Optimal dividend payments of the two-dimensional compound Poisson risk model with capital injection*

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Abstract This paper deals with the optimal dividend payment and capital injection problem for a two-dimensional compound Poisson risk model which constructs correlation among the two claims. The objective of the corporation is to maximize the discounted dividend payments minus the penalized discounted capital injections. The problem is formulated as a stochastic control problem. By solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation, we obtain the optimal dividend strategy of the problem. We solve this problem explicitly in the case of exponential claim amount distributions.

Keywords optimal dividends, capital injection, Hamilton-Jacobi-Bellman (HJB) equation, stochastic control, two-dimensional compound Poisson risk model

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带注资的二维复合泊松模型的最优分红

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摘要 研究建立两类理赔关系的二维复合泊松模型的最优分红与注资问题，目标为最大化分红减注资的折现．该问题由随机控制问题刻画，通过解相应的哈密尔顿 - 雅克比 - 贝尔曼 (HJB) 方程，得到了最优分红策略，并在指数理赔时明确地解决该问题.

关键词 最优分红，注资，哈密尔顿 - 雅克比 - 贝尔曼 (HJB) 方程，随机控制，二维复合泊松模型

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0 Introduction

In the classical risk theory, the compound Poisson risk model in one-dimensional situation plays a significant role. However, it is clear that the independence assumption does not always reflect reality. It is sometimes convenient to have a model describing situations

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where there are two (or more) classes of correlated business. This is achieved by considering the two-dimensional compound Poisson risk model. More precisely, the two-dimensional compound Poisson risk model constructs correlation among claims. As for this model, we refer readers to [1-4].

The optimal control in the two-dimensional risk model has not received much attention. Dickson and Waters[5] suggested that the shareholders should be liable to cover the deficit at ruin. This led to a new mathematical problem, the maximization of the expectation of the difference between discounted dividends and the deficit at ruin. For optimization problems faced by a company that controls its liquid reserves by paying dividends and by issuing new equity, there has been a few papers studied the problem. For the diffusion model, Sethi and Taksar[6] addressed the problem of finding an optimal financing mix of retained earnings and external equity for maximizing the value of a firm subject to random returns. Løkka and Zervos[7] studied the same problem with possibility of bankruptcy in a model of Brownian motion with drift. Depending on the relationships between the coefficients, the optimal strategy requires the consideration of two auxiliary suboptimal models. As for the compound Poisson risk model, Kulenko and Schmidli[8] found an optimal dividend strategy under the specific framework that the shareholders will inject capital to cover the deficit whatever severity it is and the ruin time of the company is infinite. This inspires us to consider a meaningful question: can we investigate the optimal dividend strategy of the two-dimensional compound Poisson risk model with capital injection?

In this paper, we consider a case that the counting process is the sum of three independent Poisson processes $\{N_1(t)\}$, $\{N_2(t)\}$ and $\{N_c(t)\}$ with intensities λ_1, λ_2 and λ_c, respectively. This model describes the problem in the situation where the insurance company has two correlated classes of business such as the auto insurance. When an accident occurs, some claims are due to the windshield glass broken, and others due to the rear lamp broken. Also, it is possible that they are both broken simultaneously. The objective is to maximize the expected discounted dividend payments minus the penalized discounted capital injections. The problem is formulated as a stochastic control problem. We derive the Hamilton-Jacobi-Bellman (HJB) equation for the problem and obtain the optimal strategy.

The paper is organized as follows. In Section 1, we formulate the problem. In Section 2, some properties of the value function are proved. By HJB equation, we obtain the optimal strategy and show the characterization of the solution to the HJB equation. In Section 3, we give an explicit procedure to obtain the optimal dividend barrier and the value function when claim sizes are exponential distributed.

1 The mathematical model

To give a rigorous mathematical formulation of the optimization problem, we start with a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathrm{P}\}$. We assume that in the absence of control, an insurance portfolio consists of two sub-portfolios with surplus processes $\{X_1^0(t)\}_{t\geqslant 0}$ and $\{X_2^0(t)\}_{t\geqslant 0}$ being defined

$$X_1^0(t) = x_1 + c_1 t - \sum_{n=1}^{N_1(t)+N_c(t)} U_n, t \geqslant 0,$$

$$X_2^0(t) = x_2 + c_2 t - \sum_{n=1}^{N_2(t)+N_c(t)} V_n, t \geqslant 0,$$

where x_1 and x_2 ($\geqslant 0$) are the initial surpluses, c_1 and c_2 (> 0) are the premium rates for the two classes of business, respectively. The counting processes $\{N_1(t)\}$, $\{N_2(t)\}$ and $\{N_c(t)\}$ are three independent Poisson processes with intensities λ_1, λ_2 and λ_c, respectively. We assume that $\{U_n\}$ and $\{V_n\}$ are independent. $X_i^0(t)(i=1,2)$ is adapted to the smallest right-continuous filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. If we want to indicate that the initial capital is x we shall write $\mathrm{P_x}$ and $\mathrm{E_x}$ for the probability measure and the expectation, respectively. Otherwise, we dismiss the letter x and write P and E.

We now enrich the model. We denote by $D_i(t)(i=1,2)$ the aggregate dividends by time t and by $Z_i(t)(i=1,2)$ the cumulative capital injections by time t. $D_i(t)$ is càdlàg, nondecreasing and adapted process with $D_i(0-)=0$. $Z_i(t)$ is càdlàg, nondecreasing and adapted process with $Z_i(0-)=0$.

