^*d\omega \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\{X^*(\omega)[jSgn(\omega)]\}d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 [jSgn(\omega)]d\omega \end{aligned}$$

对于实信号： $|X(\omega)|^2$ 为 ω 的偶函数

然而： $Sgn(\omega)$ 为 ω 的奇函数

$\Rightarrow |X(\omega)|[jSgn(\omega)]$ 为 ω 的奇函数

$$\therefore \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 [jSgn(\omega)]d\omega = 0$$

4、信号 $x(t)$ 的H变换 $\hat{x}(t)$ 的H变换为：

$$\hat{\hat{x}}(t) = -x(t)$$

证： $F[\hat{x}(t)] = F[\hat{x}(t)][-j\text{Sgn}(\omega)] = X(\omega)[-j\text{Sgn}(\omega)][-j\text{Sgn}(\omega)]$
 $= X(\omega)(-1)^2 j^2 [\text{Sgn}(\omega)]^2 = -X(\omega)[\text{Sgn}(\omega)]^2$
 $= \begin{cases} -X(\omega) \times 1 \times 1 & \omega > 0 \\ -X(\omega) \times (-1) \times (-1) & \omega < 0 \end{cases} = -X(\omega)$
 $\Rightarrow \hat{x}(t) = -x(t)$

5、卷积的H变换：

if $x(t) = x_1(t) * x_2(t) \Rightarrow \hat{x}(t) = \hat{x}_1(t) * x_2(t) = x_1(t) * \hat{x}_2(t)$

证：

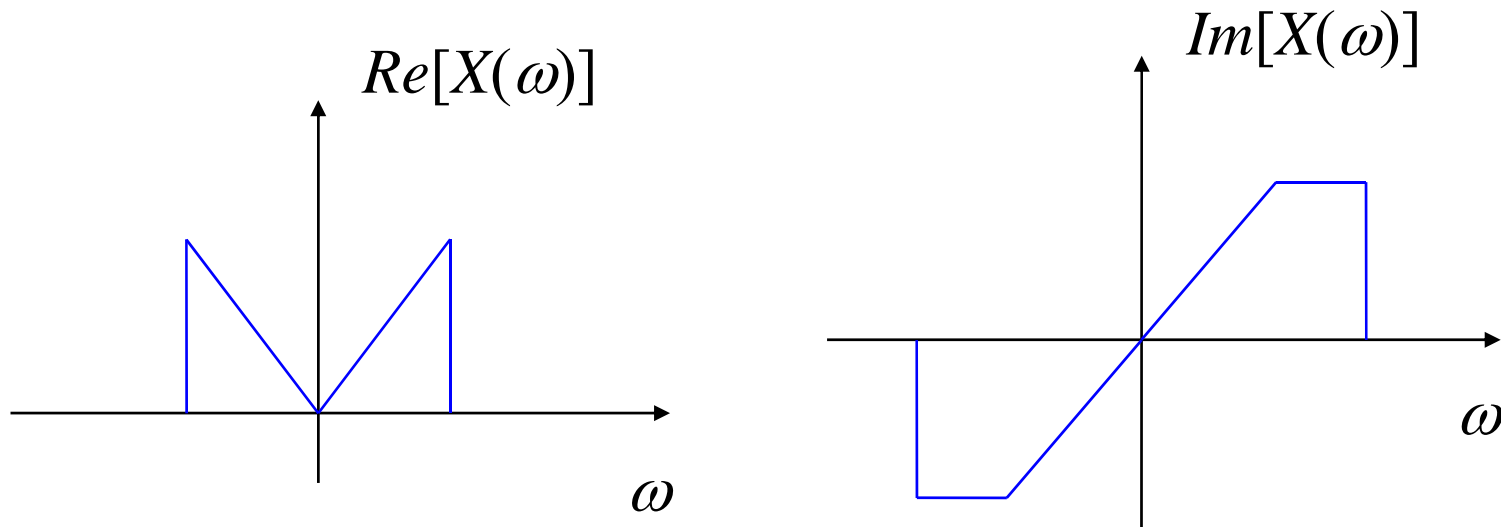
$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = [x_1(t) * x_2(t)] * \frac{1}{\pi t} = x_1(t) * \left[x_2(t) * \frac{1}{\pi t} \right]$$

$$= x_1(t) * \hat{x}_2(t) = x_2(t) * \left[x_1(t) * \frac{1}{\pi t} \right] = \hat{x}_1(t) * x_2(t)$$

