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Moments of the Riesz distribution

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Abstract

This article derives the first two moments of the two versions of the Riesz distribution in the terms of their characteristic functions.

1 Introduction

There is no doubt about the important role played by Wishart distribution in the context of multivariate statistics and random matrix theory. Based in the Riesz measure, Hassairi and Lajmi (2001) proposed a generalisation of the Wishart distribution, which they termed Riesz distribution. Some of their main properties and related distributions have been studied by Hassairi and Lajmi (2001), Faraut and Korányi (1994) and Hassairi *et al.* (2005). Recently, Díaz-García (2012) proposes two versions of the Riesz distribution for real normed division algebras and some of their properties are also being studied.

In particular, the characteristic function was obtained for both versions of the Riesz distribution, but a topic that has been disregarded, is the study of its moments. As indicated in the conclusions section, in particular, these moments can be used to study the asymptotic normality of the Riesz distribution.

This article studies the first two moments for the two versions of the Riesz distribution. Section 2 reviews some definitions and notations on the matrix algebra and some special functions with matrix arguments on real symmetric cones, also, are summarised the two definitions of the Riesz distribution, their corresponding characteristic functions and the section is finalised with obtaining the first two matrix derivatives of the characteristic functions. The main results are proposed in Section 3.

2 Preliminary results

2.1 Matrix algebra and special function with matrix argument

A detailed discussion of theory of matrices and special function with matrix argument can be found in Magnus and Neudecker (1988) and Muirhead (1982), respectively. For convenience, we shall introduce some notation, although in general, we adhere to standard notation forms.

Let $\mathfrak{A}^{m\times n}$ be the set of all $m \times n$ matrices over $\Re$. Let $\mathbf{A} \in \mathfrak{A}^{m\times n}$, then $\mathbf{A}' \in \mathfrak{A}^{n\times m}$ denotes the transpose. It is denoted by $\mathfrak{S}_m$ the real vector space of all $\mathbf{S} \in \mathfrak{A}^{m\times m}$ such that

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$\mathbf{S} = \mathbf{S}'$. In addition, let $\mathfrak{P}_m$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{A}^{m\times m}$. Thus, $\mathfrak{P}_m$ consist of all matrices $\mathbf{S} = \mathbf{X}'\mathbf{X}$, with $\mathbf{X} \in \mathfrak{A}^{n\times m}$; then $\mathfrak{P}_m$ is an open subset of $\mathfrak{S}_m$.

$\Gamma_m[a]$ denotes the multivariate *Gamma function* for the space $\mathfrak{S}_m$. This can be obtained as a particular case of the *generalised gamma function of weight* κ for the space $\mathfrak{S}_m$ with $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, taking $\kappa = (0, 0, \dots, 0)$ and which for $\mathrm{Re}(a) \geq (m-1)/2 - k_m$ is defined by (see Gross and Richards, 1987),

$$\begin{aligned}
\Gamma_m[a,\kappa] &= \int_{\mathbf{A}\in\mathfrak{P}_m} \mathrm{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)/2-1} q_\kappa(\mathbf{A})(d\mathbf{A}) &(1)\\
&= \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma[a+k_i-(i-1)/2]\\
&= [a]_\kappa \Gamma_m[a], &(2)
\end{aligned}$$

where $\mathrm{etr}(\cdot) = \exp(\mathrm{tr}(\cdot))$, $|\cdot|$ denotes the determinant, and for $\mathbf{A} \in \mathfrak{S}_m$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m}\prod_{i=1}^{m-1}|\mathbf{A}_i|^{k_i-k_{i+1}} \tag{3}$$

with $\mathbf{A}_p = (a_{rs})$, $r, s = 1, 2, \dots, p$, $p = 1, 2, \dots, m$ is termed the *highest weight vector*, see Gross and Richards (1987). Also,

$$\begin{aligned}
\Gamma_m[a] &= \int_{\mathbf{A}\in\mathfrak{P}_m} \mathrm{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)/2-1}(d\mathbf{A})\\
&= \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma[a-(i-1)/2],
\end{aligned}$$

and $\mathrm{Re}(a) > (m-1)/2$.