The dynamics of the controlled surplus process are given by

$$\begin{aligned} X_1^0(t) &= x_1 + c_1 t - \sum_{n=1}^{N_1(t)+N_c(t)} U_n - D_1(t) + Z_1(t), \quad t \geqslant 0; \\ X_2^0(t) &= x_2 + c_2 t - \sum_{n=1}^{N_2(t)+N_c(t)} V_n - D_2(t) + Z_2(t), \quad t \geqslant 0. \end{aligned}$$

Actually, we concern about the sum $D(t) = D_1(t) + D_2(t)$ and $Z(t) = Z_1(t) + Z_2(t)$. Therefore, we study the following controlled process

$$X^\pi(t) = x + ct - \sum_{n=1}^{N_1(t)+N_c(t)} U_n - \sum_{n=1}^{N_2(t)+N_c(t)} V_n - D(t) + Z(t),$$

where $x = x_1 + x_2$ and $c = c_1 + c_2$.

$\pi = \{D(t), Z(t)\}$ is called an admissible strategy or an admissible control if for any $t \geqslant 0$,

$$\mathrm{P}\big[X_t^\pi \geqslant 0 \text{ for } \text{ all } t \geqslant 0\big] = 1.$$

The class of all admissible controls is denoted by $\Pi = \{\pi: \ \pi \text{ is admissible}\}$.

With each admissible controls $\pi \in \Pi$, we associate a performance functional $V^\pi(x)$ defined by

$$V^\pi(x) = \mathrm{E}_x\left(\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}D(t) - \phi\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}Z(t)\right), \tag{1.1}$$

where $\delta > 0$ is a discounting factor and $\phi > 1$ is the penalizing factor. The objective is to find the control that maximizes the performance index.

We define the value function V by

$$V(x) := \sup_{\pi\in\Pi}\{V^\pi(x)\}. \tag{1.2}$$

The optimal policy π^* is a policy for which the following equality is true:

$$V(x) = V^{\pi^*}(x).$$

As in the argument in Kulenko and Schimidli[8], the injection process is

$$Z(t) := \max\left\{-\inf_{0\leqslant s\leqslant t}\left(x + ct - \sum_{n=1}^{N_1(t)+N_c(t)} U_n - \sum_{n=1}^{N_2(t)+N_c(t)} V_n - D(t)\right), 0\right\}.$$

Furthermore,

$$V(x) = V(0) - \phi|x| \quad \text{for } x < 0. \tag{1.3}$$

The next section is devoted to finding the optimal dividend strategy for our problem.

2 HJB equation, the verification theorem and the characterization of the value function

In this section, firstly, some properties of the value function are given. Then, we employ the dynamic programming principle to derive the HJB equation. Furthermore, we give the verification theorem and show the characterization of the solution to the HJB equation.

2.1 The value function

We start with giving some properties of $V(x)$.

Lemma 2.1 *The function $V(x)$ is increasing with $V(x) - V(y) \geqslant x - y$ for $0 \leqslant y \leqslant x$ and Lipschitz continuous on $[0, \infty)$, and therefore absolutely continuous. Furthermore, for any $x \geqslant 0$,*

$$-\frac{\phi\lambda}{\delta}\mathrm{E}[U + V] \leqslant V(x) \leqslant x + \frac{c_1 + c_2}{\delta}.$$

The proof follows the same line as in proving Lemma 2 and Lemma 5 in [8].

Lemma 2.2 *The function $V(x)$ is concave.*

Proof The proof follows the similar line as in proving Lemma 1 in [8]. Let $x, y \in R$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$. Consider the strategies (D^x, Z^x) and (D^y, Z^y) for the initial capitals x and y. Define $D(t) = \alpha D^x(t) + \beta D^y(t)$, $\bar{Z}(t) = \alpha Z^x(t) + \beta Z^y(t)$, we have

$$\alpha x + \beta y + ct - \sum_{n=1}^{N_1(t)+N_c(t)} U_n - \sum_{n=1}^{N_2(t)+N_c(t)} V_n - D(t) + \bar{Z}(t) \geqslant 0.$$

Therefore, the strategy $(D(t), \bar{Z}(t))$ is admissible and that $Z^{\alpha x+\beta y}(t) \leqslant \alpha Z^x(t) + \beta Z^y(t)$. It follows that

$$\begin{aligned} V(\alpha x + \beta y) &\geqslant \mathrm{E}_x\left(\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}D(t) - \phi\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}Z^{\alpha x+\beta y, D}(t)\right) \\ &\geqslant \mathrm{E}_x\left(\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}D(t) - \phi\left(\alpha\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}Z^{x,D}(t) + \beta\int_{0-}^{\infty} e^{-\delta t}\mathrm{d}Z^{y,D}(t)\right)\right) \\ &= \alpha V^{D^x}(x) + \beta V^{D^y}(y). \end{aligned}$$

Taking the supremum over all admissible strategies D, we obtain that

$$V(\alpha x+\beta y)\geqslant \alpha V(x)+\beta V(y).$$

2.2 The HJB equation and the optimal strategies

With reference to the standard theory of optimal control[9], the following dynamic programming principle holds:

$$V(x)=\sup_{\pi\in\Pi}\mathrm{E}_x\left[\int_{0-}^{\tau}e^{-\delta s}\mathrm{d}D(s)-\phi\int_{0-}^{\tau}e^{-\delta s}\mathrm{d}Z(s)+e^{-\delta\tau}V^{\pi}(X^{\pi}(\tau))\right],\tag{2.1}$$

for any $\mathcal{F}_t-$ stopping time τ. This principle serves us to derive the HJB equation.