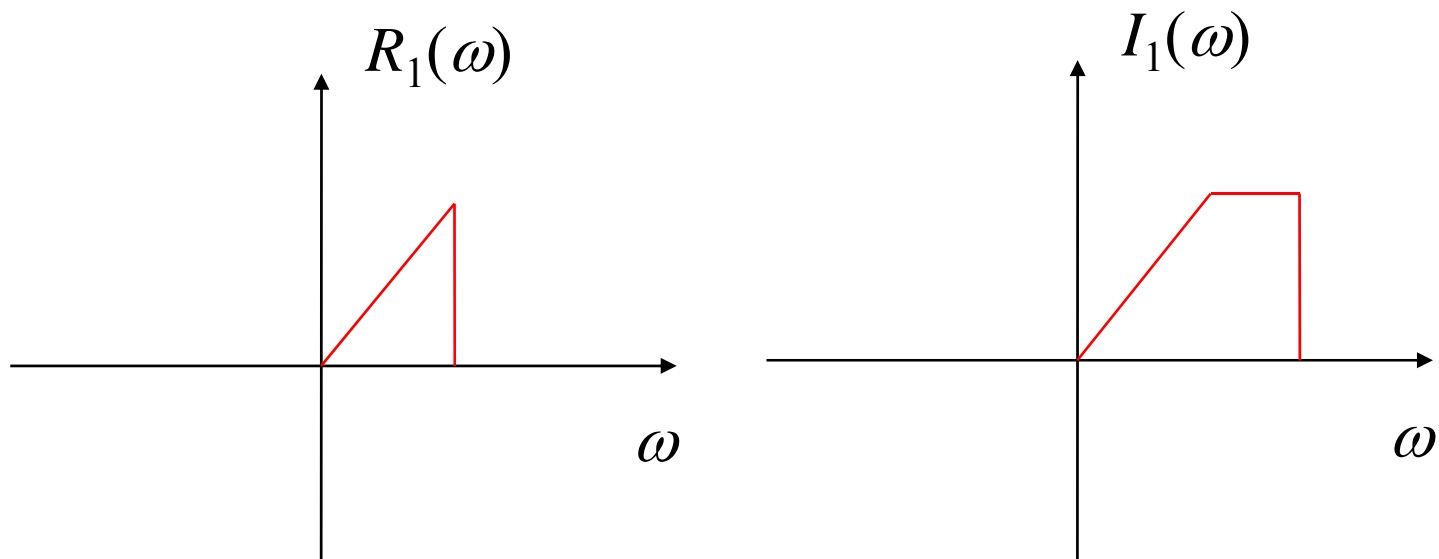
三、Hilbert变换的应用

1、实信号的单边频谱表示——解析信号

- 实信号 $x(t)$ 的频谱实部关于 ω 偶对称，虚部关于 ω 奇对称



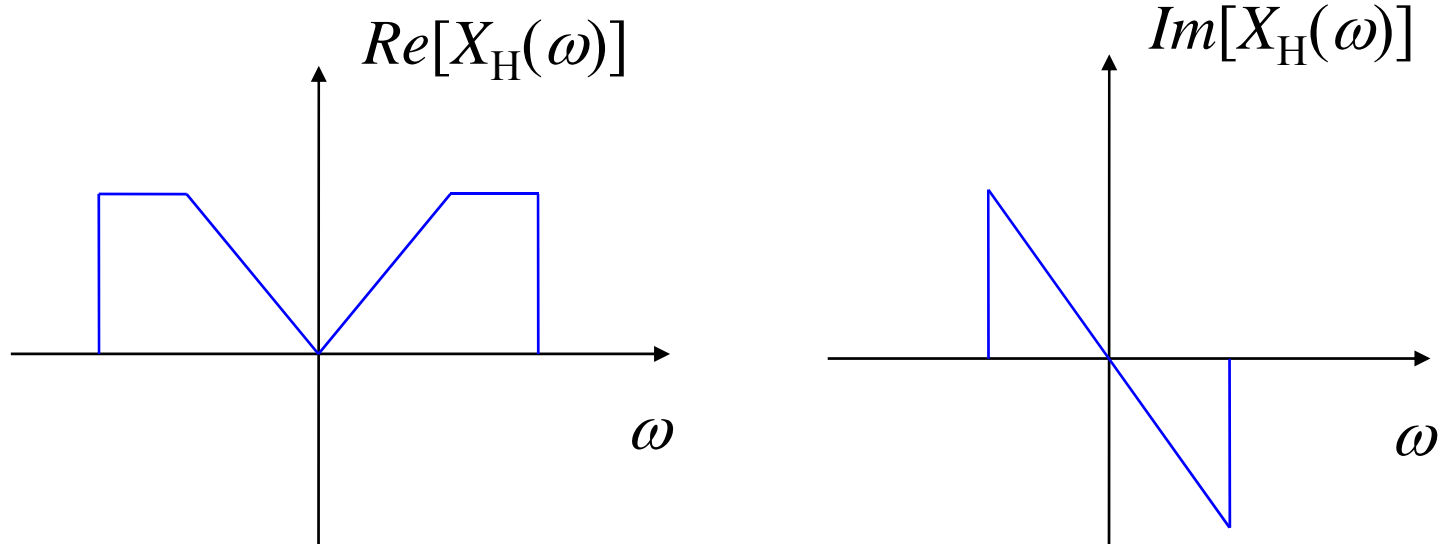
- 已知实信号 $\omega > 0$ 的频谱部分 \rightarrow 实信号的频谱 $X(\omega)$
- 有时只需讨论 $\omega > 0$ 的频谱部分 \rightarrow 单边频谱的信号



$$x(t) \xleftrightarrow{F} X(\omega) = \begin{cases} R_1(\omega) + jI_1(\omega) & \omega > 0 \\ R_1(|\omega|) - jI_1(|\omega|) & \omega < 0 \end{cases}$$