In other branches of mathematics the *highest weight vector* $q_\kappa(\mathbf{A})$ is also termed the *generalised power* of $\mathbf{A}$ and is denoted as $\Delta_\kappa(\mathbf{A})$, see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Additional properties of $q_\kappa(\mathbf{A})$, which are immediate consequences of the definition of $q_\kappa(\mathbf{A})$ and the following property 1, are:

1. if $\lambda_1, \dots, \lambda_m$, are the eigenvalues of $\mathbf{A}$, then:
$$q_\kappa(\mathbf{A}) = \prod_{i=1}^{m}\lambda_i^{k_i}. \tag{4}$$

2. $$q_\kappa(\mathbf{A}^{-1}) = q_\kappa^{-1}(\mathbf{A}) = q_{-\kappa}(\mathbf{A}), \tag{5}$$

3. if $\kappa = (p, \dots, p)$, then:
$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p, \tag{6}$$
in particular if $p = 0$, then $q_\kappa(\mathbf{A}) = 1$.

4. if $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then:
$$q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A})q_\tau(\mathbf{A}), \tag{7}$$
in particular if $\tau = (p, p, \dots, p)$, then:
$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A}). \tag{8}$$

5. Finally, for $\mathbf{B} \in \mathfrak{A}^{m\times m}$ such that $\mathbf{C} = \mathbf{B}^*\mathbf{B} \in \mathfrak{S}_m$,

$$q_\kappa(\mathbf{BAB}^*) = q_\kappa(\mathbf{C})q_\kappa(\mathbf{A}) \tag{9}$$

and

$$q_\kappa(\mathbf{B}^{-1}\mathbf{AB}^{*-1}) = (q_\kappa(\mathbf{C}))^{-1}q_\kappa(\mathbf{A}). \tag{10}$$

Remark 2.1. Let $\mathcal{P}(\mathfrak{S}_m)$ denote the algebra of all polynomial functions on $\mathfrak{S}_m$, and $\mathcal{P}_k(\mathfrak{S}_m)$ the subspace of homogeneous polynomials of degree k and let $\mathcal{P}^\kappa(\mathfrak{S}_m)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m)$ such that:

$$\mathcal{P}_k(\mathfrak{S}_m) = \sum_\kappa \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m).$$

Note that q_κ is a homogeneous polynomial of degree k, moreover $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m)$, for references see Gross and Richards (1987).

In (2), $[a]_\kappa$ denotes the generalised Pochhammer symbol of weight κ, defined as:

$$\begin{aligned}
[a]_\kappa &= \prod_{i=1}^m (a-(i-1)/2)_{k_i} \\
&= \frac{\pi^{m(m-1)/4}\displaystyle\prod_{i=1}^m \Gamma[a+k_i-(i-1)/2]}{\Gamma_m[a]} \\
&= \frac{\Gamma_m[a,\kappa]}{\Gamma_m[a]},
\end{aligned}$$

where $\text{Re}(a) > (m-1)/2 - k_m$ and

$$(a)_i = a(a+1)\cdots(a+i-1),$$

is the standard Pochhammer symbol.

An alternatively definition of the generalised gamma function of weight κ is proposed by Khatri (1966), which is defined as:

$$\begin{aligned}
\Gamma_m[a,-\kappa] &= \int_{\mathbf{A}\in\mathfrak{P}_m} \text{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)/2-1}q_\kappa(\mathbf{A}^{-1})(d\mathbf{A}) && (11)\\
&= \pi^{m(m-1)/4}\prod_{i=1}^m \Gamma[a-k_i-(m-i)/2] \\
&= \frac{(-1)^k\Gamma_m[a]}{[-a+(m-1)/2+1]_\kappa}, && (12)
\end{aligned}$$

where $\text{Re}(a) > (m-1)/2 + k_1$.

Also recall that from Magnus and Neudecker (1988);

1. Let $\mathbf{A}$ be an $m\times n$ matrix and $\mathbf{B}$ a $p\times q$ matrix. The $mp\times nq$ matrix is defined as:

$$\begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

and is termed the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ and written $\mathbf{A}\otimes\mathbf{B}$.

2. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{A}_j$ its j-th column, the $\operatorname{vec} A$ is the $mn \times 1$ vector

$$\operatorname{vec} A = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$$

3. $\operatorname{tr} \mathbf{AB} = \operatorname{vec}' \mathbf{A}' \operatorname{vec} \mathbf{B}$.

4. $\operatorname{tr} \mathbf{ABCD} = \operatorname{vec}' \mathbf{D}'(\mathbf{C}' \otimes \mathbf{A}) \operatorname{vec} \mathbf{B} = \operatorname{vec}' \mathbf{D}(\mathbf{A}' \otimes \mathbf{C}') \operatorname{vec} \mathbf{B}'$.

5. For any matrix $\mathbf{X} = (x_{ij}) \in \Re^{m \times n}$, $d\mathbf{X}$ denotes the *matrix of differentials*, $d\mathbf{X} = (dx_{ij})$.

6. In particular[1], if $F : \Re^{m \times m} \to \Re^{m \times m}$ of rank m, then:
   (a) $d|F(\mathbf{X})|^p = |F(\mathbf{X})|^p \operatorname{tr}(F(\mathbf{X}))^{-1} dF(\mathbf{X})$,
   (b) $dF(\mathbf{X})^{-1} = -F(\mathbf{X})^{-1} dF(\mathbf{X}) F(\mathbf{X})^{-1}$. Also
   (c) $d\mathbf{AXB} = \mathbf{A}d\mathbf{XB}$, and
   (d) $d \operatorname{tr} \mathbf{AXB} = \operatorname{tr} \mathbf{A}d\mathbf{XB}$.

7. If $f : \Re^{m \times n} \to \Re$, $\operatorname{vec}' \mathbf{X} = (\operatorname{vec} \mathbf{X})'$ and $d \operatorname{vec}' \mathbf{X} = (d \operatorname{vec} \mathbf{X})'$ then:
   (a) if $d \operatorname{vec} f(\mathbf{X}) = \operatorname{vec}' \mathbf{B} d \operatorname{vec} \mathbf{X}$, its obtained that:
   $$\frac{\partial \operatorname{vec} f(\mathbf{X})}{\partial \operatorname{vec}' \mathbf{X}} = \operatorname{vec}' \mathbf{B} \text{ and } \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \mathbf{B}.$$
   (b) if $d^{\,2} \operatorname{vec} f(\mathbf{X}) = d \operatorname{vec}' \mathbf{X}\ \mathbf{B}\ d \operatorname{vec} \mathbf{X}$ then:
   $$\frac{\partial^2 \operatorname{vec} f(\mathbf{X})}{\partial \operatorname{vec} \mathbf{X} \partial \operatorname{vec}' \mathbf{X}} = \frac{1}{2}(\mathbf{B} + \mathbf{B}').$$

8. Finally, note that for a matrix $\mathbf{A}$ of order $m \times m$, their submatrices
   $$\mathbf{A}_1 = a_{11},\ \mathbf{A}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \cdots \mathbf{A}_m = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix},$$
   can be written as:
   $$\mathbf{A}_i = \mathbf{E}_i' \mathbf{A} \mathbf{E}_i, \tag{13}$$
   where the columns of $\mathbf{E}_i : m \times i$ are the first i columns of the identity matrix $\mathbf{I}_m$ and $\mathbf{E}_i = [\mathbf{I}_i \ \underset{i \times m-i}{\mathbf{0}}]'$ is such that
   (a) $\mathbf{E}_i' \mathbf{E}_i = \mathbf{I}_i$
   (b) $\mathbf{E}_i^+ = \mathbf{E}_i'$, hence, $\mathbf{E}_i^{+'} = \mathbf{E}_i$, where $\mathbf{A}^+$ denotes the Moore-Penrose inverse of $\mathbf{A}$,
   (c) $\mathbf{E}_m = \mathbf{I}_m$,
   (d) and
   $$\mathbf{E}_i \mathbf{E}_i' = \begin{bmatrix} \mathbf{I}_i \\ \underset{m-i \times i}{\mathbf{0}} \end{bmatrix} [\mathbf{I}_i \ \underset{i \times m-i}{\mathbf{0}}] = \begin{bmatrix} \mathbf{I}_i & \underset{i \times m-i}{\mathbf{0}} \\ \underset{m-i \times i}{\mathbf{0}} & \underset{m-i \times m-i}{\mathbf{0}} \end{bmatrix}.$$