For $x\geqslant 0$, and any admissible strategy π, let $h>0$. The waiting time T_1 until the first claim caused by U alone has density $\lambda_1 e^{-\lambda_1 t}$, and caused by V alone has density $\lambda_2 e^{-\lambda_2 t}$, caused by both U and V together with probability $\lambda_c e^{-\lambda_c t}$. Define π by $D(t)=0$ and $Z(t)=0$ for $t<\tau$. Using the law of total probability and taking $\tau=\tau^{\pi}=T_1\wedge h$ in (2.1), we write

$$\begin{aligned}
V(x)\geqslant{}& \mathrm{E}_x\left[e^{-\delta\tau^{\pi}}V(X_{\tau^{\pi}})\right]\\
={}& e^{-(\delta+\lambda_1+\lambda_2+\lambda_c)h}V(x+ch)\\
&+\lambda_1 e^{-(\lambda_2+\lambda_c)h}\int_0^h\int_0^{\infty}e^{-(\delta+\lambda_1)s}V(x+ct-u)\mathrm{d}F_U(u)\mathrm{d}s\\
&+\lambda_2 e^{-(\lambda_1+\lambda_c)h}\int_0^h\int_0^{\infty}e^{-(\delta+\lambda_2)s}V(x+ct-v)\mathrm{d}F_V(v)\mathrm{d}s\\
&+\lambda_c e^{-(\lambda_1+\lambda_2)h}\int_0^h\int_0^{\infty}e^{-(\delta+\lambda_c)s}V(x+ct-w)\mathrm{d}F_W(w)\mathrm{d}s,
\end{aligned}$$

where $F_U(u)$ is the distribution function of U_i which arrives alone, $F_V(v)$ is the distribution function of V_i which arrives alone and $F_W(w)$ is the distribution function of W_i which expresses that U_i and V_i arrive simultaneously. Rearranging the terms, dividing h, letting $h\to 0$, we obtain

$$\begin{aligned}
cV'(x)-(\delta+\lambda)V(x)+\lambda_1\int_0^{\infty}V(x-u)\mathrm{d}F_U(u)+\lambda_2\int_0^{\infty}V(x-v)\mathrm{d}F_V(v)&\\
+\lambda_c\int_0^{\infty}V(x-w)\mathrm{d}F_W(w)\leqslant 0,&
\end{aligned}$$

where $\lambda=\lambda_1+\lambda_2+\lambda_c$.

Now, considering such strategy with an initial payout of size ε, and then following the optimal policy, we have

$$V(x)\geqslant V(x-\varepsilon)+\varepsilon.$$

Subtracting $V(x-\varepsilon)$ from both sides, dividing by ε, and letting $\varepsilon\to 0$ yields $V'(x)\geqslant 1$.

On the other hand, consider a strategy by receiving $\varepsilon>0$ from the shareholder immediately and following the optimal strategy for the capital $x+\varepsilon$ afterwards, then $V(x)\geqslant V(x+\varepsilon)-\phi\varepsilon$. Letting $\varepsilon\to 0$, we get

$$V'(x)\leqslant\phi.$$

Now we have the following HJB equation

$$\max\quad \Big\{cV'(x)-(\delta+\lambda)V(x)+\lambda_1\int_0^\infty V(x-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty V(x-v)\mathrm{d}F_V(v)$$
$$+\lambda_c\int_0^\infty V(x-w)\mathrm{d}F_W(w), V'(x)-\phi, 1-V'(x)\Big\}=0. \tag{2.2}$$

Moreover, $V(x)$ is concave, so there exists a dividend barrier $b:=\inf\{x\geqslant 0: V'(x)=1\}$.

Now we define the strategy π^* as follows:

$$\begin{aligned}
&D^*(0)=\max(x-b,0),\\
&D^*(t)=D^*(0)+\int_0^t c1_{\{X^*(s)=b\}}\mathrm{d}s \quad \text{for } t>0,\\
&Z^*(t)=\max\Big\{-\inf_{0\leqslant s\leqslant t}\Big(x+ct-\sum_{n=1}^{N(t)}W_n'-D^*(t)\Big),0\Big\}.
\end{aligned}\tag{2.3}$$

The strategy $\pi^*=\{D^*,Z^*\}$ restricts the process into the band with upper barrier b and lower barrier 0.

Theorem 2.1 *The strategy* (2.3) *is optimal, i.e.,*

$$V^{\pi^*}(x)=V(x).$$

Proof We denote by $X^*(t)$ the corresponding surplus process under the strategy (2.3). We have $V'(X^*(t))=1$ on $\{X^*(t)=b\}$ and $V'(X^*(t))>1$ on $\{X^*(t)<b\}$. Applying the Itô formula, we have