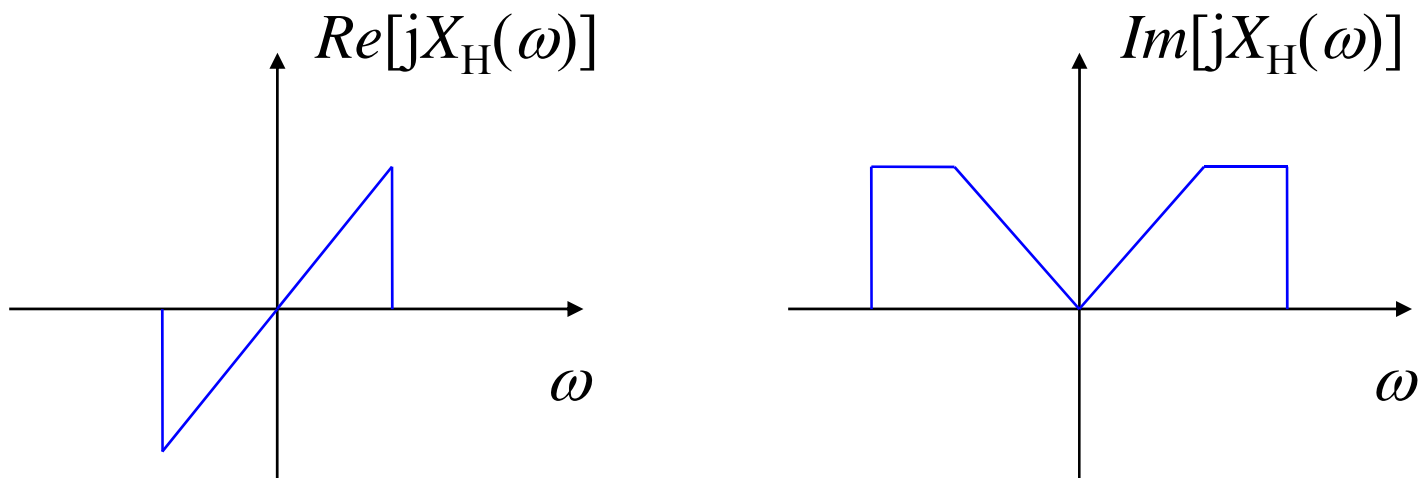
$$\therefore \hat{x}(t) \xleftrightarrow{F} X_H(\omega) = X(\omega)[-j\text{Sgn}(\omega)] = \begin{cases} -j[R_1(\omega) + jI_1(\omega)] & \omega > 0 \\ j[R_1(|\omega|) - jI_1(|\omega|)] & \omega < 0 \end{cases}$$

$$= \begin{cases} -jR_1(\omega) + I_1(\omega) & \omega > 0 \\ jR_1(|\omega|) + I_1(|\omega|) & \omega < 0 \end{cases}$$



