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Left cells in the weighted Coxeter group $\widetilde{C}_n$

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Abstract: The fixed point set of the affine Weyl group $(\widetilde{A}_{2n}, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$ can be considered as the affine Weyl group $(\widetilde{C}_n, S)$. Then the left and two-sided cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$, where $\widetilde{\ell}$ is the length function of $\widetilde{A}_{2n}$, can be given an explicit description by studying the fixed point set of the affine Weyl group $(\widetilde{A}_{2n}, \widetilde{S})$ under α. We describe the cells of $(\widetilde{C}_n, \widetilde{\ell})$ corresponding to the partitions $\mathbf{k1^{2n+1-k}}$ with $1 \leqslant k \leqslant 2n+1$ and $(2n-1, 2)$.

Key words: affine Weyl groups; left cells; quasi-split case; weighted Coxeter group

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加权的 Coxeter 群 $\widetilde{C}_n$ 的左胞腔

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摘要: 仿射 Weyl 群 $(\widetilde{A}_{2n}, \widetilde{S})$ 在某个群同构 α (其中 $\alpha(\widetilde{S}) = \widetilde{S}$) 下的固定点集合能被看作是仿射 Weyl 群 $(\widetilde{C}_n, S)$. 那么加权的 Coxeter 群 $(\widetilde{C}_n, \widetilde{\ell})$的左和双边胞腔($\widetilde{\ell}$ 是仿射 Weyl 群 $\widetilde{A}_{2n}$ 的长度函数), 就能通过研究仿射 Weyl 群 $(\widetilde{A}_{2n}, \widetilde{S})$ 在群同构 α 下的固定点集合而给出一个清晰的划分. 因此给出了加权的 Coxeter 群 $(\widetilde{C}_n, \widetilde{\ell})$ 对应于划分 $\mathbf{k1^{2n+1-k}}$ 和 $(2n-1, 2)$ 的所有左胞腔的清晰刻画, 这里对所有的 $1 \leqslant k \leqslant 2n+1$.

关键词: 仿射 Weyl 群; 左胞腔; 拟分裂; 加权的 Coxeter 群

0 Introduction

In this article, we will discuss some results about left cells in a weighted Coxeter group (W, L) as defined by Lusztig in [1], which is, by definition, a Coxeter system (W, S) together with a weight function $L : W \longrightarrow \mathbb{Z}$. When $L = \widetilde{\ell}$, (W, L) is called in split case by Lusztig, where $\widetilde{\ell}$ is the length function of W. And when W can be realized as the fixed point set of a finite or an affine Coxeter system $(\widetilde{W}, \widetilde{S})$ under a group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$, where

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the weight function L is the restriction to W of the length function $\widetilde{\ell}$ of $\widetilde{W}$, (W, L) is called in quasi-split case by Lusztig (see [1, Ch. 16]).

The affine Weyl group $W = \widetilde{C}_n$ can be realized as the fixed point set of the affine Weyl group $\widetilde{W} = \widetilde{A}_{2n}$ under the group automorphism α determined by $\alpha(s_i) = s_{2n-i}$ for $0 \leqslant i < 2n+1$, where the Coxeter generator set $\widetilde{S} = \{s_i \mid 0 \leqslant i < 2n+1\}$ of $\widetilde{A}_{2n}$ satisfies $s_i^2 = 1, s_i s_j = s_j s_i$ ($j \not\equiv i \pm 1 \bmod 2n+1$) and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for any $0 \leqslant i, j < 2n+1$ (we stipulate $s_{2n+1} = s_0$). Following the ideas of Shi in [2], we will study certain left cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$, where $\widetilde{\ell}$ is the length function of $\widetilde{A}_{2n}$.

1 Cells theories of a weighted Coxeter group

We assemble in this section some basic concepts and results of a weighted Coxeter group, which follow from Lusztig in [1] except for 1.7 from Shi in [2].

1.1 Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley order on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called reduced if $r = \ell(w)$. By a *weight function* on W, we mean a map $L : W \longrightarrow \mathbb{Z}$ satisfying that $L(s) = L(t)$ for any $s, t \in S$ conjugate in W and that $L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r \in W$. Call (W, L) is a *weighted Coxeter group*.

Suppose that there exists a group automorphism $\alpha : W \longrightarrow W$ with $\alpha(S) = S$. Let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J in S, let $w_J \in W^\alpha$ be the longest element in the subgroup W_J of W generated by J. Let S_α be the set of elements w_J with J ranging over all α-orbits on S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function $\ell : W \longrightarrow \mathbb{N}$ is a weight function on W^α. The weighted Coxeter group (W^α, ℓ) is called in the *quasi-split* case.

1.2 Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminant v with integer coefficients. Denote $v_w = v^{L(w)}$ for any $w \in W$. Define a ring involution $a \longrightarrow \overline{a}$ of $\mathcal{A}$ by setting $\overline{\sum a_i v^i} = \sum a_i v^{-i}$ where $a_i \in \mathbb{Z}$ in the sum. Define $\mathcal{A}_{<m} = \{f \in \mathcal{A} \mid \deg f < m\}$ for any $m \in \mathbb{Z}$.

1.3 Let (W, S) be a Coxeter system. For any $w, x, y, z \in W$ and $s \in S$ with $sx < x < y < sy$, define $p_{z,w}, M_{x,y}^s \in \mathcal{A}$ recurrently by the following requirements:

$$p_{z,w} = 0 \ \text{ if } \ z \not\leqslant w, p_{w,w} = 1 \ \text{ and } \ p_{z,w} \in \mathcal{A}_{<0} \ \text{ if } \ z < w. \tag{1.3.1}$$

$p_{z,w} = v_s^{\epsilon} p_{z,sw} + p_{sz,sw} - \sum\limits_{\substack{z \leqslant z' < sw \\ sz' < z'}} M_{z',sw}^s p_{z,z'}$ for $z < w$ and $sw < w$, where $\epsilon = 1$ if $sz < z$, and -1 if $sz > z$ (see [1, The proof of Theorem 6.6]). (1.3.2)

$$\sum_{\substack{x \leqslant z < y \\ sz < z}} M_{z,y}^s p_{x,z} \equiv v_s p_{x,y} (\bmod \mathcal{A}_{<0}), \tag{1.3.3}$$

$$\overline{M_{x,y}^s} = M_{x,y}^s. \tag{1.3.4}$$

The condition (1.3.3) determines the coefficients of v^k in $M_{x,y}^s$ for all $k \geqslant 0$; then (1.3.4) determines all the other coefficients (see [1, Proposition 6.3]).

1.4 Let (W,S) be a Coxeter system. For $y \neq w \in W$, if there exists $s \in S$ with $w < sw$, such that either $y = sw$ or $M^s_{y,w} \neq 0$, then we denote $y \underset{L}{\leftarrow} w$; if there exists $s \in S$ with $w < ws$, such that either $y = ws$ or $M^s_{y^{-1},w^{-1}} \neq 0$, then we denote $y \underset{R}{\leftarrow} w$. Let $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$) be the preorder on W which is transitively generated by the relation $y \underset{L}{\leftarrow} w$ (resp., $y \underset{R}{\leftarrow} w$). The equivalence relation associated to this preorder is denoted by $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$). The corresponding equivalence classes in W are called *left cells* (resp., *right cells*) of W. Write $y \underset{LR}{\leqslant} w$ in W, if there exists a sequence $y_0 = y, y_1, \ldots, y_r = w$ in W with some $r \geqslant 0$ such that for every $1 \leqslant i \leqslant r$, either $y_{i-1} \underset{L}{\leqslant} y_i$ or $y_{i-1} \underset{R}{\leqslant} y_i$ holds. The equivalence relation associated to the preorder $\underset{LR}{\leqslant}$ is denoted by $\underset{LR}{\sim}$ and the corresponding equivalence classes in W are called *two-sided cells* of W.