$$\begin{aligned}
&V(X^*(t))e^{-\delta t}\\
=\ &V(X^*(0))+\sum_{i=1}^{N(t)}e^{-\delta T_i}[V(X^*(T_i))-V(X^*(T_i-))]\\
&+\sum_{i=1}^{N(t)}[V(X^*(T_i-))e^{-\delta T_i-}-V(X^*(T_{i-1}))e^{-\delta T_{i-1}}]\\
&+V(X^*(t))e^{-\delta t}-V(X^*(T_{N(t)}))e^{-\delta T_{N(t)}}\\
=\ &V(X(0))+\phi Z_0+\sum_{i=1}^{N_1(t)}e^{-\delta s}[V(X^*_{T_i-}-U_i)-V(X^*_{T_i-})]\\
&+\sum_{i=1}^{N_2(t)}e^{-\delta s}[V(X^*_{T_i-}-V_i)-V(X^*_{T_i-})]+\sum_{i=1}^{N_c(t)}e^{-\delta s}[V(X^*_{T_i-}-W_i)-V(X^*_{T_i-})]\\
&+\phi\sum_{i=1}^{N(t)}e^{-\delta T_i}Z_i+\sum_{i=1}^{N(t)}[V(X^*(T_i-))e^{-\delta T_i-}-V(X^*(T_{i-1}))e^{-\delta T_{i-1}}]\\
&+V(X^*(t))e^{-\delta t}-V(X^*(T_{N(t)}))e^{-\delta T_{N(t)}}
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{N_1(t)} e^{-\delta s}[V(X^*_{T_i-}-U_i)-V(X^*_{T_i-})]+\sum_{i=1}^{N_2(t)} e^{-\delta s}[V(X^*_{T_i-}-V_i)-V(X^*_{T_i-})] \\
& +\sum_{i=1}^{N_c(t)} e^{-\delta s}[V(X^*_{T_i-}-W_i)-V(X^*_{T_i-})]+\phi\int_{0-}^{t} e^{-\delta s}\mathrm{d}Z_s \\
& +V(x)+\sum_{i=1}^{N(t)}\int_{T_{i-1}}^{T_i-}[cV'(X^*(s))-\delta V(X^*(s))]\mathbf{I}_{\{X^*(s)<b\}}e^{-\delta s}\mathrm{d}s \\
& +\int_{T_{N(t)}}^{t}[cV'(X^*(s))-\delta V(X^*(s))]\mathbf{I}_{\{X^*(s)<b\}}e^{-\delta s}\mathrm{d}s \\
& -\sum_{i=1}^{N(t)}\int_{T_{i-1}}^{T_i-}\delta V(X^*(s))\mathbf{I}_{\{X^*(s)=b\}}e^{-\delta s}\mathrm{d}s-\int_{T_{N(t)}}^{t}\delta V(X^*(s))\mathbf{I}_{\{X^*(s)=b\}}e^{-\delta s}\mathrm{d}s.
\end{aligned}
$$

When the claim arrives,

$$
\sum_{i=1}^{N_1(t)} e^{-\delta s}[V(X^*_{T_i-}-U_i)-V(X^*_{T_i-})]-\lambda_1\int_0^t e^{-\delta s}\left[\int_0^\infty V(X^*_{T_i-}-u)\mathrm{d}F_U(u)-V(X^*_{T_i-})\right]\mathrm{d}s,
$$

$$
\sum_{i=1}^{N_2(t)} e^{-\delta s}[V(X^*_{T_i-}-V_i)-V(X^*_{T_i-})]-\lambda_2\int_0^t e^{-\delta s}\left[\int_0^\infty V(X^*_{T_i-}-v)\mathrm{d}F_V(v)-V(X^*_{T_i-})\right]\mathrm{d}s,
$$

$$
\sum_{i=1}^{N_c(t)} e^{-\delta s}[V(X^*_{T_i-}-W_i)-V(X^*_{T_i-})]-\lambda_6\int_0^t e^{-\delta s}\left[\int_0^\infty V(X^*_{T_i-}-w)\mathrm{d}F_W(w)-V(X^*_{T_i-})\right]\mathrm{d}s,
$$

are martingales with mean value 0, or equivalently, the process

$$
\begin{aligned}
& \bigg\{V(X^*(t))e^{-\delta t}-V(x)-\phi\int_{0-}^{t} e^{-\delta s}\mathrm{d}Z_s-\int_0^t\bigg[cV'(X^*(s)) \\
& +\lambda_1\int_0^\infty V(X^*_{s-}-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty V(X^*_{s-}-v)\mathrm{d}F_V(v) \\
& +\lambda_c\int_0^\infty V(X^*_{s-}-w)\mathrm{d}F_W(w)-(\lambda+\delta)V(X^*(s-))\bigg]\mathbf{I}_{\{X^*(s)<b\}}e^{-\delta s}\mathrm{d}s \\
& -\int_0^t\bigg[\lambda_1\int_0^\infty V(X^*_{s-}-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty V(X^*_{s-}-v)\mathrm{d}F_V(v) \\
& +\lambda_c\int_0^\infty V(X^*_{s-}-w)\mathrm{d}F_W(w)-(\lambda+\delta)V(X^*(s-))\bigg]\mathbf{I}_{\{X^*(s-)=b\}}e^{-\delta s}\mathrm{d}s\bigg\}
\end{aligned}
$$

is a martingale. Because $V(x)$ is concave, the derivatives of $V(x)$ from left and right exist. Moreover, we have assumed that $F_U(u)$, $F_V(v)$ and $F_W(w)$ are continuous, so $V(x)$ fulfills (2.2) and is continuously differentiable. For $V'(X^*_s)>1$ on $\{X^*_s<b\}$, the first term on the left-hand side of (2.2) is 0, thus the integral over $\{X^*_s<b\}$ on the expression above is 0. Furthermore, from $V'(X^*_s)=1$ on $\{X^*_s=b\}$ and (2.2), it follows that

$$
\begin{aligned}
& -(\delta+\lambda)V(X^*(s-))+\lambda_1\int_0^\infty V(X^*_{s-}-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty V(X^*_{s-}-v)\mathrm{d}F_V(v) \\
& +\lambda_c\int_0^\infty V(X^*_{s-}-w)\mathrm{d}F_W(w)=-c.
\end{aligned}
$$

Thus, we get that

$$\left\{V(X^*(t))e^{-\delta t}-V(x)-\phi\int_{0-}^{t}e^{-\delta s}\mathrm{d}Z_s+\int_0^t c\mathbf{I}_{\{X^*(s)=b\}}e^{-\delta s}\mathrm{d}s\right\}$$

is a martingale with expected value 0. From the martingale property we get that

$$V(x)=\mathrm{E}_x\left[V(X^*(t))e^{-\delta t}-\phi\int_{0-}^{t}e^{-\delta s}\mathrm{d}Z_s+\int_0^t c\mathbf{I}_{\{X_s^*=b\}}e^{-\delta s}\mathrm{d}s\right].$$