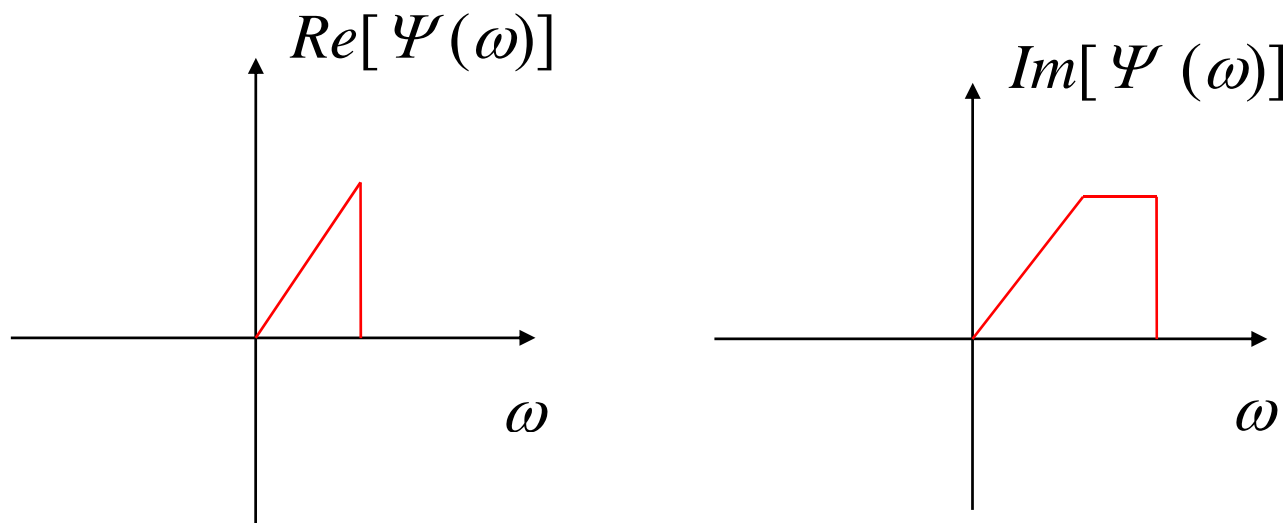
$$\hat{x}(t) \xleftrightarrow{F} X_H(\omega) = \begin{cases} -jR_1(\omega) + I_1(\omega) & \omega > 0 \\ jR_1(|\omega|) + I_1(|\omega|) & \omega < 0 \end{cases}$$

$$\Rightarrow j\hat{x}(t) \xleftrightarrow{F} jX_H(\omega) = \begin{cases} R_1(\omega) + jI_1(\omega) & \omega > 0 \\ -R_1(|\omega|) + jI_1(|\omega|) & \omega < 0 \end{cases}$$



$$\therefore \psi(t) = x(t) + j\hat{x}(t) \xrightarrow{F} \Psi(\omega) = \begin{cases} [R_1(\omega) + jI_1(\omega)] + [R_1(\omega) + jI_1(\omega)] & \omega > 0 \\ [R_1(|\omega|) - jI_1(|\omega|)] + [-R_1(|\omega|) + jI_1(|\omega|)] & \omega < 0 \end{cases}$$

$$= \begin{cases} 2[R_1(\omega) + jI_1(\omega)] = 2X(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases}$$



