第23卷 第4期 哈尔滨工业大学学报 Vol. 23, No. 4
1991年8月 JOURNAL OF HARBIN INSTITUTE OF TECHNOLOGY Aug. 1991

一类非线性 Volterra 积分方程的非负解

数学系 余乃忠 何风兰*

Nonnegative Solutions of A Class of Nonlinear Volterra Integral Equation

Yu Naizhong He Fenglan

(Department of Mathematics)

Key words Volterra integral equation; integral inequality

In some problems of mathematics and physics and physics, the following nonlinear Volterra equation of convolution

$$u(x)=\int_0^x a(x-s)g(u(s))ds+f(x) \tag{1}$$

was given[1-3].

Because of requirement of real existence, the solution of (1) ought to satisfy $u(0)=0$; $u(x)>0$, $x>0$ and be continuous. All known functions of (1) are nonnegative. The existence of the solutions (1) has been studied[4-5]. In this paper, we consider more general equation as follows

$$u(x)=\int_0^x g(x,s,u(s))ds+f(x) \tag{2}$$

The theorem of existence of continuous solutions has been obtained, where solution $u(x)$ satisfies $u(x)>0$, $x>0$.

Lemma Let u, f be nonnegative, bounded and measurable functions on R^*, g be measurable on $D=T\times R^*$, and $g(x,s,u)$ be a nonnegative, continuous and nondecreasing function of u on R^+ for each $(x,t)\in T$ as well as bounded on T for

Received Date: 1990-11-12
*牡丹江师范学院

each $u \in R^+$, $T=\{(x, s),\ 0< s< x<\infty\}$. Let $\bar{g}(x, u)$ be a strictly upper convex and derivable function of u, for each $x \in R^+$, where $g(x, u) = \sup\limits_{0<s<t<x} g(t, s, u)$ $g(x, s, 0)=0, f(x)>0, x>0$.

If $u(x)$ satisfies the following inequality

$$u(x)< \int_0^x g(x, s, u(s))ds+f(x) \tag{3}$$

then $u(x)< \bar{f}(x)+\varphi_x^{-1}(x)$, $x \in R^+$, where, $\bar{f}(x) = \sup\limits_{0<t<x} f(t)$

$$\varphi_x(r)=\int_0^r \frac{ds}{\bar{g}(x, \bar{f}(x)+s)}$$

φ_x^{-1} is the inverse function of φ_x.

Proof We define the functions $\bar{f}$, $\bar{g}$ by

$$\bar{f}(x)= \sup_{0<t<x} f(t),\ \bar{g}(x, u)= \sup_{0<s<t<x} f(t, s, u).$$

Then $\bar{f}(x)$ and $\bar{g}(x, u)$ are nondecreasing in x. By (3), we have

$$u(x)< \bar{f}(x)+\int_0^x \bar{g}(x, u(s))ds.$$

For arbitrary fixed $x \in R^+$, we have

$$u(x)< \bar{f}(x)+U(x), \qquad 0< x< X \tag{4}$$

here

$$U(x)=\int_0^x \bar{g}(X, u(s))ds.$$

Then

$$U'(x)=\bar{g}(X, u(x))< \bar{g}(X, \bar{f}(X)+U(x)), \qquad 0< x< X,$$

$$\frac{U'(x)}{\bar{g}(X, \bar{f}(X)+U(x))}< 1, \quad 0< x< X \tag{5}$$

Set

$$\Phi_X(r)=\int_0^r \frac{ds}{\bar{g}(X, \bar{f}(X)+s)}$$

Then, by (5), we have

$$\frac{d\Phi_X(U(x))}{dx}< 1, \quad 0< x< X,$$

and

$$\varphi_X(U(x))< x, 0< x< X.$$

In particular, for $x=X$, also have

$$\Phi_X(U(X))< X$$

Since X is arbitrary, replacing X by x, we have

$$\Phi_X(U(x)) < x$$

Let Φ_x^{-1} denote the inverse function of Φ_X, then we obtain

$$U(x) < \varphi_x(x), \quad x \in \mathrm{Dom}(\Phi_x^{-1})$$

By the known conditions, $\bar{g}(x, u)$ is strictly upper convex and derivable in x and $\bar{g}(x, 0) = 0$, so we see that for arbitrary $u_0 > 0$, there exists $a > 0$, such that $g(x, u) < au$, for $u > u_0$. Then, for arbitrary fixed $x > 0$, we have