1.5 For $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. If $y, w \in W$ satisfy $y \underset{L}{\leqslant} w$ (resp., $y \underset{R}{\leqslant} w$), then $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ (resp., $\mathcal{L}(y) \supseteq \mathcal{L}(w)$). In particular, if $y \underset{L}{\sim} w$ (resp., $y \underset{R}{\sim} w$), then $\mathcal{R}(y) = \mathcal{R}(w)$ (resp., $\mathcal{L}(y) = \mathcal{L}(w)$) (see [1, Lemma 8.6]).

1.6 In [1, Chapter 13], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ in terms of structural coefficients of the Hecke algebra associated to W.

In [1, Chapters 14-16], Lusztig proved the following results when W is either a finite or an affine Coxeter group and when (W,L) is either in the split case or in the quasi-split case.

(1) $y \underset{LR}{\leqslant} w$ in W implies $a(w) \leqslant a(y)$. Hence $y \underset{LR}{\sim} w$ in W implies $a(w) = a(y)$.

(2) If $w, y \in W$ satisfy $a(w) = a(y)$ and $y \underset{L}{\leqslant} w$ (resp., $y \underset{R}{\leqslant} w$, $y \underset{LR}{\leqslant} w$), then $y \underset{L}{\sim} w$ (resp., $y \underset{R}{\sim} w$, $y \underset{LR}{\sim} w$).

For any $X \subset W$, write $X^{-1} := \{x^{-1} \mid x \in X\}$.

Lemma 1.7 (see [2, Lemma 1.7]) *Suppose that W is either a finite or an affine Coxeter group and that (W,L) is either in the split case or in the quasi-split case.*

Let E be a non-empty subset of W satisfying the following conditions:

(a) *There exists some $k \in \mathbb{N}$ with $a(x) = k$ for any $x \in E$*;

(b) *E is a union of some left cells of W*;

(c) *$E^{-1} = E$.*

Then E is a union of some two-sided cells of W.

2 The weighted Coxeter groups $(\widetilde{A}_{2n}, \widetilde{\ell})$ and $(\widetilde{C}_n, \widetilde{\ell})$

2.1 The affine Weyl group $\widetilde{A}_{2n}$ can be realized as the following permutation group on the integer set $\mathbb{Z}$ (see [3, Subsection 3.6] and [4, Subsection 4.1]:

$$\widetilde{A}_{2n} = \Big\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+2n+1)w = (i)w + 2n + 1, \sum_{i=1}^{2n+1} (i)w = \sum_{i=1}^{2n+1} i \Big\}.$$

The Coxeter generator set $\widetilde{S} = \{s_i \mid 0 \leqslant i < 2n+1\}$ of $\widetilde{A}_{2n}$ is given by

$$(t)s_i = \begin{cases} t, & \text{if } t \not\equiv i, i+1 \pmod{2n+1}, \\ t+1, & \text{if } t \equiv i \pmod{2n+1}, \\ t-1, & \text{if } t \equiv i+1 \pmod{2n+1}, \end{cases}$$

for any $t \in \mathbb{Z}$ and $0 \leqslant i < 2n+1$. Any $w \in \widetilde{A}_{2n}$ can be realized as a $\mathbb{Z} \times \mathbb{Z}$ monomial matrix $A_w = (a_{ij})_{i,j\in\mathbb{Z}}$, where a_{ij} is 1 if $j = (i)w$ and 0 if otherwise. The row (resp., column) indices of A_w are increasing from top to bottom (resp., from left to right). We can conveniently use some familiar operations in linear algebra on the matrix A_w. For example, $A_{w^{-1}}$ is just the transposed matrix of A_w; $A_{s_i w}$ (resp., A_{ws_i}) can be obtained from A_w by transposing the $(2nq+q+i)$th and the $(2nq+q+i+1)$th rows (resp., columns) for all $q \in \mathbb{Z}$.

Let $\alpha : \widetilde{A}_{2n} \longrightarrow \widetilde{A}_{2n}$ be the group automorphism determined by $\alpha(s_i) = s_{2n-i}$ for $0 \leqslant i < 2n+1$. Then the affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_{2n}$ under α, which can also be described as a permutation group on $\mathbb{Z}$ as follows.

$$\widetilde{C}_n = \{w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+2n+1)w = (i)w + 2n + 1, (-i)w = -(i)w, \forall i \in \mathbb{Z}\}$$

with the Coxeter generator set $S = \{t_i \mid 0 \leqslant i \leqslant n\}$, where $t_i = s_i s_{2n-i}$ for $1 < i < n$, $t_0 = s_0 s_{2n} s_0$ and $t_n = s_n$. For any $w \in \widetilde{C}_n$, we can see that $(k(2n+1))w = k(2n+1)$, for any $k \in \mathbb{Z}$. For the sake of convenience, we define s_i for any $i \in \mathbb{Z}$ by setting $s_{2qn+q+b}$ to be s_b for any $q \in \mathbb{Z}$ and $0 \leqslant b < 2n+1$.

2.2 Denote the set $\{1, 2, \cdots, k\}$ by $[k]$ for any $k \in \mathbb{N}$. By a partition of a positive integer n, we mean an r-tuple $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_r)$ of weakly decreasing positive integers $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ with $\sum\limits_{k=1}^{r} \lambda_k = n$ for some $r \geqslant 1$. λ_i is called a *part* of λ. We usually denote λ in the form $\mathbf{j_1^{k_1} j_2^{k_2} \cdots j_m^{k_m}}$ (boldfaced) with $j_1 > j_2 > \cdots > j_m \geqslant 1$ if j_i is a part of λ with multiplicity $k_i \geqslant 1$ for $i \geqslant 1$. For example, $\mathbf{85^2 2^3 1^2}$ stands for the partition $(8, 5, 5, 2, 2, 2, 1, 1)$ of 26.

Fix $w \in \widetilde{A}_{2n}$. For any $i \neq j$ in $[2n+1]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that both inequalities $2pn+p+i > 2qn+q+j$ and $(2pn+p+i)w < (2qn+q+j)w$ hold. In terms of matrix entries of w, this means that the entry 1 at the position $(2qn+q+j, (2qn+q+j)w)$ is located to the northeast of the entry 1 at the position $(2pn+p+i, (2pn+p+i)w)$ (see Fig. 1). This defines a partial order $\prec_w$ on the set $[2n+1]$.

$$\begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix} \begin{array}{l} \text{——— } (2qn+q+j)\text{-th row} \\ \\ \text{——— } (2pn+p+i)\text{-th row} \end{array}$$

Fig. 1 Illustration of $i \prec_w j$

A sequence $a_1, a_2, \cdots, a_r$ in $[2n+1]$ is called a *w-chain*, if $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r$. We identify a w-chain $a_1, a_2, \cdots, a_r$ with the corresponding set $\{a_1, a_2, \cdots, a_r\}$. For any $k \geqslant 1$, a k-w-chain-family is by definition a disjoint union $X = \bigcup\limits_{i=1}^{k} X_i$ of k w-chains $X_1, \cdots, X_k$ in $[2n+1]$. Let d_k be the maximally possible cardinal of a k-w-chain-family for any $k \geqslant 1$. Then there exists some $r \geqslant 1$ such that $d_1 < d_2 < \cdots < d_r = 2n+1$. Let $\lambda_1 = d_1$ and $\lambda_k = d_k - d_{k-1}$ for any $1 < k \leqslant r$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ by a result of Curtis Greene in [2n+1]. Let Λ_{2n+1} be the set of partitions of $2n+1$. Hence $w \mapsto \psi(w) = (\lambda_1, \cdots, \lambda_r)$ defines a map from the set $\widetilde{A}_{2n}$ to the set Λ_{2n+1}.

2.3 Let $\widetilde{\ell}$, ℓ be the length functions on the Coxeter systems $(\widetilde{A}_{2n},\widetilde{S})$, $(\widetilde{C}_n,S)$, respectively. By the definition in 1.1, we see that the weighted Coxeter group $(\widetilde{A}_{2n},\widetilde{\ell})$ is in the split case, while $(\widetilde{C}_n,\widetilde{\ell})$ is in the quasi-split case (see [1, Lemma 16.2]).