[1]These results exist for more general conditions, see Magnus and Neudecker (1988, Theorem 1, pp. 149)

Closely related to the differentiation are the commutation matrix $\mathbf{K}_{mn}$ and the matrix $\mathbf{N}_m = \frac{1}{2}(\mathbf{I}_{m^2} + \mathbf{K}_m)$.

The commutation matrix $\mathbf{K}_{mn}$ is such that for a $m \times n$ matrix $\mathbf{A}$,

$$\mathbf{K}_{mn} \operatorname{vec} \mathbf{A} = \operatorname{vec} \mathbf{A}'.$$

with, $\mathbf{K}'_{mn} = \mathbf{K}^{-1}_{mn} = \mathbf{K}_{nm}$. If $m = n$, is written as $\mathbf{K}_m$ instead of $\mathbf{K}_{mn}$. The main property of this matrix is: Let $\mathbf{A}$ be an $m \times n$ matrix, $\mathbf{B}$ a $p \times q$ matrix. Then:

$$\mathbf{K}_{pm}(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{B} \otimes \mathbf{A})\mathbf{K}_{qn}.$$

By other hand, the matrix $\mathbf{N}_m$ is such that:

$$\mathbf{N}_m = \mathbf{N}'_m = \mathbf{N}^2_m = \mathbf{K}_m\mathbf{N}_m = \mathbf{N}_m\mathbf{K}_m.$$

Note that for $\mathbf{A}$ an $m \times m$ symmetric matrix,

$$\operatorname{vec} \mathbf{A} = \mathbf{N}_m \operatorname{vec} \mathbf{A}. \tag{14}$$

Finally consider the following matrix factorisation.

Proposition 2.1. *If $\mathbf{A}$ is a non-negative definite $m \times m$ matrix then there exist a non-negative definite $m \times m$ matrix, written as $\mathbf{A}^{1/2}$ such that $\mathbf{A} = \mathbf{A}^{1/2}\mathbf{A}^{1/2}$.*

2.2 Riesz distributions

This section summarise the densities and their corresponding characteristic functions for the two versions of the Riesz distribution. From Díaz-García (2012),

Definition 2.1. Let $\mathbf{\Sigma} \in \mathfrak{P}_m$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.

1. Then it is said that $\mathbf{X}$ has a Riesz distribution of type I if its density function is:

$$\frac{1}{\Gamma_m[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)/2-1} q_\kappa(\mathbf{X})(d\mathbf{X}) \tag{15}$$

   for $\mathbf{X} \in \mathfrak{P}_m$ and $\operatorname{Re}(a) \geq (m-1)/2 - k_m$; denoting this fact as $\mathbf{X} \sim \mathfrak{R}^I_m(a, \kappa, \mathbf{\Sigma})$.

2. Then it is said that $\mathbf{X}$ has a Riesz distribution of type II if its density function is:

$$\frac{q_\kappa(\mathbf{\Sigma})}{\Gamma_m[a,-\kappa]|\mathbf{\Sigma}|^a} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)/2-1} q_\kappa(\mathbf{X}^{-1})(d\mathbf{X}) \tag{16}$$

   for $\mathbf{X} \in \mathfrak{P}_m$ and $\operatorname{Re}(a) > (m-1)/2 + k_1$; denoting this fact as $\mathbf{X} \sim \mathfrak{R}^{II}_m(a, \kappa, \mathbf{\Sigma})$.