Since $V(X^*(t))e^{-\delta t}$ is bounded and converges to 0 when $t\to\infty$. By the bounded convergence theorem, we have

$$\lim_{t\to\infty}\mathrm{E}_x[V(X^*(t))e^{-\delta t}]=0.$$

Because $\mathrm{d}D^*(t)=0$ on $\{X^*(t)<b\}$ and $\mathrm{d}D^*(t)=c\mathrm{d}t$ on $\{X^*(t)=b\}$, by the monotone convergence theorem, we finally get that

$$\begin{aligned}V(x)&=\lim_{t\to\infty}\mathrm{E}_x\left[\int_0^t c\mathbf{I}_{\{X^*(s)=b\}}e^{-\delta s}\mathrm{d}s-\phi\int_{0-}^{t}e^{-\delta s}\mathrm{d}Z_s\right]\\&=\mathrm{E}_x\left[\int_0^\infty e^{-\delta s}\mathrm{d}D^*(s)-\phi\int_{0-}^{\infty}e^{-\delta s}\mathrm{d}Z_s\right]=V^*(x).\end{aligned}$$

2.3 Characterization of the solution

Because we have neither an explicit solution nor an initial value, we need to characterize the solution $V(x)$ among other possible solutions.

Theorem 2.2 *$V(x)$ is the minimal non-negative solution to* (2.2).

Proof If f is a solution to the HJB equation, then $f(x)$ is increasing. Let X^* be the process under the optimal strategy. From Theorem 2.1,

$$\begin{aligned}&\left\{f(X^*(t))e^{-\delta t}-f(x)-\phi\int_{0-}^{t}e^{-\delta s}\mathrm{d}Z_s-\int_0^t\left[cf'(X^*(s))\right.\right.\\&+\lambda_1\int_0^\infty f(X_{s-}^*-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty f(X_{s-}^*-v)\mathrm{d}F_V(v)\\&\left.+\lambda_c\int_0^\infty f(X_{s-}^*-w)\mathrm{d}F_W(w)-(\lambda+\delta)f(X^*(s-))\right]\mathbf{I}_{\{X^*(s)<b\}}e^{-\delta s}\mathrm{d}s\\&-\int_0^t\left[\lambda_1\int_0^\infty f(X_{s-}^*-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty f(X_{s-}^*-v)\mathrm{d}F_V(v)\right.\\&\left.\left.+\lambda_c\int_0^\infty f(X_{s-}^*-w)\mathrm{d}F_W(w)-(\lambda+\delta)f(X^*(s-))\right]\mathbf{I}_{\{X^*(s-)=b\}}e^{-\delta s}\mathrm{d}s\right\}\end{aligned}$$

is a martingale with expected value 0. By (2.2),

$$\begin{aligned}&cf'(X^*(s))+\lambda_1\int_0^\infty f(X_{s-}^*-u)\mathrm{d}F_U(u)+\lambda_2\int_0^\infty f(X_{s-}^*-v)\mathrm{d}F_V(v)\\&+\lambda_c\int_0^\infty f(X_{s-}^*-w)\mathrm{d}F_W(w)-(\lambda+\delta)f(X^*(s))\leqslant 0.\end{aligned}$$

Because $f'(x) \geqslant 1$,

$$\lambda_1 \int_0^\infty f(X^*_{s-} - u)\mathrm{d}F_U(u) + \lambda_2 \int_0^\infty f(X^*_{s-} - v)\mathrm{d}F_V(v)$$
$$+\lambda_c \int_0^\infty f(X^*_{s-} - w)\mathrm{d}F_W(w) - (\lambda+\delta)f(X^*_s) \leqslant -cf'(X^*(s)) \leqslant -c.$$

This yields

$$f(x) \geqslant \mathrm{E}_x\left[f(X^*(t))e^{-\delta t} - \phi \int_{0-}^t e^{-\delta s}\mathrm{d}Z_s + \int_0^t c1_{\{X^*(s)=b\}}e^{-\delta s}\mathrm{d}s\right]$$
$$\geqslant \mathrm{E}_x\left[\int_0^t e^{-\delta s}\mathrm{d}D^*(s) - \phi \int_{0-}^t e^{-\delta s}\mathrm{d}Z_s\right].$$

Letting $t \to \infty$ gives $f(x) \geqslant V(x)$.

3 Solution to the problem

In order to obtain an explicit solution to the HJB equation (2.2) and an optimal dividend payment policy, we assume that the claim size distribution is given by $\mathrm{d}F_U(u) = \beta_1 e^{-\beta_1 u}\mathrm{d}u$, $\mathrm{d}F_V(v) = \beta_2 e^{-\beta_2 v}\mathrm{d}v$. Since we assume that U_i and V_i are independent, $\mathrm{d}F_W(w) = \frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2 w} - e^{-\beta_1 w})\mathrm{d}w$. To solve this HJB equation, we start with the following equation

$$cV'(x) - (\delta+\lambda)V(x) + \lambda_1 \int_0^\infty V(x-u)\mathrm{d}F_U(u)$$
$$+ \lambda_2 \int_0^\infty V(x-v)\mathrm{d}F_V(v) + \lambda_c \int_0^\infty V(x-w)\mathrm{d}F_W(w) = 0.$$

Combining with

$$\mathrm{d}F_U(u) = \beta_1 e^{-\beta_1 u}\mathrm{d}u, \quad \mathrm{d}F_V(v) = \beta_2 e^{-\beta_2 v}\mathrm{d}v, \quad \mathrm{d}F_W(w) = \frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2 w} - e^{-\beta_1 w})\mathrm{d}w,$$

we have

$$cV'(x) + \lambda_1 \int_0^\infty V(x-u)\beta_1 e^{-\beta_1 u}\mathrm{d}u + \lambda_2 \int_0^\infty V(x-v)\beta_2 e^{-\beta_2 v}\mathrm{d}v$$
$$+\lambda_c \int_0^\infty V(x-w)\frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2 w} - e^{-\beta_1 w})\mathrm{d}w - (\lambda+\delta)V(x) = 0. \tag{3.1}$$