For any $x\in\widetilde{A}_{2n}$ and $k\in\mathbb{Z}$, let $m_k(x)={}^{\#}\,\{i\in\mathbb{Z}\mid i<k \text{ and } (i)x>(k)x\}$ and $m^n(x)={}^{\#}\,\{i\in\mathbb{Z}\mid i<n+1 \text{ and } (i)x>n\}$. Then the formulae for the functions $\widetilde{\ell}$ and ℓ are as follows.

Proposition 2.4 *For any $w\in\widetilde{A}_{2n}$ and $x\in\widetilde{C}_n$, we have*

(1) $\widetilde{\ell}(w)=\sum\limits_{1\leqslant i<j\leqslant 2n+1}\left|\left\lfloor\frac{(j)w-(i)w}{2n+1}\right\rfloor\right|=\sum\limits_{k=1}^{2n+1}m_k(w)$;

(2) $\ell(x)=\frac{1}{2}(\widetilde{\ell}(x)-m_0(x)+m^n(x))$,

where $\lfloor a\rfloor$ stands for the largest integer not larger than a and $|a|$ stands for the absolute value of a for any $a\in\mathbb{Q}$.

Proof (1) follows from Shi in [8, Proposition 2.4].

(2) When $\ell(x)=0$, $\widetilde{\ell}(x)=m_0(x)=m^n(x)=0$.

When $\ell(x)=1$, if $x=t_i$ with $0<i<n$, then $\widetilde{\ell}(x)=2$ and $m_0(x)=m^n(x)=0$; if $x=t_0$, then $\widetilde{\ell}(x)=3$, $m_0(x)=1$ and $m^n(x)=0$; if $x=t_n$, then $\widetilde{\ell}(x)=1$, $m_0(x)=0$ and $m^n(x)=1$. These imply that the equality is right when $\ell(x)\leqslant 1$.

Now suppose that the equality is right when $\ell(x)\leqslant k$ with $k\in\mathbb{N}$.

When $\ell(x)=k+1$, if $t_i\in\mathcal{L}(x)$ with $0<i<n$, then $\widetilde{\ell}(t_ix)=\widetilde{\ell}(x)-2$, $m_0(t_ix)=m_0(x)$ and $m^n(t_ix)=m^n(x)$; if $t_0\in\mathcal{L}(x)$, then $\widetilde{\ell}(t_0x)=\widetilde{\ell}(x)-3$, $m_0(t_0x)=m_0(x)-1$ and $m^n(t_0x)=m^n(x)$; if $t_n\in\mathcal{L}(x)$, then $\widetilde{\ell}(t_nx)=\widetilde{\ell}(x)-1$, $m_0(t_nx)=m_0(x)$ and $m^n(t_nx)=m^n(x)-1$. These imply that the equality is right when $\ell(x)=k+1$. Hence (2) is obtained.

2.5 Let $\leqslant,\leqslant_C$ be the Bruhat-Chevalley orders on the Coxeter systems $(\widetilde{A}_{2n},\widetilde{S})$, $(\widetilde{C}_n,S)$, respectively. Since the condition $x\leqslant_C y$ is equivalent to $x\leqslant y$ for any $x,y\in\widetilde{C}_n$, it will cause no confusion if we use the notation $\leqslant$ in the place of $\leqslant_C$. Hence from now on we shall use $\leqslant$ instead of both $\leqslant$ and $\leqslant_C$.

Let $\widetilde{\mathcal{L}}(x)=\{s\in\widetilde{S}\mid sx<x\}$ and $\widetilde{\mathcal{R}}(x)=\{s\in\widetilde{S}\mid xs<x\}$ for $x\in\widetilde{A}_{2n}$ and let $\mathcal{L}(y)=\{t\in S\mid ty<y\}$ and $\mathcal{R}(y)=\{t\in S\mid yt<y\}$ for $y\in\widetilde{C}_n$.

Corollary 2.6 *For any $x\in\widetilde{C}_n$ and $0\leqslant i\leqslant n$,*

$s_i\in\widetilde{\mathcal{L}}(x)\Longleftrightarrow s_{2n-i}\in\widetilde{\mathcal{L}}(x)\Longleftrightarrow t_i\in\mathcal{L}(x)\Longleftrightarrow (i)x>(i+1)x\Longleftrightarrow(2n+1-i)x<(2n-i)x$,

$s_i\in\widetilde{\mathcal{R}}(x)\Longleftrightarrow s_{2n-i}\in\widetilde{\mathcal{R}}(x)\Longleftrightarrow t_i\in\mathcal{R}(x)\Longleftrightarrow (i)x^{-1}>(i+1)x^{-1}\Longleftrightarrow(2n+1-i)x^{-1}<(2n-i)x^{-1}$.

Proof The results follow from Shi in [2, Corollary 2.6].

2.7 For any $a\in\mathbb{Z}$, denote by $\langle a\rangle$ the unique integer in $[2n+1]$ satisfying $a\equiv\langle a\rangle \bmod 2n+1$. It is known that every $w\in\widetilde{C}_n$ is determined uniquely by the n-tuple $((1)w,(2)w,\cdots,(n)w)$. Hence we shall identify w with the n-tuple $((1)w,(2)w,\cdots,(n)w)$ and denote the latter by $[(1)w,(2)w,\cdots,(n)w]$ in such a sense. Let $w=[a_1,a_2,\cdots,a_n]$, $w'=t_iw=[a'_1,a'_2,\cdots,a'_n]$ and $w''=wt_i=[a''_1,a''_2,\cdots,a''_n]$ be in $\widetilde{C}_n$. When $i\in[n-1]$, we have $a'_j=a_j$ for $j\in[n]\setminus\{i,i+1\}$, $a'_i=a_{i+1}$ and $a'_{i+1}=a_i$; when $i=0$, we have $a'_j=a_j$ for $1<j\leqslant n$ and $a'_1=-a_1$; when $i=n$, we have $a'_j=a_j$ for $j\in[n-1]$ and $a'_n=2n+1-a_n$. On the other

hand, when $i \in [n-1]$, we have $a''_j = a_j$ if $\langle a_j \rangle \notin \{i, i+1, 2n-i, 2n+1-i\}$, $a''_j = a_j + 1$ if $\langle a_j \rangle \in \{i, 2n-i\}$ and $a''_j = a_j - 1$ if $\langle a_j \rangle \in \{i+1, 2n+1-i\}$; when $i = 0$, we have $a''_j = a_j$ if $\langle a_j \rangle \notin \{1, 2n\}$, $a''_j = a_j + 2$ if $\langle a_j \rangle = 2n$ and $a''_j = a_j - 2$ if $\langle a_j \rangle = 1$; when $i = n$, we have $a''_j = a_j$ if $\langle a_j \rangle \notin \{n, n+1\}$, $a''_j = a_j + 1$ if $\langle a_j \rangle = n$ and $a''_j = a_j - 1$ if $\langle a_j \rangle = n+1$.

The following results provide some information from $w = [a_1, a_2, \cdots, a_n] \in \widetilde{C}_n$.