From Díaz-García (2012) and using (7) it is obtained that:

Lemma 2.1. *Let $\mathbf{\Sigma} \in \mathfrak{P}_m$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.*

1. *Then if $\mathbf{X} \sim \mathfrak{R}^I_m(a, \kappa, \mathbf{\Sigma})$ its characteristic function is:*

$$\phi_{\mathbf{X}}(\mathbf{T}) = q_{\kappa+a}\left(\left(\mathbf{I}_m - i\mathbf{\Sigma}^{1/2}\mathbf{T}\mathbf{\Sigma}^{1/2}\right)^{-1}\right) \tag{17}$$

   for $\operatorname{Re}(a) \geq (m-1)/2 - k_m$.

2. *Then if $\mathbf{X} \sim \mathfrak{R}^{II}_m(a, \kappa, \mathbf{\Sigma})$ its characteristic function is:*

$$\phi_{\mathbf{X}}(\mathbf{T}) = q_{\kappa-a}\left(\mathbf{I}_m - i\mathbf{\Sigma}^{1/2}\mathbf{T}\mathbf{\Sigma}^{1/2}\right) \tag{18}$$

   for $\operatorname{Re}(a) > (m-1)/2 + k_1$.

2.3 Differentiation

Finally, consider the following result about differentiation.

Lemma 2.2. *Let* $\mathbf{\Sigma} \in \mathfrak{P}_m$ *and* $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$. *Then if* $\mathbf{X} \sim \mathfrak{R}^I_m(a, \kappa, \mathbf{\Sigma})$ *it is obtained:*

1.
$$\frac{\partial \operatorname{vec} \phi_{\mathbf{X}}(\mathbf{T})}{\partial \operatorname{vec}' \mathbf{T}} = i \left[\sum_{i=1}^{m} (t_i - t_{i+1}) \operatorname{vec}' \mathbf{A}_i \right] \mathbf{N}_m q_\tau \left(\left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right)^{-1} \right),$$

2. and
$$\begin{aligned} \frac{\partial^2 \operatorname{vec} \phi_{\mathbf{X}}(\mathbf{T})}{\partial \operatorname{vec} \mathbf{T} \partial \operatorname{vec}' \mathbf{T}} &= i^2 \mathbf{N}_m \left\{ \sum_{i=1}^{m} (t_i - t_{i+1})^2 \operatorname{vec} \mathbf{A}_i \operatorname{vec}' \mathbf{A}_i \right. \\ &+ \frac{1}{2} \left[\sum_{i \neq j}^{m} \sum (t_i - t_{i+1})(t_j - t_{j+1}) \operatorname{vec} \mathbf{A}_i \operatorname{vec}' \mathbf{A}_j \right. \\ &+ \left. \sum_{i \neq j}^{m} \sum (t_i - t_{i+1})(t_j - t_{j+1}) \operatorname{vec} \mathbf{A}_j \operatorname{vec}' \mathbf{A}_i \right] \\ &+ \left. \sum_{i=1}^{m} (t_i - t_{i+1}) (\mathbf{A}_i \otimes \mathbf{A}_i) \right\} \mathbf{N}_m q_\tau \left(\left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right)^{-1} \right), \end{aligned}$$

where
$$\mathbf{A}_\alpha = \mathbf{\Sigma}^{1/2} \mathbf{E}_\alpha \left(\mathbf{E}'_\alpha \left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right) \mathbf{E}_\alpha \right)^{-1} \mathbf{E}'_\alpha \mathbf{\Sigma}^{1/2}, \quad \alpha = i, j,$$
$\tau = \kappa + a = (t_1, \ldots, t_m)$ *and* $\operatorname{Re}(a) \geq (m-1)/2 - k_m$.

Proof. By 6(a), 6(b) and Lemma 2.1, defining $\kappa + a = \tau = (t_1, \ldots, t_m)$, with $t_{m+1} = 0$, $\mathbf{E}_m = \mathbf{I}_m$ and
$$\mathbf{A}_\alpha = \mathbf{\Sigma}^{1/2} \mathbf{E}_\alpha \left(\mathbf{E}'_\alpha \left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right) \mathbf{E}_\alpha \right)^{-1} \mathbf{E}'_\alpha \mathbf{\Sigma}^{1/2}, \quad \alpha = i, j,$$
it is obtained that
$$d\phi_{\mathbf{X}}(\mathbf{T}) = i \left[\sum_{i=1}^{m} (t_i - t_{i+1}) \operatorname{tr} \mathbf{A}_i d\mathbf{T} \right] q_\tau \left(\left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right)^{-1} \right).$$
Hence, vectorising,
$$d \operatorname{vec} \phi_{\mathbf{X}}(\mathbf{T}) = i \left[\sum_{i=1}^{m} (t_i - t_{i+1}) \operatorname{vec}' \mathbf{A}_i \right] q_\tau \left(\left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right)^{-1} \right) d \operatorname{vec} \mathbf{T},$$
from where applying 7(a) and (14), the desired result is obtained.