$$\bar{g}(x, \bar{f}(x) + s) < a(\bar{f}(x) + s), \quad s > 0,$$

so

$$\Phi_x(r) = \int_0^r \frac{ds}{\bar{g}(x, \bar{f}(x) + s)} > \int_0^r \frac{ds}{a\bar{f}(x) + as}$$

and further

$$\int_0^\infty \frac{ds}{a\bar{f}(x) + as} = \infty.$$

Then,

$$\Phi_X(r) = \infty, \quad r \to \infty.$$

Since $\Phi_X(r)$ is increasing and continuous in r, the domain of value of $\Phi_x(r)$ is $[0, \infty)$, $x \in R^+$.

$$\text{i.e. } \mathrm{Dom}(\Phi_x^{-1}) = R^+, \quad x \in R^+.$$

Thus

$$U(x) < \Phi_x^{-1}(x), \quad x \in R^+.$$

By (4), we have

$$u(x) < \bar{f}(x) + U(x)$$

so

$$u(x) < \bar{f}(x) + \Phi_x^{-1}(x), \quad x \in R^+.$$

The proof is completed.

Theorem Under the hypotheses of the lemma, suppose $g(x, s, u)$ is a uniformly continuous function of u for $(x, u) \in (R^+)^2$ and $g(x, s, 0) = 0$, $(x, s) \in T$; f is a continuous function on R^+ and $f(0) = 0$; $f(x) > 0$, $x > 0$. Then the equation (2) has a continuous solution $u(x)$ on R^+ and $u(0) = 0$; $u(x) > 0$, $x > 0$.

Proof Define

$$Tu(x) = \int_0^{\alpha x} g(x, s, u(s))\,ds + f(x)$$

$$u_0 = f(x)$$

$$u_{n+1} = T u_n(x), \quad n = 0, 1, 2, \cdots \tag{6}$$

We obtain a sequence $\{u_n(x)\}$. Now study the sequence

$$u_1 = T(u_0(x)) = \int_0^x g(x, s, f(s))ds + f(x) > f(x) = u_0$$

$$u_2 = T(u_1(x)) = \int_0^x g(x, s, u_1(x))ds + f(x) > \int_0^x g(x, s, u_0(s))ds + f(x) = u_1(x)$$

By induction, we have

$$u_{n+1}(x) > u_n(x), \quad n = 0, 1, 2, \cdots, \quad x \in R^+.$$

$$\text{i. e.} \quad u_n(x) < \int_0^x g(x, s, u_n(s))ds + f(x)$$

By the lemma, we see that

$$u_n(x) < \bar{f}(x) + \Phi_x^{-1}(x), \quad x \in R^+.$$

It follows that $\{u_n(x)\}$ is monotonic and bounded.
Thus, there exists a $u(x)$ such that

$$u_n(x) \to u(x), \quad n \to \infty.$$

By Levin's theorem and (6),

$$u(x) = \int_0^x g(x, s, u(s))ds + f(x).$$

This means that $u(x)$ is a solution of (2).
Since $g(x, s, u)$ is uniformly continuous in x for (s, u), we get that $\int_0^x g(x, s, u(s))ds$ is continuous function, and $u(x)$ is continuous. According to the nonnegative of g and $f(x) > 0$, $x > 0$; $f(0) = 0$, we have $u(x) > 0$, $x > 0$; $u(0) = 0$. Thus, it follows that $u(x)$ is the nonnegative continuous solution of (2).

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