Proposition 2.8 *Let $w = [a_1, a_2, \cdots, a_n] \in \widetilde{C}_n$. Then*

(1) *Let $0 \leqslant k \leqslant n$. $t_k \in \mathcal{L}(w)$ if and only if $a_k > a_{k+1}$, with the convention that $a_0 = 0$ and $a_{n+1} = n$.*

(2) *Let $\langle a_i \rangle, \langle a_j \rangle \in \{k, k+1, 2n-k, 2n+1-k\}$ for some $i \neq j$ in $[n]$. Then $t_k \in \mathcal{R}(w)$, $0 < k \leqslant n$, if one of the following conditions holds*:

(i) *either $a_j - a_i > 2n+1$, or $i > j$ and $a_j > a_i$ if $(\langle a_i \rangle, \langle a_j \rangle) \in \{(k, k+1), (2n-k, 2n+1-k)\}$*;

(ii) *$a_i + a_j < 0$ if $(\langle a_i \rangle, \langle a_j \rangle) = (k, 2n-k)$*;

(iii) *$a_i + a_j > 2n+1$ if $(\langle a_i \rangle, \langle a_j \rangle) = (2n+1-k, k+1)$.*

Let $\langle a_i \rangle \in \{1, 2n\}$. Then $t_0 \in \mathcal{R}(w)$ if one of the following conditions holds:

(iv) *$a_i < 0$ if $\langle a_i \rangle = 2n$*;

(v) *$a_i > 2n+1$ if $\langle a_i \rangle = 1$.*

Proof The results follow from Shi in [2, Proposition 2.8], by 2.7 and Corollary 2.6.

2.9 For any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_t)$ in Λ_{2n+1}, we write $\lambda \leqslant \mu$ if $\lambda_1 + \cdots + \lambda_k \leqslant \mu_1 + \cdots + \mu_k$ for any $1 \leqslant k \leqslant \min\{r, t\}$. This defines a partial order on Λ_{2n+1}. It is well known that if $x \in \widetilde{A}_{2n}$, $s \in \widetilde{\mathcal{L}}(x)$ and $t \in \widetilde{\mathcal{R}}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$ (see [4, Lemma 5.5 and Corollary 5.6]). This implies by Corollary 2.6 that if $x \in \widetilde{C}_n$, $s \in \mathcal{L}(x)$ and $t \in \mathcal{R}(x)$, then $\psi(sx), \psi(xt) \leqslant \psi(x)$.

Let $\widetilde{a}, a$ be the a-functions of the weighted Coxeter groups $(\widetilde{A}_{2n}, \widetilde{\ell})$ $(\widetilde{C}_n, \widetilde{\ell})$, respectively (see 2.3 and 1.6).

Lemma 2.10(see [1, Lemma 16.5]) *$a(z) = \widetilde{a}(z)$ for any $z \in \widetilde{C}_n$.*

Lemma 2.11(see [1, Lemma 16.14]) *Let $x, y \in \widetilde{C}_n$. Then $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) in $\widetilde{C}_n$ if and only if $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) in $\widetilde{A}_{2n}$.*

By Lemma 2.11, we can just use the notation $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to the group $\widetilde{A}_{2n}$ or $\widetilde{C}_n$.

For any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in \Lambda_{2n+1}$, define $\mu = (\mu_1, \mu_2, \cdots, \mu_t) \in \Lambda_{2n+1}$ by setting $\mu_j =^{\#} \{k \geqslant 1 \mid \lambda_k \geqslant j\}$, for any $j \geqslant 1$, and call μ the *dual partition* of λ.

Lemma 2.12 *Let $x, y \in \widetilde{A}_{2n}$.*

(1) *$x \underset{LR}{\leqslant} y$ if and only if $\psi(y) \leqslant \psi(x)$. In particular, $x \underset{LR}{\sim} y$* if and only if $\psi(x) = \psi(y)$ (*see* [5, *Theorem* 6], [4, *Theorem* 17.4] *and* [6, *Theorem* B]).

(2) *$\widetilde{a}(x) = \sum_{i=1}^{t} (i-1)\mu_i$, where $(\mu_1, \mu_2, \cdots, \mu_t)$ is the dual partition of $\psi(x)$ (see* [7, *Subsection* 6.27]).

2.13 A non-empty subset E of a Coxeter group $W = (W, S)$ is said *left-connected*, (resp., *right-connected*) if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, \cdots, x_r = y$ in E such that $x_{i-1}x_i^{-1} \in S$ (resp., $x_i^{-1}x_{i-1} \in S$) for every $i \in [r]$. E is said *two-sided-connected* if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, \cdots, x_r = y$ in E such that either $x_{i-1}x_i^{-1}$ or $x_i^{-1}x_{i-1}$ is in S for every $i \in [r]$.

Let $F \subseteq E$ in W. Call F a *left-connected component* of E, if F is a maximal left-connected subset of E. One can define a right-connected component and a two-sided-connected component of E similarly.

For any $\lambda \in \Lambda_{2n+1}$, define $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

Lemma 2.14 *Let* $\lambda \in \Lambda_{2n+1}$.

(1) *Any left-connected (resp., right-connected, two-sided-connected) set of* $\psi^{-1}(\lambda)$ *is contained in some left (resp., right, two-sided) cell of* $\widetilde{A}_{2n}$.

(2) *Any left-connected (resp., right-connected, two-sided-connected) set of* E_λ *is contained in some left (resp., right, two-sided) cell of* $\widetilde{C}_n$.

(3) *The set* E_λ *is either empty or a union of some two-sided cells of* $\widetilde{C}_n$.

Proof The results follow from Shi in [8, Corollary 2.18].

Corollary 2.15 *Let* $x, y \in \widetilde{A}_{2n}$ *satisfy* $x, y \in \psi^{-1}(\lambda)$ *for some* $\lambda \in \Lambda_{2n+1}$.

(1) *If* $\widetilde{\ell}(y) = \widetilde{\ell}(x) + \widetilde{\ell}(yx^{-1})$, *then* x, y *are in the same left-connected component of* $\psi^{-1}(\lambda)$ *and hence* $x \underset{L}{\sim} y$.

(2) *If* $\widetilde{\ell}(y) = \widetilde{\ell}(x) + \widetilde{\ell}(x^{-1}y)$, *then* x, y *are in the same right-connected component of* $\psi^{-1}(\lambda)$ *and hence* $x \underset{R}{\sim} y$.

Let $x, y \in \widetilde{C}_n$ *satisfy* $x, y \in \psi^{-1}(\lambda)$ *for some* $\lambda \in \Lambda_{2n+1}$.

(3) *If* $\ell(y) = \ell(x) + \ell(yx^{-1})$, *then* x, y *are in the same left-connected component of* E_λ *and hence* $x \underset{L}{\sim} y$.

(4) *If* $\ell(y) = \ell(x) + \ell(x^{-1}y)$, *then* x, y *are in the same right-connected component of* E_λ *and hence* $x \underset{R}{\sim} y$.

Proof The results follow from Shi in [2, Corollary 2.19].

3 Partial order $\preceq_w$ on $[2n+1]$ determined by element w

In this section, we introduce two technical tools following Shi in [2, Section 3]. One is a transformation on an element in 3.3, which is a crucial step in proving the left-connectedness of a left cell and in finding a representative set for the left cells of $\widetilde{C}_n$ in the set E_λ, $\lambda \in \Lambda_{2n+1}$. The other is the generalized tabloids in 3.5, by which we can check whether two elements of $\widetilde{C}_n$ are in the same left cell.

3.1 $i, j \in [2n]$ are said $2n$-*dual*, if $i + j = 2n + 1$; in this case, we denote $j = \overline{i}$ (hence $i = \overline{j}$ also). Recall the partial order $\preceq_w$ on $[2n+1]$ defined in 2.2 for any $w \in \widetilde{A}_{2n}$ and $\widetilde{C}_n$ can be regarded as a subset of $\widetilde{A}_{2n}$ (see 2.1). Fix $w \in \widetilde{A}_{2n}$. $i \neq j$ in $[2n+1]$ are said *w-comparable* if either $i \prec_w j$ or $j \prec_w i$, and *w-uncomparable* if otherwise. When $w \in \widetilde{C}_n$, $i \in [2n]$ is said *w-wild* if i and $\overline{i}$ are w-comparable and *w-tame* if otherwise. $i \in [2n]$ is said a *w-wild head* (resp., a *w-tame head*), if i is w-wild (resp., w-tame) with $(\overline{i})w < (i)w$.

It is easily seen that $i < j$ in $[2n+1]$ are w-uncomparable if and only if $(i)w < (j)w < (i)w + 2n + 1$.