Analogously, differentiating again,
$$\begin{aligned} d^2 \phi_{\mathbf{X}}(\mathbf{T}) &= i^2 \left\{ \sum_{i=1}^{m} (t_i - t_{i+1})^2 \operatorname{tr} \mathbf{A}_i d\mathbf{T} \operatorname{tr} \mathbf{A}_i d\mathbf{T} \right. \\ &+ \sum_{i \neq j}^{m} \sum (t_i - t_{i+1})(t_j - t_{j+1}) \operatorname{tr} \mathbf{A}_i d\mathbf{T} \operatorname{tr} \mathbf{A}_j d\mathbf{T} \\ &+ \left. \sum_{i=1}^{m} (t_i - t_{i+1}) \operatorname{tr} \mathbf{A}_i d\mathbf{T} \mathbf{A}_i d\mathbf{T} \right\} q_\tau \left(\left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right)^{-1} \right). \end{aligned}$$

And vectorising using 4, it is got;

$$d^2 \operatorname{vec} \phi_{\mathbf{X}}(\mathbf{T}) = i^2 \left\{ \sum_{i=1}^{m} (t_i - t_{i+1})^2 d \operatorname{vec}' \mathbf{T} \operatorname{vec} \mathbf{A}_i \operatorname{vec}' \mathbf{A}_i d \operatorname{vec} \mathbf{T} \right.$$

$$+ \sum_{i \neq j}^{m}\sum (t_i - t_{i+1})(t_j - t_{j+1}) d \operatorname{vec}' \mathbf{T} \operatorname{vec} \mathbf{A}_i \operatorname{vec}' \mathbf{A}_j d \operatorname{vec} \mathbf{T}$$

$$\left. + \sum_{i=1}^{m} (t_i - t_{i+1}) d \operatorname{vec}' \mathbf{T} (\mathbf{A}_i \otimes \mathbf{A}_i) d \operatorname{vec} \mathbf{T} \right\} q_\tau \left(\left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right)^{-1} \right).$$

From where, by applying 7(b) and (14), the outcome sought is obtained. □ □

Analogously, one has:

Lemma 2.3. *Let $\mathbf{\Sigma} \in \mathfrak{P}_m$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$. Then if $\mathbf{X} \sim \mathfrak{R}_m^{II}(a, \kappa, \mathbf{\Sigma})$ it is obtained:*

1.
$$\frac{\partial \operatorname{vec} \phi_{\mathbf{X}}(\mathbf{T})}{\partial \operatorname{vec}' \mathbf{T}} = -i \left[\sum_{i=1}^{m} (t_i - t_{i+1}) \operatorname{vec}' \mathbf{A}_i \right] \mathbf{N}_m q_\tau \left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right),$$

2. *and*
$$\frac{\partial^2 \operatorname{vec} \phi_{\mathbf{X}}(\mathbf{T})}{\partial \operatorname{vec} \mathbf{T} \partial \operatorname{vec}' \mathbf{T}} = i^2 \mathbf{N}_m \left\{ \sum_{i=1}^{m} (t_i - t_{i+1})^2 \operatorname{vec} \mathbf{A}_i \operatorname{vec}' \mathbf{A}_i \right.$$
$$+ \frac{1}{2} \left[\sum_{i \neq j}^{m}\sum (t_i - t_{i+1})(t_j - t_{j+1}) \operatorname{vec} \mathbf{A}_i \operatorname{vec}' \mathbf{A}_j \right.$$
$$\left. + \sum_{i \neq j}^{m}\sum (t_i - t_{i+1})(t_j - t_{j+1}) \operatorname{vec} \mathbf{A}_j \operatorname{vec}' \mathbf{A}_i \right]$$
$$\left. - \sum_{i=1}^{m} (t_i - t_{i+1}) (\mathbf{A}_i \otimes \mathbf{A}_i) \right\} \mathbf{N}_m q_\tau \left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right),$$