In view of (1.3), we have

$$\int_0^\infty V(x-u)\beta_1 e^{-\beta_1 u}\mathrm{d}u$$
$$= \int_0^x V(x-u)\beta_1 e^{-\beta_1 u}\mathrm{d}u + \int_x^\infty [V(0) + \phi(x-u)]\beta_1 e^{-\beta_1 u}\mathrm{d}u$$
$$= \int_0^x V(u)\beta_1 e^{-\beta_1(x-u)}\mathrm{d}u + \left(V(0) - \frac{\phi}{\beta_1}\right)e^{-\beta_1 x}.$$

Similarly,

$$\int_0^\infty V(x-v)\beta_2 e^{-\beta_2 v}\mathrm{d}v = \int_0^x V(v)\beta_2 e^{-\beta_2(x-v)}\mathrm{d}v + \Big(V(0)-\frac{\phi}{\beta_2}\Big)e^{-\beta_2 x}.$$

$$\begin{aligned}
&\int_0^\infty V(x-w)\frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2 w}-e^{-\beta_1 w})\mathrm{d}w\\
= &\int_0^x V(x-w)\frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2 w}-e^{-\beta_1 w})\mathrm{d}w\\
&+\int_x^\infty [V(0)+\phi(x-w)]\frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2 w}-e^{-\beta_1 w})\mathrm{d}w\\
= &\int_0^x V(w)\frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2(x-w)}-e^{-\beta_1(x-w)})\mathrm{d}w + (V(0)+\phi x)\frac{\beta_1\beta_2}{\beta_1-\beta_2}\Big(\frac{e^{-\beta_2 x}}{\beta_2}-\frac{e^{-\beta_1 w}}{\beta_1}\Big)\\
&-\frac{\phi\beta_1\beta_2}{\beta_1-\beta_2}\Big(\frac{xe^{-\beta_2 x}}{\beta_2}+\frac{e^{-\beta_2 x}}{\beta_2^2}-\frac{xe^{-\beta_1 x}}{\beta_1}-\frac{e^{-\beta_1 x}}{\beta_1^2}\Big).
\end{aligned}$$

Therefore, putting all the pieces together, we have

$$\begin{aligned}
&cV'(x)+\lambda_1\int_0^x V(u)\beta_1 e^{-\beta_1(x-u)}\mathrm{d}u+\lambda_1\Big(V(0)-\frac{\phi}{\beta_1}\Big)e^{-\beta_1 x}\\
&+\lambda_2\int_0^x V(v)\beta_2 e^{-\beta_2(x-v)}\mathrm{d}v+\lambda_2\Big(V(0)-\frac{\phi}{\beta_2}\Big)e^{-\beta_2 x}\\
&+\lambda_c\int_0^x V(w)\frac{\beta_1\beta_2}{\beta_1-\beta_2}(e^{-\beta_2(x-w)}-e^{-\beta_1(x-w)})\mathrm{d}w+\frac{\lambda_c\beta_2}{\beta_1-\beta_2}\Big(\frac{\phi}{\beta_1}-V(0)\Big)e^{-\beta_1 x}\\
&+\frac{\lambda_c\beta_1}{\beta_1-\beta_2}\Big(V(0)-\frac{\phi}{\beta_2}\Big)e^{-\beta_2 x}-(\lambda+\delta)V(x)=0. \qquad (3.2)
\end{aligned}$$

We use the notation $\mathcal{I}$ for the identity operator and $\mathcal{D}$ for the differentiation operator with respect to the function on which the operator is performed. Applying the operator $\mathcal{D}(\mathcal{D}+\beta_1\mathcal{I})+\beta_2(\mathcal{D}+\beta_1\mathcal{I})$ to both sides of (3.2) yields

$$\begin{aligned}
&cV'''(x)+(c\beta_1+c\beta_2-\lambda-\delta)V''(x)+[\lambda_1\beta_1+\lambda_2\beta_2\\
&-(\lambda+\delta)(\beta_1+\beta_2)+c\beta_1\beta_2]V'(x)-\delta\beta_1\beta_2 V(x)=0. \qquad (3.3)
\end{aligned}$$

In order to find the solution of the above ordinary differential equation, we consider the roots of the characteristic equation

$$\begin{aligned}
k(x) \quad :\equiv \quad & cx^3+(c\beta_1+c\beta_2-\lambda-\delta)x^2+[\lambda_1\beta_1+\lambda_2\beta_2\\
&-(\lambda+\delta)(\beta_1+\beta_2)+c\beta_1\beta_2]x-\delta\beta_1\beta_2=0. \qquad (3.4)
\end{aligned}$$

Note that

$$k(0)=-\delta\beta_1\beta_2, \quad \lim_{x\to+\infty}k(x)\to+\infty,$$

we can conclude that (3.4) has at least one positive root. By calculation, we have

$$k(0)=-\delta\beta_1\beta_2, \qquad \lim_{x\to-\infty}k(x)\to-\infty.$$

$$k(-\beta_1) = -\beta_1(\lambda_1(\beta_1 - \beta_2) - \lambda_c\beta_2),$$
$$k(-\beta_2) = -\beta_2(\lambda_2(\beta_2 - \beta_1) - \lambda_c\beta_1).$$