- 信号 $\psi(t)$ 只存在 $\omega > 0$ 的频谱(单边频谱)
- 信号 $\psi(t)$ 称为实信号 $x(t)$ 的解析信号(analytic signal)

$$\psi(t) = x(t) + j\hat{x}(t) = x(t) + j \left[x(t) * \frac{1}{\pi t} \right]$$

解析信号的虚部为其实部的H变换 解析信号的傅里叶变换总是因果的

- 常见实信号 $x(t)$ 的解析信号 $\psi(t)$

$$x(t) = \cos \omega_0 t \quad \rightarrow \quad \psi(t) = x(t) + j\hat{x}(t) = \cos \omega_0 t + j \sin \omega_0 t = e^{j\omega_0 t}$$

$$x(t) = \cos \omega_0 t \xleftrightarrow{\text{F}} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$\psi(t) = e^{j\omega_0 t} \xleftrightarrow{\text{F}} 2\pi\delta(\omega - \omega_0)$$

$$\begin{aligned} x(t) = \sin \omega_0 t \quad \rightarrow \quad \psi(t) &= x(t) + j\hat{x}(t) = \sin \omega_0 t - j \cos \omega_0 t \\ &= -j(\cos \omega_0 t + j \sin \omega_0 t) = -je^{j\omega_0 t} = e^{j(\omega_0 t - \pi/2)} \end{aligned}$$

$$x(t) = \sin \omega_0 t \xleftrightarrow{\text{F}} -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$$

$$\psi(t) = -je^{j\omega_0 t} \xleftrightarrow{\text{F}} -j[2\pi\delta(\omega - \omega_0)] = -2j\pi\delta(\omega - \omega_0)$$

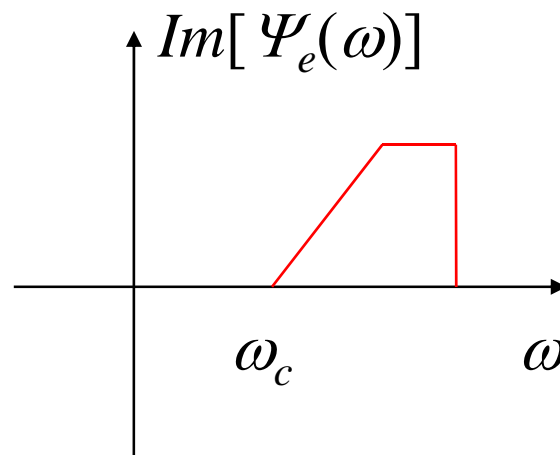
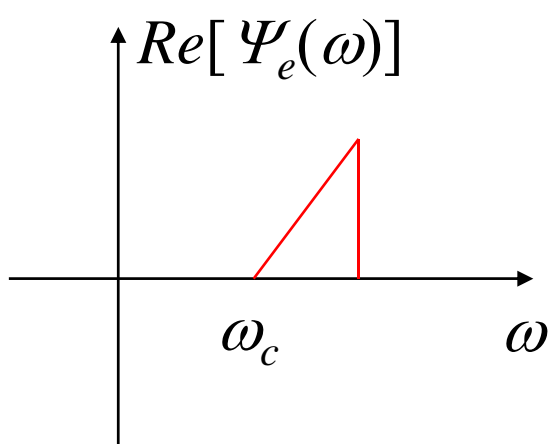
- 实信号 $x(t)$ 的解析信号 $\psi(t)$ 也称为 $x(t)$ 的**预包络(per-envelope)**
- 实信号 $x(t)$ 的**复合包络(complex-envelope)**为:

$$\psi_e(t) = \psi(t)e^{-j\omega_c t}, \quad \omega_c > 0$$

复合包络也是解析信号

$$\Psi(\omega) = \begin{cases} 2X(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases} \quad e^{-j\omega_c t} \xleftrightarrow{F} 2\pi\delta(\omega - \omega_c)$$

$$\Rightarrow \Psi_e(\omega) = \begin{cases} 2X(\omega - \omega_c) & \omega > \omega_c \\ 0 & \omega < \omega_c \end{cases}$$



复合包络 $\psi_e(t)$ 只有 $\omega > 0$ 的频谱 \rightarrow

$$\psi_e(t) = I(t) + jQ(t) = I(t) + j\hat{I}(t)$$

$I(t)$ 和 $Q(t)$ 分别称为 $x(t)$ 在频率 ω_c 上的 **同相分量** 和 **正交分量**

实信号 $x(t)$ 可用其在频率 ω_c 上的复合包络来描述:

$$x(t) = \text{Re}[\psi(t)] = \text{Re}[\psi(t)e^{-j\omega_c t} e^{j\omega_c t}] = \text{Re}[\psi_2(t)e^{j\omega_c t}]$$