where:
$$\mathbf{A}_\alpha = \mathbf{\Sigma}^{1/2} \mathbf{E}_\alpha \left(\mathbf{E}'_\alpha \left(\mathbf{I}_m - i \mathbf{\Sigma}^{1/2} \mathbf{T} \mathbf{\Sigma}^{1/2} \right) \mathbf{E}_\alpha \right)^{-1} \mathbf{E}'_\alpha \mathbf{\Sigma}^{1/2}, \quad \alpha = i, j,$$

$\tau = \kappa - a = (t_1, \dots, t_m)$ and $\operatorname{Re}(a) > (m-1)/2 + k_1$.

Proof. This is a verbatim copy of the proof of Lemma 2.2. □ □

3 Moments of Riesz distributions

This section proposed the main result.

Theorem 3.1. *Let $\mathbf{\Sigma} \in \mathfrak{P}_m$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.*

1. *Then if $\mathbf{X}$ has a Riesz distribution of type I,*

(a) $\mathrm{E}(\mathbf{X}) = (k_m + a)\mathbf{\Sigma} + \sum_{i=1}^{m-1}(k_i - k_{i-1})\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}$.

(b) *and* $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X})$ *is*

$$(k_m + a)\mathbf{N}_m(\mathbf{\Sigma}\otimes\mathbf{\Sigma}) + \sum_{i=1}^{m-1}(k_i - k_{i-1})\mathbf{N}_m\left(\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\otimes\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\right).$$

for $\mathrm{Re}(a) \geq (m-1)/2 - k_m$.

2. *Then if* $\mathbf{X}$ *has a Riesz distribution of type II,*

(a) $\mathrm{E}(\mathbf{X}) = -(k_m - a)\mathbf{\Sigma} - \sum_{i=1}^{m-1}(k_i - k_{i-1})\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}$.

(b) *and* $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X})$ *is*

$$-(k_m - a)\mathbf{N}_m(\mathbf{\Sigma}\otimes\mathbf{\Sigma}) - \sum_{i=1}^{m-1}(k_i - k_{i-1})\mathbf{N}_m\left(\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\otimes\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\right).$$

for $\mathrm{Re}(a) > (m-1)/2 + k_1$.

Proof. Results are immediately from lemmas 2.2 and 2.3, remembering that:

$$\mathrm{E}(\mathrm{vec}\,\mathbf{X}) = \left.\frac{\partial\,\mathrm{vec}\,\phi_{\mathbf{X}}(\mathbf{T})}{i\;\partial\,\mathrm{vec}\,\mathbf{X}}\right|_{\mathbf{T}=0}$$

and

$$\mathrm{Cov}(\mathrm{vec}\,\mathbf{X}) = \mathrm{E}(\mathrm{vec}\,\mathbf{X}\,\mathrm{vec}'\,\mathbf{X}) - \mathrm{E}(\mathrm{vec}\,\mathbf{X})\,\mathrm{E}(\mathrm{vec}'\,\mathbf{X}) \tag{19}$$

where

$$\mathrm{E}(\mathrm{vec}\,\mathbf{X}\,\mathrm{vec}'\,\mathbf{X}) = \left.\frac{\partial^2\,\mathrm{vec}\,\phi_{\mathbf{X}}(\mathbf{T})}{i^2\;\partial\,\mathrm{vec}\,\mathbf{X}\partial\,\mathrm{vec}'\,\mathbf{X}}\right|_{\mathbf{T}=0}.$$

