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参量化的 Hardy 型积分不等式

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摘要: 应用权函数与实分析方法,引入两对共轭指数与一个独立参数,建立若干推广的 Hardy 型积分不等式,并证明其常数因子为最佳值;考虑等价式的情形及一些特殊结果,包括若干基本的不含参数的 Hardy 型积分不等式.

关键词: 权函数;Hardy 型积分不等式;最佳值;等价式

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Hardy-Type Integral Inequalities with Multiple Parameters

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Abstract: Using the method of weight function and the technique of real analysis, and introducing two pairs of conjugate exponents and an independent parameter, some extended Hardy-type integral inequalities are given. The equivalent forms and some particular results including the basic Hardy-type integral inequalities without a parameter are considered.

Key words: weight function; Hardy-type integral inequality; best value; equivalent form

设 $L^2(0,\infty)$ 为实空间,$f,g\in L^2(0,\infty)$,且 $0<\int_0^\infty f^2(t)\,\mathrm{d}t<\infty$,$0<\int_0^\infty g^2(t)\,\mathrm{d}t<\infty$,有如下经典的 Hilbert 积分不等式[1]成立:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi\left(\int_0^\infty f^2(t)\,\mathrm{d}t\int_0^\infty g^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}}, \tag{1}$$

式中,常数因子 π 为最佳值.

1925 年,Hardy[2]引入一对共轭指数 (p,q),即 $\frac{1}{p}+\frac{1}{q}=1$,推广式(1)为

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\pi/p)}\cdot\left(\int_0^\infty f^p(t)\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(t)\,\mathrm{d}t\right)^{\frac{1}{q}}, \tag{2}$$

式中,常数因子 $\frac{\pi}{\sin(\pi/p)}(p>1)$ 为最佳值.

Hardy[1]还建立了式(2)的如下等价式:

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x\right)^p\mathrm{d}y < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\int_0^\infty f^p(t)\,\mathrm{d}t. \tag{3}$$

适当变化式(3)的核及积分区间, Hardy[3]建立了如下具有最佳常数因子的 Hardy 积分不等式:

$$\int_0^\infty\left(\frac{1}{y}\int_0^y f(x)\,\mathrm{d}x\right)^p\mathrm{d}y < q^p\int_0^\infty f^p(t)\,\mathrm{d}t, \tag{4}$$

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$$\int_0^\infty\left(\int_x^\infty \frac{1}{y}g(y)\,\mathrm{d}y\right)^q\mathrm{d}x < q^q\int_0^\infty g^q(t)\,\mathrm{d}t. \tag{5}$$

式(1)~(5)自诞生之日起至20世纪90年代末,因在分析学及相关领域应用甚广而倍受瞩目[4],然其自身却无大的推广变化. 1998年,Yang[5-6]引入独立参数 $\lambda>0$ 及 Beta 函数,推广式(1)为

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\cdot \left(\int_0^\infty t^{1-\lambda}f^2(t)\,\mathrm{d}t\int_0^\infty t^{1-\lambda}g^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}}, \tag{6}$$

式中,常数因子 $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ 为最佳值. 2004—2005年,杨必成[7-8]引入两对共轭指数 (p,q),(r,s) 及独立参数 $\lambda>0$,推广式(2)如下:设 $p,r>1$,在右边积分收敛于正数的情况下,有

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\lambda\sin\left(\frac{\pi}{r}\right)}\left\{\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{7}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\left\{\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty t^{q\left(1-\frac{\lambda}{r}\right)-1}g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{8}$$

式中,常数因子 $\frac{\pi}{\lambda\sin\left(\frac{\pi}{r}\right)}$ 及 $B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)$ 均为最佳值.

参考文献[7-8]的参量化思想,本研究拟应用权函数与实分析的方法,建立旨在推广式(4)和(5)的引入两对共轭指数与一个独立参数的积分不等式,并证明其常数因子为最佳值;还考虑其等价式的情形,并导出一些有趣的特殊结果,其中包括若干基本的不含参数的 Hardy 型积分不等式.

定理1　设 $\lambda>0,p>1,r>0(r\neq1),\frac{1}{p}+\frac{1}{q}=1,\frac{1}{r}+\frac{1}{s}=1$, $f(t),g(t)$ 为 $(0,\infty)$ 上的非负可测函数,且 $0<\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,\mathrm{d}t<\infty$, $0<\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}\cdot g^q(t)\,\mathrm{d}t<\infty$,则有如下重积分不等式:

$$I_\lambda=\int_0^\infty\int_0^y \frac{1}{y^\lambda}f(x)g(y)\,\mathrm{d}x\mathrm{d}y=$$

$$\int_0^\infty\int_x^\infty \frac{1}{y^\lambda}f(x)g(y)\,\mathrm{d}y\mathrm{d}x < \frac{r}{\lambda}\left\{\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{9}$$

式中,常数因子 $\frac{r}{\lambda}$ 为最佳值. 特别地,当 $r=p,r=q$ 时,有如下特殊不等式:

$$I_\lambda<\frac{p}{\lambda}\left\{\int_0^\infty t^{p-\lambda-1}f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty t^{q-\lambda-1}g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{10}$$

$$I_\lambda<\frac{q}{\lambda}\left\{\int_0^\infty t^{(p-1)(1-\lambda)}f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty t^{(q-1)(1-\lambda)}g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}. \tag{11}$$