$x(t)$ 也可用其在频率 ω_c 上的同相分量和正交分量来描述:

$$\begin{aligned} x(t) &= \text{Re}[\psi_e(t)e^{j\omega_c t}] = \text{Re}\{[I(t) + jQ(t)][\cos \omega_c t + j \sin \omega_c t]\} \\ &= I(t) \cos \omega_c t - Q(t) \sin \omega_c t \end{aligned}$$

- $\psi(t)$ 或 $\psi_e(t)$ 的幅度称为 $x(t)$ 的 **自然包络** (natural envelope)

$$A(t) = |\psi(t)| = |\psi_e(t)|$$

自然包络 $A(t)$ 是正实信号

例2 求信号 $x(t)=\cos\omega t$ 的预包络、复合包络和自然包络

解： $\because \cos\omega t \xrightarrow{H} \sin\omega t$

$x(t)$ 的预包络为：

$$\psi(t) = x(t) + j\hat{x}(t) = \cos\omega t + j\sin\omega t = e^{j\omega t}$$

$x(t)$ 在频率 ω_c 上的复合包络为：

$$\psi_e(t) = \psi(t)e^{-j\omega_c t} = e^{j\omega t}e^{-j\omega_c t} = e^{j(\omega-\omega_c)t}$$

$x(t)$ 的自然包络为： $A(t) = |\psi_e(t)| = |\psi(t)| = 1$

验证： $I(t) = \cos(\omega - \omega_c)t$, $Q(t) = \sin(\omega - \omega_c)t$

$$\begin{aligned} I(t)\cos\omega_c t - Q(t)\sin\omega_c t &= \cos(\omega - \omega_c)t \cos\omega_c t - \sin(\omega - \omega_c)t \sin\omega_c t \\ &= \cos[(\omega - \omega_c)t + \omega_c t] = \cos\omega t = x(t) \end{aligned}$$

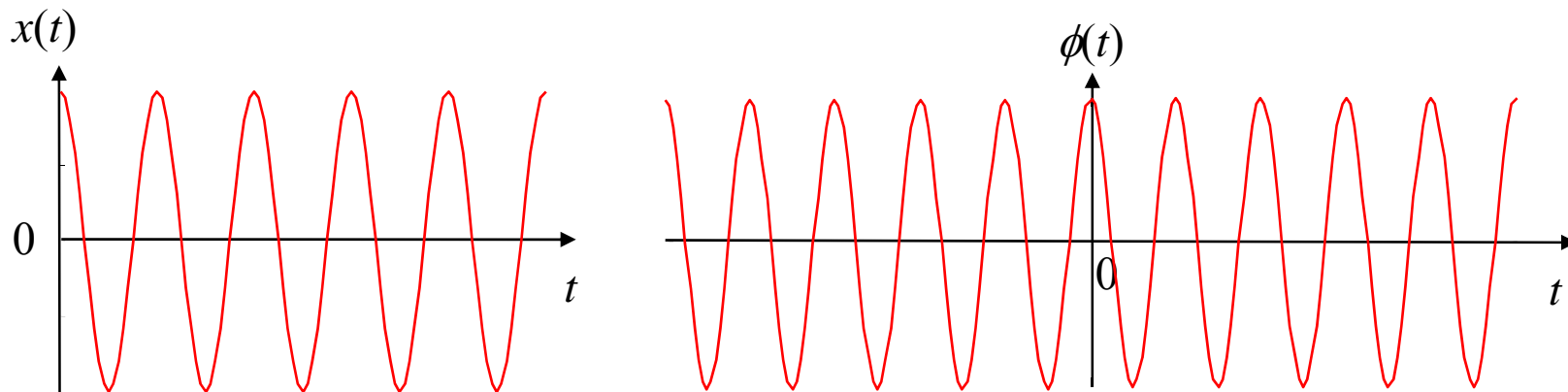
2、实因果信号频谱实部与虚部的关系

- 对于实因果信号 $x(t)$ ： $x(t) = x^*(t)$ and $\begin{cases} x(t) \neq 0 & t > 0 \\ x(t) = 0 & t < 0 \end{cases}$