If $\beta_1 > \beta_2$, then $k(-\beta_2) > 0$. If $\beta_1 < \beta_2$, then $k(-\beta_1) > 0$. So we have the conclusion that (3.4) has two negative roots s_1, s_2 and one positive root s_3. Moreover, the two negative roots s_1, s_2 satisfy that

$$s_1 < -(\beta_1 \wedge \beta_2) < s_2 < 0.$$

From the above analysis, the solution to this ordinary differential equation is given by

$$V(x) = B_1 e^{s_1 x} + B_2 e^{s_2 x} + B_3 e^{s_3 x}, \tag{3.5}$$

where s_1, s_2 and s_3 are the roots of the characteristic equation (3.4). Substituting (3.5) into equation (3.2) and equating the coefficients of terms involving $e^{-\beta_1 x}$ to zero gives

$$\frac{B_1 s_1}{s_1+\beta_1} + \frac{B_2 s_2}{s_2+\beta_1} + \frac{B_3 s_3}{s_3+\beta_1} - \frac{\phi}{\beta_1} = 0, \tag{3.6}$$

and equating the coefficients of terms involving $e^{-\beta_2 x}$ to zero gives

$$\frac{B_1 s_1}{s_1+\beta_2} + \frac{B_2 s_2}{s_2+\beta_2} + \frac{B_3 s_3}{s_3+\beta_2} - \frac{\phi}{\beta_2} = 0. \tag{3.7}$$

Additionally, from $V'(b) = 1$, we have

$$B_1 s_1 e^{s_1 b} + B_2 s_2 e^{s_2 b} + B_3 s_3 e^{s_3 b} = 1. \tag{3.8}$$

Solving (3.6), (3.7) and (3.8)

$$\begin{aligned}
B_1 = {} & \frac{e^{-s_1 b}}{s_1} + \bigg\{ e^{-s_1 b + s_3 b} s_3 \bigg[\bigg(-\frac{e^{s_2 b} s_1 s_2}{s_1+\beta_1} + \frac{e^{s_1 b} s_1 s_2}{s_2+\beta_1} \bigg) \bigg(-\frac{e^{s_1 b}\phi s_1}{\beta_2} + \frac{s_1}{s_1+\beta_2} \bigg) \\
& - \bigg(\frac{e^{s_1 b} s_1 s_2}{s_2+\beta_2} - \frac{e^{s_2 b} s_1 s_2}{s_1+\beta_2} \bigg) \bigg(\frac{s_1}{s_1+\beta_1} - \frac{e^{s_1 b}\phi s_1}{\beta_1} \bigg) \bigg] \bigg\} \bigg/ \bigg\{ s_1 \bigg[-\bigg(\frac{e^{s_1 b} s_1 s_3}{s_3+\beta_1} \\
& - \frac{e^{s_3 b} s_1 s_3}{s_1+\beta_1} \bigg) \bigg(\frac{e^{s_1 b} s_1 s_2}{s_2+\beta_2} - \frac{e^{s_2 b} s_1 s_2}{s_1+\beta_2} \bigg) + \bigg(\frac{e^{s_1 b} s_1 s_2}{s_2+\beta_1} - \frac{e^{s_2 b} s_1 s_2}{s_1+\beta_1} \bigg) \\
& \times \bigg(\frac{e^{s_1 b} s_1 s_3}{s_3+\beta_2} - \frac{e^{s_3 b} s_1 s_3}{s_1+\beta_2} \bigg) \bigg] \bigg\} + \frac{1}{s_1} \bigg\{ e^{-s_1 b + s_2 b} s_2 \bigg\{ \bigg(\frac{s_1}{s_1+\beta_1} - \frac{e^{s_1 b}\phi s_1}{\beta_1} \bigg) \\
& \times \bigg(\frac{e^{s_1 b} s_1 s_2}{\beta_1+s_2} - \frac{e^{s_2 b} s_1 s_2}{\beta_1+s_1} \bigg)^{-1} - \bigg[\bigg(\frac{e^{s_1 b} s_1 s_3}{\beta_1+s_3} - \frac{e^{s_3 b} s_1 s_3}{\beta_1+s_1} \bigg) \bigg(\bigg(\frac{e^{s_1 b} s_1 s_2}{\beta_1+s_2} - \frac{e^{s_2 b} s_1 s_2}{\beta_1+s_1} \bigg) \\
& \times \bigg(\frac{s_1}{\beta_2+s_1} - \frac{e^{s_1 b} s_1 \phi}{\beta_2} \bigg) - \bigg(\frac{e^{s_1 b} s_1 s_2}{\beta_2+s_2} - \frac{e^{s_2 b} s_1 s_2}{\beta_2+s_1} \bigg) \bigg(\frac{s_1}{\beta_1+s_1} - \frac{e^{s_1 b} s_1 \phi}{\beta_1} \bigg) \bigg) \bigg] \\
& \bigg/ \bigg\{ \bigg(\frac{e^{s_1 b} s_1 s_2}{\beta_1+s_2} - \frac{e^{s_2 b} s_1 s_2}{\beta_1+s_1} \bigg) \bigg[-\bigg(\frac{e^{s_1 b} s_1 s_3}{\beta_1+s_3} - \frac{e^{s_3 b} s_1 s_3}{\beta_1+s_1} \bigg) \bigg(\frac{e^{s_1 b} s_1 s_2}{\beta_2+s_2} - \frac{e^{s_2 b} s_1 s_2}{\beta_2+s_1} \bigg) \\
& + \bigg(\frac{e^{s_1 b} s_1 s_2}{\beta_1+s_2} - \frac{e^{s_2 b} s_1 s_2}{\beta_1+s_1} \bigg) \bigg(\frac{e^{s_1 b} s_1 s_3}{\beta_2+s_3} - \frac{e^{s_3 b} s_1 s_3}{\beta_2+s_1} \bigg) \bigg] \bigg\} \bigg\} \bigg\},
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
B_2=&\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)^{-1}+\left[\left(\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}\right)\right.\\
&\times\left(\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\left(\frac{s_1}{s_1+\beta_2}-\frac{e^{s_1b}\phi s_1}{\beta_2}\right)-\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\right.\\
&\left.\left.\times\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\right)\right]\bigg/\left\{\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\left[\left(\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}\right)\right.\right.\\
&\left.\left.\times\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)-\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)\left(\frac{e^{s_1b}s_1s_3}{\beta_2+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_2+s_1}\right)\right]\right\}, \qquad (3.10)\\
B_3=&\left[\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)\left(\frac{s_1}{s_1+\beta_2}-\frac{e^{s_1b}\phi s_1}{\beta_2}\right)-\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\right.\\
&\left.\times\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\right]\bigg/\left[\left(\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}\right)\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\right.\\
&\left.-\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)\left(\frac{e^{s_1b}s_1s_3}{\beta_2+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_2+s_1}\right)\right]. \qquad (3.11)
\end{aligned}
$$