In order to ensure that $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X}) = \mathrm{Cov}'(\mathrm{vec}\,\mathbf{X})$ it is necessary that

$$\mathrm{E}(\mathrm{vec}\,\mathbf{X})\,\mathrm{E}'(\mathrm{vec}\,\mathbf{X})$$

be a symmetric matrix. Then, proceeding as in the case of $\mathrm{E}(\mathrm{vec}\,\mathbf{X}\,\mathrm{vec}'\,\mathbf{X}) = (\mathbf{B}+\mathbf{B}')/2$, consider the following equivalent definition;

$$\mathrm{Cov}(\mathrm{vec}\,\mathbf{X}) = \mathrm{E}(\mathrm{vec}\,\mathbf{X}\,\mathrm{vec}'\,\mathbf{X}) - \frac{1}{2}\{\mathrm{E}(\mathrm{vec}\,\mathbf{X})\,\mathrm{E}(\mathrm{vec}'\,\mathbf{X}) + [\mathrm{E}(\mathrm{vec}\,\mathbf{X})\,\mathrm{E}(\mathrm{vec}'\,\mathbf{X})]'\},$$

which alternative definition, coincides with (19) when $\mathrm{E}(\mathrm{vec}\,\mathbf{X})\,\mathrm{E}'(\mathrm{vec}\,\mathbf{X})$ is a symmetric matrix and at the same time ensuring that $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X})$ is a symmetric matrix. □ □

Observe that in Theorem 3.1.1 and 3.1.2, are defined as $a = n/2$, $\mathbf{\Sigma}\to 2\mathbf{\Sigma}$ and $\kappa = (0,\dots,0)$ the Wishart case is obtained. Moreover,

1. $\mathrm{E}(\mathbf{X}) = n\mathbf{\Sigma}$,
2. $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X}) = 2n\mathbf{N}_m(\mathbf{\Sigma}\otimes\mathbf{\Sigma}) = n(\mathbf{I}_{m^2} + \mathbf{K}_m)(\mathbf{\Sigma}\otimes\mathbf{\Sigma})$,

see Muirhead (1982, p. 90) and Magnus and Neudecker (1988, p. 253).

4 Conclusions

Observe that if $\mathbf{Y}_1, \cdots, \mathbf{Y}_N$, are independent p-dimensional random vectors, with $N \geq p$, such that the random matrix $\mathbf{X}$, is defined as:

$$\mathbf{X} = \sum_{i=1}^{N} \mathbf{Y}_i \mathbf{Y}_i' = \mathbf{Y}'\mathbf{Y}, \quad \mathbf{Y}' = (\mathbf{Y}_1, \cdots, \mathbf{Y}_N)$$

has a Riesz distribution type I, then by multivariate central limit theorem, Muirhead (1982, p. 15) and Theorem 3.1, if

$$\bar{\mathbf{Y}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{Y}_i$$

and

$$\mathbf{S}(n) = \frac{1}{n}\sum_{i=1}^{N} (\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})', \quad N = n+1$$

the asymptotic distribution as $n \to \infty$ of

$$n^{1/2}\left[\operatorname{vec}\mathbf{S}(n) - \frac{(k_m + a)}{n}\operatorname{vec}\mathbf{\Sigma} + \sum_{i=1}^{m-1}\frac{(k_i - k_{i-1})}{n}\operatorname{vec}\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\right]$$

is:

$$\mathcal{N}_{m^2}\left(\mathbf{0}, \frac{(k_m + a)}{n}\mathbf{N}_m(\mathbf{\Sigma}\otimes\mathbf{\Sigma}) + \sum_{i=1}^{m-1}\frac{(k_i - k_{i-1})}{n}\mathbf{N}_m\left(\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\otimes\mathbf{\Sigma}^{1/2}\mathbf{E}_i\mathbf{E}_i'\mathbf{\Sigma}^{1/2}\right)\right).$$

Note that this asymptotic multivariate normal distribution is singular, moreover, its rank is $m(m+1)/2$. Also, observe that if $a = n/2$, $\mathbf{\Sigma} \to 2\mathbf{\Sigma}$ and $\kappa = (0, \dots, 0)$ the asymptotic result is obtained by Muirhead (1982, pp. 90-91) for the Wishart case. The author is currently studying in detail the distribution of the random matrix $\mathbf{Y}$ and some of their basic properties.

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