Lemma 3.2 *Fix* $w \in \widetilde{C}_n$.

(i) *For any* $j \neq k$ *in* $[2n]$, $j \prec_w k$ *if and only if* $\overline{k} \prec_w \overline{j}$.

Now suppose that $j \neq k$ *in* $[2n]$ *are* w*-wild heads and* $i \in [2n]$ *is a* w*-tame head.*

(ii) $\overline{j} \prec_w k$ *if and only if* $j, \overline{k}$ *are* w*-comparable.*

(iii) *If* $j, \overline{k}$ *are* w*-uncomparable then so are* j, k (*resp.*, $\overline{j}, \overline{k}$).

(iv) i *and* k *are* w*-comparable if and only if* $i \prec_w k$.

(v) $j, i, \overline{j}$ *is a* w*-chain if and only if* j *is* w*-comparable with both* i *and* $\overline{i}$.

(vi) $\{j, k, \overline{j}, \overline{k}\}$ *is a* w*-chain if and only if* j, k *are* w*-comparable.*

(vii) *If* $(j)w > 2n+1$, *then* $\overline{j} \prec_w 2n+1 \prec_w j$; *or else*, $j, \overline{j}$ *are* w*-uncomparable with* $2n+1$.

(viii) $i, \overline{i}$ *are* w-uncomparable *with* $2n+1$.

Proof The results (i)–(vi) follow by [2, Lemma 3.2] and (vii)–(viii) can be checked directly.

3.3 Let

$$t'_k = \begin{cases} t_{\langle k\rangle}, & \text{if } \langle k\rangle \in [n], \\ t_{\overline{\langle k\rangle}-1}, & \text{if } \langle k\rangle \in [2n] \setminus [n], \\ 1, & \text{if } \langle k\rangle = 2n+1, \end{cases}$$

and

$$t_{i,j} = t'_{i+j-1} t'_{i+j-2} \cdots t'_{i+1} t'_i. \tag{3.3.1}$$

for any $i, j, k \in \mathbb{Z}$ with $j > 0$. Suppose $x \in \widetilde{C}_n$ and $i \in \mathbb{Z}$ satisfy $(i)x - 2n - 1 > (j)x$ for any $i < j \leqslant i + a$ with some $a \in [2n]$. Let $x' = t_{i,a}x$. Then $\ell(x') = \ell(x) - \ell(t_{i,a})$ and $\psi(x) = \psi(x')$. Moreover, if $(i)x - 2n - 1 > (j)x$ for any $i < j < i + 2n + 1$, let $x'' = t_{i,2n+1}x$, then

$$(m)x'' = \begin{cases} (m)x - 2n - 1, & \text{if } \langle m\rangle = \langle i\rangle, \\ (m)x + 2n + 1, & \text{if } \langle m\rangle = \langle 2n+1-i\rangle, \\ (m)x, & \text{otherwise}, \end{cases}$$

for any $m \in \mathbb{Z}$, where x'' satisfies $\ell(x'') = \ell(x) - 2n - 1$ and $\psi(x) = \psi(x'')$.

Fix $w \in \widetilde{C}_n$. Suppose that $E_1 = \{i_1, i_2, \cdots, i_a\}$ and $E_2 = \{j_1, j_2, \cdots, j_b\}$ are two subsets of $[2n]$ satisfying that

(i) $i_1 < i_2 < \cdots < i_a$ and $j_1 < j_2 < \cdots < j_b$ with $a > 0$, $b \geqslant 0$ and $a + b = n$;

(ii) the elements of $E_1 \cup E_2$ are pairwise not $2n$-dual;

(iii) $(\overline{k})w < (k)w$ for any $k \in E_1 \cup E_2$;

(iv) $(i)w - (j)w > l(2n+1)$ for any $i \in E_1$ and $j \in E_2 \cup \{2n+1\}$, where l is nonnegative integer.

By repeatedly left-multiplying the elements $t_{i,j}, i, j \in \mathbb{Z}$, to w, we can obtain some $w' \in \widetilde{C}_n$ such that there are some $1 \leqslant k_1 < k_2 < \cdots < k_b \leqslant 2b$ satisfying that

(1) $\ell(w') = \ell(w) - \ell(ww'^{-1})$;

(2) if $b > 0$, then $[2b] = \{k_1, k_2, \cdots, k_b, 2b+1-k_1, 2b+1-k_2, \cdots, 2b+1-k_b\}$ and the map $\phi : \{j_1, j_2, \cdots, j_b, \overline{j}_1, \overline{j}_2, \cdots, \overline{j}_b\} \longrightarrow [2b]$ given by $\phi(j_m) = k_m$ and $\phi(\overline{j}_m) = 2b+1-k_m$ for $m \in [b]$ is an order-preserving bijection.

(3) $(p)w' = (i_p)w - l'(2n+1)$ and $(a+k_q)w' = (j_q)w$ for any $p \in [a]$ and $q \in [b]$, where l' is some integer $\geqslant l$;

(4) $(\overline{c})w' < (c)w'$ for any $c \in [a] \cup \{a + k_m \mid m \in [b]\}$;

(5) if $b > 0$, then $0 < \min\{(c)w' - (a+k_m)w' \mid c \in [a], m \in [b]\} \leqslant 2n+1$.

We see by Lemma 3.2 that $\psi(w') = \psi(w)$, where $\lambda = \psi(w') = \psi(w)$, and by Corollary 2.15 that w, w' are in the same left-connected component of E_λ.

Example 3.4 Let $x = [19, 11, 20] \in \widetilde{C}_3$. Then $E_1 = \{1, 3\}$ and $E_2 = \{2\}$ satisfy the conditions (i)–(iv) in 3.3 with $n = 3$ and $(a, b, l) = (2, 1, 1)$. Let $y = t_{2,5}t_{3,4}x$. Then $y = [12, 13, 11] \in \widetilde{C}_3$. Hence y satisfies the conditions (1)–(5) in 3.3 and $\psi(x) = \psi(y) = 52$.

3.5 By a *composition* of $2n+1$, we mean an r-tuple $(a_1, a_2, \cdots, a_r)$ of positive integers $a_1, a_2, \cdots, a_r$ with some $r \geqslant 1$ such that $\sum\limits_{i=1}^{r} a_i = 2n+1$. Let $\widetilde{\Lambda}_{2n+1}$ be the set of all compositions of $2n+1$. Clearly, $\Lambda_{2n+1} \subseteq \widetilde{\Lambda}_{2n+1}$.

A generalized tabloid Y of rank $2n+1$ is, by definition, an r-tuple $T = (T_1, T_2, \cdots, T_r)$ with some $r \in \mathbb{N}$ such that $[2n+1]$ is a disjoint union of its non-empty subsets $T_j, j \in [r]$. We have $\xi(T) := (|T_1|, |T_2|, \cdots, |T_r|) \in \widetilde{\Lambda}_{2n+1}$, where $|T_i|$ denotes the cardinal of the set T_i. Let $i_1, i_2, \cdots, i_r$ be a permutation of $1, 2, \cdots, r$ such that $|T_{i_1}| > |T_{i_2}| > \cdots > |T_{i_r}|$. Then $\zeta(T) := (|T_{i_1}|, |T_{i_2}|, \cdots, |T_{i_r}|) \in \Lambda_{2n+1}$. Two generalized tabloids $T = (T_1, T_2, \cdots, T_r)$ and $T' = (T'_1, T'_2, \cdots, T'_r)$ of $2n+1$ are said *equal*, if $r = t$ and $T_i = T'_i$ for any $i \in [r]$. Let $\mathcal{C}_{2n+1}$ be the set of all generalized tabloids of rank $2n+1$. Then both $\xi : \mathcal{C}_{2n+1} \longrightarrow \widetilde{\Lambda}_{2n+1}$ and $\zeta : \mathcal{C}_{2n+1} \longrightarrow \Lambda_{2n+1}$ are surjective maps.