其频谱 $X(\omega)$ 为：

$$x(t) \xrightarrow{F} X(\omega) = R(\omega) + jI(\omega), \quad I(\omega) = -R(\omega) * \frac{1}{\pi\omega}$$

证： 设偶信号 $\phi(t)$ 为： $\phi(t) = \begin{cases} x(t) & t > 0 \\ x(-t) & t < 0 \end{cases}$



$$\therefore x(t) = \phi(t)u(t)$$

$$\begin{aligned} \Rightarrow X(\omega) &= \frac{1}{2\pi} \left\{ \Phi(\omega) * \left[\pi\delta(\omega) + \frac{1}{j\omega} \right] \right\} = \frac{1}{2} \Phi(\omega) * \delta(\omega) + \frac{1}{2\pi} \Phi(\omega) * \frac{1}{j\omega} \\ &= \frac{1}{2} \Phi(\omega) + \frac{1}{2j} \Phi(\omega) * \frac{1}{\pi\omega} = \frac{1}{2} \Phi(\omega) + j \left[-\frac{1}{2} \Phi(\omega) * \frac{1}{\pi\omega} \right] \end{aligned}$$

而 $\Phi(\omega) = \int_{-\infty}^{\infty} \phi(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} \phi(t) \cos \omega t dt - j \int_{-\infty}^{\infty} \phi(t) \sin \omega t dt$

$\phi(t)$ 、 $\cos \omega t$ 为偶信号； $\sin \omega t$ 为奇信号 \rightarrow

$\phi(t) \cos \omega t$ 为偶信号； $\phi(t) \sin \omega t$ 为奇信号

$$\therefore \Phi(\omega) = 2 \int_0^{\infty} \phi(t) \cos \omega t dt = 2 \int_0^{\infty} x(t) \cos \omega t dt$$

而 $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) \cos \omega t dt - j \int_{-\infty}^{\infty} x(t) \sin \omega t dt$

$$50 \quad = R(\omega) + jI(\omega) \quad \Rightarrow \quad R(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t dt$$

$$x(t) \text{ 为因果信号} \rightarrow R(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t dt = \int_0^{\infty} x(t) \cos \omega t dt$$

$$\therefore \Phi(\omega) = 2 \int_0^{\infty} x(t) \cos \omega t dt = 2R(\omega)$$

$$\Rightarrow X(\omega) = \frac{1}{2} \Phi(\omega) + j \left[-\frac{1}{2} \Phi(\omega) * \frac{1}{\pi\omega} \right]$$

$$= R(\omega) + j \left[-R(\omega) * \frac{1}{\pi\omega} \right] = R(\omega) + jI(\omega)$$

$$\therefore I(\omega) = -R(\omega) * \frac{1}{\pi\omega}$$

实因果信号频谱的实部和虚部不独立，两者为H变换关系

例3 讨论阶跃信号 $u(t)$ 频谱实部与虚部的关系

解： $u(t) \xrightarrow{F} \pi\delta(\omega) + \frac{1}{j\omega} = \pi\delta(\omega) + j \left[-\frac{1}{\omega} \right]$

$$R(\omega) = \pi\delta(\omega) \quad I(\omega) = -\frac{1}{\omega}$$

$$-R(\omega) * \frac{1}{\pi\omega} = -\pi\delta(\omega) * \frac{1}{\pi\omega} = -\delta(\omega) * \frac{1}{\omega} = -\frac{1}{\omega}$$