From $V''(b)=0$, i.e.

$$
B_1s_1^2e^{s_1b}+B_2s_2^2e^{s_2b}+B_3s_3^2e^{s_3b}=0, \qquad (3.12)
$$

we obtain the equation that b satisfies:

$$
\begin{aligned}
&\left\{\frac{e^{-s_1b}}{s_1}+\left\{e^{-s_1b+s_3b}s_3\left[\left(-\frac{e^{s_2b}s_1s_2}{s_1+\beta_1}+\frac{e^{s_1b}s_1s_2}{s_2+\beta_1}\right)\left(-\frac{e^{s_1b}\phi s_1}{\beta_2}+\frac{s_1}{s_1+\beta_2}\right)\right.\right.\right.\\
&\left.\left.-\left(\frac{e^{s_1b}s_1s_2}{s_2+\beta_2}-\frac{e^{s_2b}s_1s_2}{s_1+\beta_2}\right)\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\right]\right\}\bigg/\left\{s_1\left[-\left(\frac{e^{s_1b}s_1s_3}{s_3+\beta_1}-\frac{e^{s_3b}s_1s_3}{s_1+\beta_1}\right)\right.\right.\\
&\left.\left.\times\left(\frac{e^{s_1b}s_1s_2}{s_2+\beta_2}-\frac{e^{s_2b}s_1s_2}{s_1+\beta_2}\right)+\left(\frac{e^{s_1b}s_1s_2}{s_2+\beta_1}-\frac{e^{s_2b}s_1s_2}{s_1+\beta_1}\right)\left(\frac{e^{s_1b}s_1s_3}{s_3+\beta_2}-\frac{e^{s_3b}s_1s_3}{s_1+\beta_2}\right)\right]\right\}\\
&+\frac{1}{s_1}\left\{e^{-s_1b+s_2b}s_2\left\{\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)^{-1}\right.\right.\\
&-\left[\left(\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}\right)\left(\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\left(\frac{s_1}{\beta_2+s_1}-\frac{e^{s_1b}s_1\phi}{\beta_2}\right)\right.\right.\\
&\left.\left.-\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\left(\frac{s_1}{\beta_1+s_1}-\frac{e^{s_1b}s_1\phi}{\beta_1}\right)\right)\right]\\
&\bigg/\left\{\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\left[-\left(\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}\right)\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\right.\right.\\
&\left.\left.\left.\left.\left.+\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\left(\frac{e^{s_1b}s_1s_3}{\beta_2+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_2+s_1}\right)\right]\right\}\right\}\right\}\right\}s_1^2e^{s_1b}\\
&+\left\{\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)^{-1}+\left[\left(\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}\right)\right.\right.\\
&\times\left(\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\left(\frac{s_1}{s_1+\beta_2}-\frac{e^{s_1b}\phi s_1}{\beta_2}\right)-\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)\right.
\end{aligned}
$$

$$\times\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\right)\Bigg]\Bigg/\Bigg\{\left(\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}\right)\Bigg[\left(\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}\right)$$
$$\times\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)-\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)\left(\frac{e^{s_1b}s_1s_3}{\beta_2+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_2+s_1}\right)\Bigg]\Bigg\}\Bigg\}$$
$$\times s_2^2e^{s_2b}+\Bigg\{\Bigg[\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)\left(\frac{s_1}{s_1+\beta_2}-\frac{e^{s_1b}\phi s_1}{\beta_2}\right)-\left(\frac{s_1}{s_1+\beta_1}-\frac{e^{s_1b}\phi s_1}{\beta_1}\right)$$
$$\times\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)\Bigg]\Bigg/\Bigg[\left(\frac{e^{s_3b}s_1s_3}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_3}{\beta_1+s_3}\right)\left(\frac{e^{s_1b}s_1s_2}{\beta_2+s_2}-\frac{e^{s_2b}s_1s_2}{\beta_2+s_1}\right)$$
$$-\left(\frac{e^{s_2b}s_1s_2}{\beta_1+s_1}-\frac{e^{s_1b}s_1s_2}{\beta_1+s_2}\right)\left(\frac{e^{s_1b}s_1s_3}{\beta_2+s_3}-\frac{e^{s_3b}s_1s_3}{\beta_2+s_1}\right)\Bigg]\Bigg\}s_3^2e^{s_3b}=0. \tag{3.13}$$

From the above analysis, the solution to the problem is given by

$$V(x)=\begin{cases} B_1e^{s_1x}+B_2e^{s_2x}+B_3e^{s_3x}, & 0\leqslant x<b;\\ x-b+V(b), & b\leqslant x;\end{cases} \tag{3.14}$$

where s_1, s_2 and s_3 are the roots of the characteristic equation (3.4), B_1, B_2 and B_3 are determined by (3.9), (3.10) and (3.11), b by (3.13).

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