Let Ω be the set of all elements w of $\widetilde{A}_{2n}$ such that there is a generalized tabloid $T = (T_1, T_2, \cdots, T_r) \in \mathcal{C}_{2n+1}$ satisfying:

(i) For any $i < j$ in $[r]$, we have $a \prec_w b$ for any $a \in T_i$ and $b \in T_j$;

(ii) For any $i \in [r]$, a and b are w-uncomparable for any $a \neq b$ in T_i.

Clearly, T is determined entirely by $w \in \Omega$, denote T by $T(w)$. The map $T : \Omega \longrightarrow \mathcal{C}_{2n+1}$ is surjective by [4, Proposition 19.1.2]. By a result of Curtis Greene in [8], we see that the partition $\zeta(T(w))$ is the dual of $\psi(w)$.

The following known result will be crucial in the proof of Lemmas 4.4 and 5.4.

Lemma 3.6(see [4, Lemma 19.4.6]) *Suppose that $y, w \in \widetilde{A}_{2n}$ are two elements in Ω with $\xi(T(y)) = \xi(T(w))$. Then $y \underset{L}{\sim} w$ if and only if $T(y) = T(w)$.*

4 The set $E_{\mathbf{k1^{2n+1-k}}}$

Recall that in 2.13 we defined the set E_λ for any $\lambda \in \Lambda_{2n+1}$. We have $E_\lambda^{-1} = E_\lambda$. In the present section, we shall describe all the cells of $\widetilde{C}_n$ in the set $E_{\mathbf{k1^{2n+1-k}}}$ for all $k \in [2n+1]$. Evidently, $E_{\mathbf{1^{2n+1}}}$ consists of the identity element of $\widetilde{C}_n$. Hence in the subsequent discussion of this section, we shall always assume $k > 1$.

4.1 Assume $k = 2m+1$ odd (resp., $k = 2m$ even) and $l = n - m$. Then $2n+1-k = 2l$ (resp., $2n+1-k = 2l+1$). By Lemma 3.2, we see that $w \in \widetilde{C}_n$ is in the set $E_{\mathbf{k1^{2n+1-k}}}$, $k = 2m+1$ (resp., $k = 2m$) if and only if w satisfies the conditions (4.1.1) (i), (ii) (resp., the conditions (4.1.1) (i), (ii)$'$) below.

(4.1.1) There exist $i_1, i_2, \cdots, i_l, j_1, j_2, \cdots, j_m \in [2n]$ which are pairwise not $2n$-dual such that

(i) $i_1, i_2, \cdots, i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; $j_1, j_2, \cdots, j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$;

(ii) $2n+1 \prec_w j_1$ or $i_1, \bar{i}_1 \prec_w j_1$ (resp., (ii)$'$ $2n+1 \not\prec_w j_1$ and either $i_1 \not\prec_w j_1$ or $\bar{i}_1 \not\prec_w j_1$).

Let F_1 (resp., F_2) be the set of all $w \in \widetilde{C}_n$ satisfying the conditions (4.1.2) (i), (ii), (iii) (resp., the conditions (4.1.2) (i), (ii) (iii)$'$).

(4.1.2) There exist $i_1, i_2, \cdots, i_l, j_1, j_2, \cdots, j_m \in [2n]$ which are pairwise not $2n$-dual such that

(i) $i_1, i_2, \cdots, i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; $j_1, j_2, \cdots, j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$;

(ii) $(\bar{i}_l, \bar{i}_{l-1}, \cdots, \bar{i}_1, j_m, j_{m-1}, \cdots, j_1) = (1, 2, \cdots, n)$;

(iii) $3n+1 < (j_1)w < 5n+3$ and $0 < (j_{a+1})w - (j_a)w < 2n+1$, for any $a \in [m-1]$ (resp., (iii)$'$ $n < (j_1)w < (i_1)w$, $4n+2 < (j_2)w < 6n+3$ and $0 < (j_{a+2})w - (j_{a+1})w < 2n+1$, for any $a \in [m-2]$).

By 3.3 and 4.1, it is easily seen that

Lemma 4.2 *For any $w \in E_{k1^{2n+1-k}}, k = 2m+1$ (resp., $k = 2m$), there exists some $w' \in F_1$ (resp., $w' \in F_2$) such that w', w are in the same left-connected component of $E_{k1^{2n+1-k}}, k = 2m+1$* (resp., $k = 2m$).

Lemma 4.3 *The set F_1 (resp., F_2) in Lemma 4.2 is contained in a right-connected component of $E_{k1^{2n+1-k}}, k = 2m+1$ (resp., $k = 2m$).*

Proof Let $w_J = [1, 2, \cdots, l, 2n-l, 2n-l-1, \cdots, n+1]$ with $J = \{t_{l+1}, t_{l+2}, \cdots, t_n\}$. Then $w_J \in E_{k1^{2n+1-k}}, k = 2m$. Let $w \in F_2$. We have $\mathcal{L}(w) = J$ by Corollary 2.6. Then $w = w_J y$ with $\ell(w) = \ell(w_J) + \ell(y)$ for some $y \in \widetilde{C}_n$. Hence w, w_J are contained in a right-connected component of $E_{k1^{2n+1-k}}, k = 2m$ by Corollary 2.15. This implies that all the elements of F_2 are contained in a right-connected component of $E_{k1^{2n+1-k}}, k = 2m$.

Next we consider $w \in F_1$. Let $x = t_l \cdots t_1 t_0 t_1 \cdots t_n$ and the set $F_1' = \{x^m y \mid y \in F_1\}$. We see that $w' \in \widetilde{C}_n$ is in the set F_1' if and only if w' satisfies the conditions (4.1.2) (i), (ii) and $n < (j_1)w' < 3n+2$, $0 < (j_{a+1})w' - (j_a)w' < 2n+1$, for any $a \in [m-1]$. Then $w_J \in F_1'$ and $x^m w_J \in F_1$. For any $w' \in F_1'$, by Corollary 2.6, we have $\mathcal{L}(w') = J$. Then for any $w' \in F_1'$, $w' = w_J y$ with $\ell(w') = \ell(w_J) + \ell(y)$ for some $y \in \widetilde{C}_n$. We see that $w = x^m w'$ satisfies $\ell(w) = \ell(x^m) + \ell(w')$ for some $w' \in F_1'$. This implies that $w = x^m w_J y$ satisfies $\ell(w) = \ell(x^m w_J) + \ell(y)$ for some $y \in \widetilde{C}_n$. Hence $w, x^m w_J$ are contained in a right-connected component of $E_{k1^{2n+1-k}}, k = 2m+1$ by Corollary 2.15. This implies that all the elements of F_1 are contained in a right-connected component of $E_{k1^{2n+1-k}}, k = 2m+1$.

Lemma 4.4 (1) *No two elements of F_1 (resp., F_2) in Lemma 4.2 are in the same left cell of $\widetilde{C}_n$.*

(2) $|F_1| = n!2^m/(n-m)!$ *and* $|F_2| = n!2^{m-1}/(n-m+1)!$.

Proof In $\widetilde{A}_{2n}$, we have $F_1 \subseteq \Omega$ and $T(F_1) = \{(T_1, T_2, \cdots, T_{2m+1}) \in \mathcal{C}_{2n+1} \mid |T_i| = 1, T_{2m+2-i} = \{\overline{j} \mid j \in T_i\}, i \in [m], 2n+1 \in T_{m+1}\}$ (see 3.5). Then for any $x, y \in F_1$, we see that $\xi(T(x)) = \xi(T(y))$ and $T(x) \neq T(y)$. So for any $x, y \in F_1$, x, y are not in the same left cell of $\widetilde{A}_{2n}$ by Lemma 3.6. Hence no two elements of F_1 are in the same left cell of $\widetilde{C}_n$ by Lemma 2.11 and $|F_1| = |T(F_1)| = n!2^m/(n-m)!$.