$$\therefore I(\omega) = -R(\omega) * \frac{1}{\pi\omega}$$

3、窄带信号的H变换

- 窄带信号 $x(t)$ 的表达式：

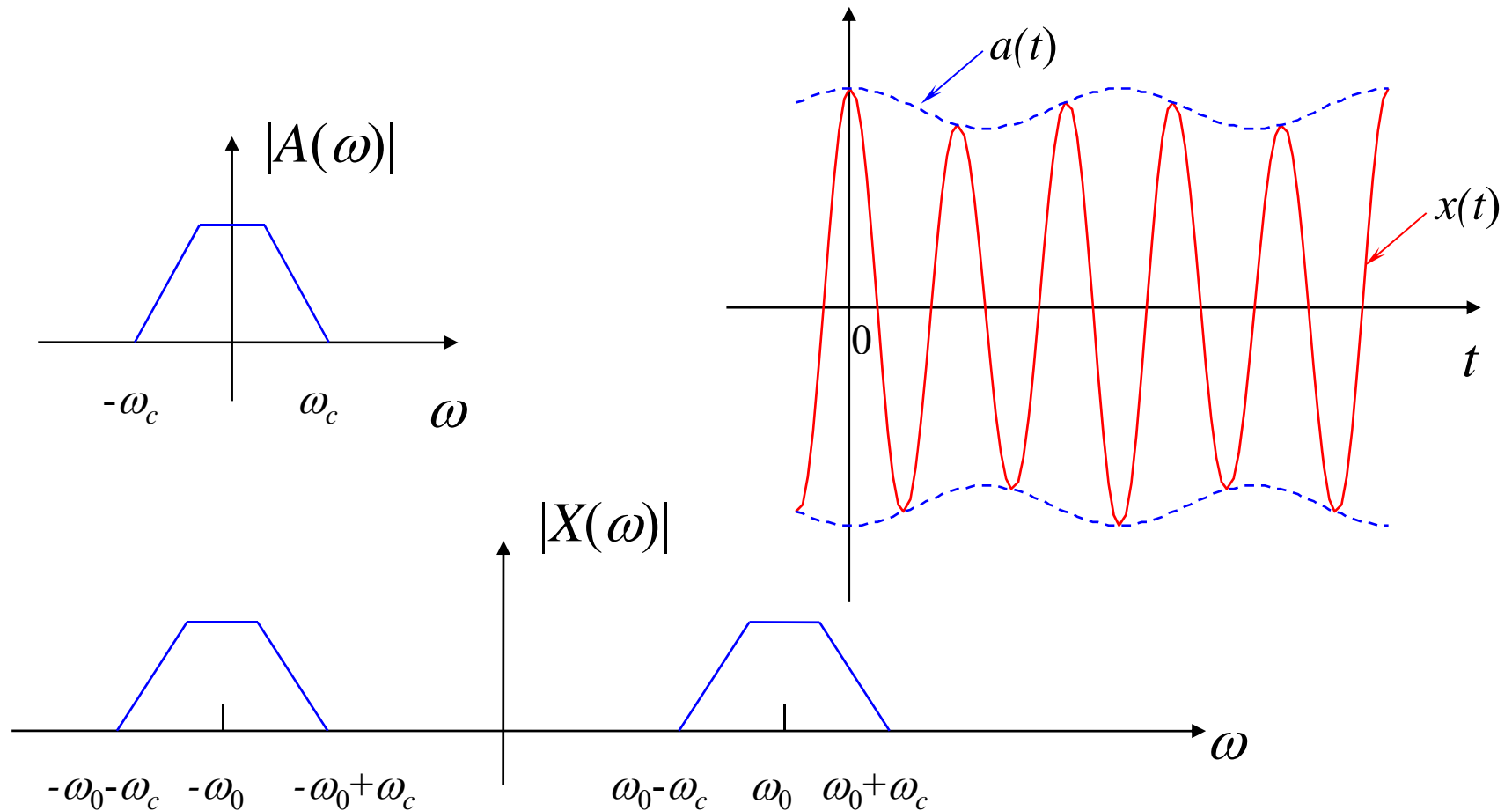
$$x(t) = a(t) \cos[\omega_0 t + \theta]$$

其中 $a(t)$ 相对 $\cos(\omega_0 t + \theta)$ 来说，是慢变化的信号

即 $a(t)$ 的频谱为低通型的：

$$a(t) \xrightarrow{F} A(\omega) = \begin{cases} \neq 0 & |\omega| \leq \omega_c \ll \omega_0 \\ = 0 & \text{else} \end{cases}$$

ω_0 是窄带信号的载频, $\omega_0 t + \theta$ 是窄带信号的相位



• 窄带信号 $x(t)$ 的H变换为：

$$\hat{x}(t) = a(t) \sin(\omega_0 t + \theta)$$