In $\widetilde{A}_{2n}$, let $z = (s_n \cdots s_1 s_0 s_{2n} s_{2n-1} \cdots s_{n+m})^m$ and $F_2' = \{zw \mid w \in F_2\}$. Let $w \in F_2$, keep the notation in (4.1.2) (i) (ii) (iii)$'$. We see that $(j)zw = (j)w, j \in [2n+1] \setminus \{\overline{j}_a \mid a \in [m]\}$ and $(\overline{j}_a)zw = (\overline{j}_a)w - 2n - 1$, for any $a \in [m]$. This implies that $\psi(zw) = \psi(w)$. Then we have $zw \underset{L}{\sim} w$ by Lemma 2.14 (1). We see that $F_2' \subseteq \Omega$. And for $w' \in F_2'$, $T(w') = (T_1, T_2, \cdots, T_{2m})$, where $T_b = \{\langle(\overline{j}_{m+1-b})w'\rangle\}$, for any $b \in [m]$, $T_{m+1} = \{\langle(j_1)w'\rangle, 2n+1, \langle(\overline{i}_a)w'\rangle, \langle(i_a)w'\rangle \mid a \in [l]\}$, $T_c = \{\langle(j_{c-m})w'\rangle\}$, for $m+2 \leqslant c \leqslant 2m$. For any $x, y \in F_2'$, we see that $\xi(T(x)) = \xi(T(y))$ and $T(x) \neq T(y)$. So for any $x, y \in F_2'$, x, y are not in the same left cell of $\widetilde{A}_{2n}$ by Lemma 3.6. Hence no two elements of F_2 are in the same left cell of $\widetilde{C}_n$ by Lemma 2.11 and $|F_2| = |F_2'| = |T(F_2')| = n!2^{m-1}/(n-m+1)!$.

Theorem 4.5 (1) $E_{k1^{2n+1-k}}, k = 2m+1$ (*resp.*, $k = 2m$) *is a two-sided cell of* $\widetilde{C}_n$ *containing* $n!2^m/(n-m)!$ (*resp.*, $n!2^{m-1}/(n-m+1)!$) *left cells and each left cell of* $\widetilde{C}_n$ *in* $E_{k1^{2n+1-k}}, k = 2m+1$ (*resp.*, $k = 2m$) *is left-connected.*

(2) $E_{k1^{2n+1-k}}$ *is infinite unless* $k = 1, 2$.

Proof By Lemma 2.14, we see that E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$ for any $\lambda \in \Lambda_{2n+1}$. Hence the assertion (1) follows by Lemmas 4.2–4.4. For the assertion (2), we see that if $k > 2$, then the number of the choices for the integer $(j_m)w$ in the condition (4.1.1) (i) is infinite. On the other hand, we have $E_{1^{2n+1}} = \{1\}$ and $E_{21^{2n-1}} = \{t_n\}$. This proves (2).

5 The set $E_{(2n-1,2)}$

In the present section, we shall describe all the cells of $\widetilde{C}_n$ in the set $E_{(2n-1,2)}$ for $n \geqslant 3$.

Lemma 5.1 $w \in \widetilde{C}_n$ is in the set $E_{(2n-1,2)}$ if and only if w satisfies the condition below.

(5.1.1) There exist $j_1, j_2, \cdots, j_n \in [2n]$ which are pairwise not $2n$-dual such that $j_1, j_2, \cdots, j_n$ are all w-wild heads with $2n+1 \prec_w j_2 \prec_w j_3 \prec_w \cdots \prec_w j_n$ and there is $j_k \in [2n]$ which is w-uncomparable with j_1 for some $k \in [n]$.

Proof Let $w \in E_{(2n-1,2)}$. We claim that for any $j \in [2n]$, j is w-wild. For otherwise, there exists some $i \in [2n]$ which is w-tame, then by Lemma 3.2 (viii), $w \in E_\lambda$ with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ and $r \geqslant 3$, which is a contradiction. If X, Y are both w-chains with $|X| = 2n-1$, $|Y| = 2$ and $[2n+1] = X \cup Y$, we claim that $2n+1 \in X$. Otherwise, let $Y = \{2n+1, j\}$ for some $j \in [2n]$. Then $\overline{j} \in X$, and $\{j\} \cup X$ is a w-chain by Lemma 3.2 (i) (vi), which is a contradiction. Hence the result follows.

Hence by Lemma 3.2, we see that $w \in \widetilde{C}_n$ is in the set $E_{(2n-1,2)}$ if and only if w satisfies the condition (5.1.1).

Let $x_m = (t_m, \cdots, t_{n-1}t_nt_{n-1}, \cdots, t_1t_0)^m$ for $m \in [n]$ and $F = \{w \in \widetilde{C}_n \mid 3n+1 < (n-1)w < 5n+3,\ (n-1)w < (n)w < (n-1)w+2n+1,\ (n)w < (n-2)w < (n)w+2n+1,\ (i+1)w < (i)w < (i+1)w+2n+1,\ i \in [n-3]\}$.

Lemma 5.2 *For any $w \in E_{(2n-1,2)}$, there exists some $w' \in F$ such that w', w are in the same left-connected component of $E_{(2n-1,2)}$.*

Proof By Lemma 5.1, we know that $w \in E_{(2n-1,2)}$ if and only if w satisfies the condition (5.1.1). Let $w \in E_{(2n-1,2)}$, keep the notation in (5.1.1). Let k is the smallest integer of $[n]$ such that j_k is w-uncomparable with j_1. Then $j_{k-1} \prec_w j_1$ if $k > 2$.

By repeatedly left-multiplying the elements $t_i, i \in \mathbb{Z}$, to w, we can obtain some $w^{(1)} \in E_{(2n-1,2)}$ satisfying $(j_n^{(1)}, \cdots, j_{k+1}^{(1)}, j_1^{(1)}, j_k^{(1)}, j_{k-1}^{(1)}, \cdots, j_2^{(1)}) = (1, 2, \cdots, n)$ if $(j_k)w > (j_1)w$, or $(j_n^{(1)}, \cdots, j_{k+1}^{(1)}, j_k^{(1)}, j_1^{(1)}, j_{k-1}^{(1)}, \cdots, j_2^{(1)}) = (1, 2, \cdots, n)$ if $(j_1)w > (j_k)w$ and $(j_m^{(1)})w^{(1)} = (j_m)w$ for $m \in [n]$. We see that $j_{k-1}^{(1)} \prec_w j_1^{(1)}$ if $k > 2$ and $j_1^{(1)} \prec_w j_{k+1}^{(1)}$ if $(j_k)w > (j_1)w$.

If $(j_1^{(1)})w^{(1)} < 2n+1$, then $(j_m^{(1)})x_n w^{(1)} = (j_m^{(1)})w^{(1)} + 2n + 1$ for $m \in [n]$. We see that $x_n w^{(1)}, w^{(1)}$ are in the same left-connected component of $E_{(2n-1,2)}$. Hence without loss of generality, let $(j_1^{(1)})w^{(1)} > 2n+1$.

Let $w^{(2)} = x_{n-k}^i w^{(1)}$ for some $i \in \mathbb{N}$ such that $(j_{k+1}^{(1)})w^{(2)} > (j_1^{(1)})w^{(2)}$. Let $w^{(3)} = t_{n+k-2,2n+1}^m w^{(2)}$ for some $m \in \mathbb{N}$ such that $(n-k+3)w^{(3)} < (n-k+2)w^{(3)} < (n-k+3)w^{(3)} + 2n+1$. Let $w^{(4)} = t_{n-k+1} w^{(3)}$ if $(n-k+3)w^{(3)} > (n-k+1)w^{(3)}$ and $w^{(4)} = x_{n-k+1} t_{n-k+2} w^{(3)}$ if $(n-k+3)w^{(3)} < (n-k+1)w^{(3)}$.

Fig. 2 displays the matrix forms of $w^{(i)}$ for $i = 1, 2, 3, 4$, where the symbol $\overset{p}{\cdot\cdot\cdot}$ stands for a rectangular submatrix with p rows for some $p \in [n]$ each row has a unique non-zero entry 1 Well, the non-zero entries of the matrix are gonging down to the left.

We can see that $w^{(i)}, i = 1, 2, 3, 4$ are in the same left-connected component of $E_{(2n-1,2)}$. By repeatedly repeating the above process, we see that there exists some $w^{(r)} \in \widetilde{C}_n$ for some $r \in \mathbb{Z}$ with the matrix form displayed in Fig. 3, such that $w^{(1)}, w^{(r)}$ are in the same left-connected component of $E_{(2n-1,2)}$.

Hence from the Fig. 3, we see that there exists some $w' \in F$ such that $w', w^{(r)}, w$ are in the same left-connected component of $E_{(2n-1,2)}$ by 3.3.

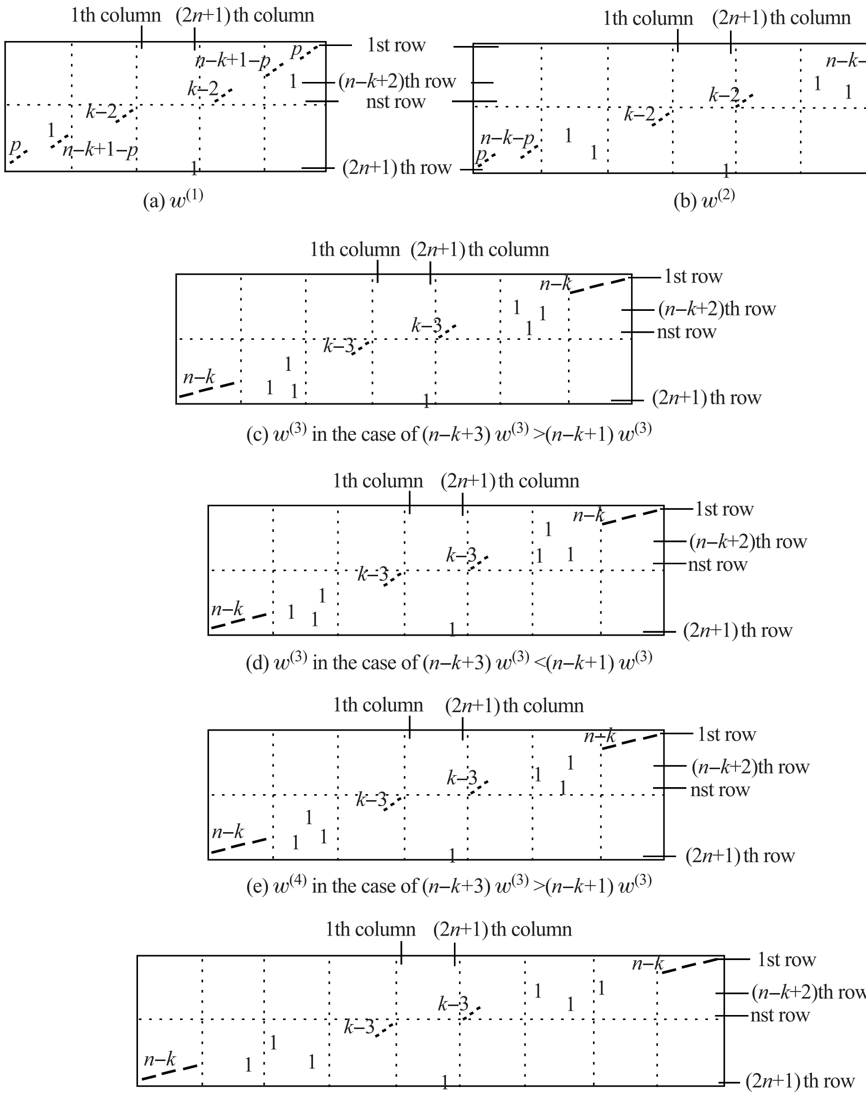
Lemma 5.3 *The set F is contained in a right-connected component of $E_{(2n-1,2)}$.*

Proof Let $w_J = [2n, \cdots, n+2, n+1]$ with $J = \{t_1, t_2, \cdots, t_n\}$ and $F' = \{x_n t_{n-1} w \mid w \in F\}$. Then $F' = \{w \in \widetilde{C}_n \mid n < (n)w < 3n+2,\ (i+1)w < (i)w < (i+1)w + 2n+1,\ i \in [n-1]\}$ and $w_J \in F'$. For any $w' \in F'$, by Corollary 2.6, we have $\mathcal{L}(w') = J$. Then for any $w' \in F'$, $w' = w_J y$ with $\ell(w') = \ell(w_J) + \ell(y)$ for some $y \in \widetilde{C}_n$. Let $w \in F$. Then $w = t_{n-1} x_n^{-1} w_J y$ with $\ell(w) = \ell(t_{n-1} x_n^{-1} w_J) + \ell(y)$ for some $y \in \widetilde{C}_n$. This implies that $t_{n-1} x_n^{-1} w_J, w$ are contained in a right-connected component of $E_{(2n-1,2)}$ by Corollary 2.15. Hence all the elements of F are contained in a right-connected component of $E_{(2n-1,2)}$.

Lemma 5.4 (1) $|F| = n! 2^{n-2}$.

(2) *No two elements of F are in the same left cell of $\widetilde{C}_n$.*

Proof In $\widetilde{A}_{2n}$, we have $F \subseteq \Omega$ and $T(F) = \{(T_1, T_2, \cdots, T_{2n-1}) \in \mathcal{C}_{2n+1} \mid |T_i| = 1$ with $i \in [n-2]$, $|T_{n-1}| = 2, T_n = \{2n+1\}, T_{2n-m} = \{\overline{j} \mid j \in T_i\}$ with $m \in [n-1]\}$ (see 3.5). Then for any $x, y \in F$, we see that $\xi(T(x)) = \xi(T(y))$ and $T(x) \neq T(y)$. So for any $x, y \in F$, x, y are not in the same left cell of $\widetilde{A}_{2n}$ by Lemma 3.6 and $|F| = |T(F)| = n! 2^{n-2}$. Hence no two elements of F are in the same left cell of $\widetilde{C}_n$ by Lemma 2.11.



(a) $w^{(1)}$ (b) $w^{(2)}$

(c) $w^{(3)}$ in the case of $(n-k+3)\,w^{(3)} > (n-k+1)\,w^{(3)}$

(d) $w^{(3)}$ in the case of $(n-k+3)\,w^{(3)} < (n-k+1)\,w^{(3)}$

(e) $w^{(4)}$ in the case of $(n-k+3)\,w^{(3)} > (n-k+1)\,w^{(3)}$

(f) $w^{(4)}$ in the case of $(n-k+3)\,w^{(3)} < (n-k+1)\,w^{(3)}$

Fig. 2 The matrix forms of $w^{(i)}$ for $i = 1, 2, 3, 4$

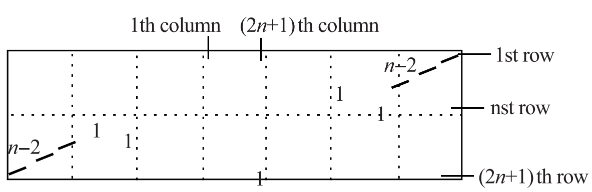


Fig. 3 The matrix form of $w^{(r)}$

Theorem 5.5 (1) $E_{(2n-1,2)}$ *is a two-sided cell of* $\widetilde{C}_n$ *containing* $n!2^{n-2}$ *left cells and each left cell of* $\widetilde{C}_n$ *in* $E_{(2n-1,2)}$ *is left-connected.*

(2) $E_{(2n-1,2)}$ *is infinite.*

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Proof The assertion (1) follows by Lemmas 5.2–5.4. For the assertion (2), we see that the number of the choices for the integer $(j_n)w$ in the condition (5.1.1) is infinite. This proves (2).

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