证明　设 $k_\lambda(x,y)=\frac{1}{y^\lambda},x\leqslant y;k_\lambda(x,y)=0,x>y$,配方并由带权的 Hölder 不等式[9],有

$$I_\lambda=\int_0^\infty\int_0^\infty k_\lambda(x,y)\left[\frac{x^{\left(1-\frac{\lambda}{r}\right)/q}}{y^{\left(1-\frac{\lambda}{s}\right)/p}}f(x)\right]\cdot \left[\frac{y^{\left(1-\frac{\lambda}{s}\right)/p}}{x^{\left(1-\frac{\lambda}{r}\right)/q}}g(y)\right]\mathrm{d}x\mathrm{d}y\leqslant$$

$$\left\{\int_0^\infty\int_0^\infty k_\lambda(x,y)\frac{x^{\left(1-\frac{\lambda}{r}\right)(p-1)}}{y^{1-\lambda/s}}f^p(x)\,\mathrm{d}y\mathrm{d}x\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty\int_0^\infty k_\lambda(x,y)\frac{y^{\left(1-\frac{\lambda}{s}\right)(q-1)}}{x^{1-\lambda/r}}g^q(y)\,\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{q}}=$$

$$\left\{\int_0^\infty\left[\int_x^\infty \frac{1}{y^\lambda}\cdot\frac{x^{\left(1-\frac{\lambda}{r}\right)(p-1)}}{y^{1-\lambda/s}}\mathrm{d}y\right]f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}}\cdot \left\{\int_0^\infty\left[\int_0^y \frac{1}{y^\lambda}\cdot\frac{y^{\left(1-\frac{\lambda}{s}\right)(q-1)}}{x^{1-\lambda/r}}\mathrm{d}x\right]g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}=$$

$$\frac{r}{\lambda}\left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q\left(1-\frac{\lambda}{s}\right)-1}g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{12}$$

若式(12)中间取等号,则有不全为零的数 A 与 B[9],使

$$A\frac{x^{\left(1-\frac{\lambda}{r}\right)(p-1)}}{y^{1-\lambda/s}}f^p(x)=B\frac{y^{\left(1-\frac{\lambda}{s}\right)(q-1)}}{x^{1-\lambda/r}}g^q(y),$$

a. e. 于 $(0,\infty)\times(0,\infty)$.

于是有常数 C,使 $Ax^{p\left(1-\frac{\lambda}{r}\right)}f^p(x)=By^{q\left(1-\frac{\lambda}{s}\right)}g^q(y)=C$,a. e. 于 $(0,\infty)$. 设 $A\neq0$,有 $x^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(x)=C/(Ax)$,a. e. 于 $(0,\infty)$,这与 $0<\int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(x)\cdot\mathrm{d}x<\infty$ 构成矛盾,故式(12)取严格不等号,式(9)

成立.

设 $n\in\mathbf{N}$, $f_n(x)=g_n(x)=0, x\in(0,1)$, $f_n(x)=x^{-1+\frac{\lambda}{r}-\frac{1}{np}}$, $g_n(x)=x^{-1+\frac{\lambda}{s}-\frac{1}{nq}}, x\in[1,\infty)$, 若有正常数 $k\left(k\leqslant\frac{r}{\lambda}\right)$ 替代常数因子 $\frac{r}{\lambda}$ 后, 式(9)依然成立, 特别代入 f_n, g_n, 有

$$k=\frac{k}{n}\left\{\int_0^\infty t^{p(1-\frac{\lambda}{r})-1}f_n^p(t)\,dt\right\}^{\frac{1}{p}}\left\{\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}g_n^q(t)\,dt\right\}^{\frac{1}{q}}>$$
$$\frac{1}{n}\int_0^\infty\int_x^\infty\frac{1}{y^\lambda}f_n(x)g_n(y)\,dy\,dx=$$
$$\frac{1}{n}\int_1^\infty x^{-1+\frac{\lambda}{r}-\frac{1}{np}}\left(\int_x^\infty y^{-1-\frac{\lambda}{r}-\frac{1}{nq}}dy\right)dx=$$
$$\frac{1}{n\left(\frac{\lambda}{r}+\frac{1}{nq}\right)}\int_1^\infty x^{-1-\frac{1}{n}}dx=\frac{1}{\frac{\lambda}{r}+\frac{1}{nq}}.\tag{13}$$

式(13)取极限, 令 $n\to\infty$, 由保号性, 有 $k\geqslant\frac{r}{\lambda}$, 故 $k=\frac{r}{\lambda}$ 为式(9)的最佳值, 证毕.

定理 2 设 $\lambda>0, p>1, r>0\,(r\neq1)$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $f(t), g(t)$ 为 $(0,\infty)$ 上的非负可测函数, 且 $0<\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,dt<\infty$, $0<\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}\cdot g^q(t)\,dt<\infty$, 则有如下不等式成立:

$$J_\lambda=\int_0^\infty\frac{1}{y^{\frac{p\lambda}{r}+1}}\left(\int_0^y f(x)\,dx\right)^p dy<\left(\frac{r}{\lambda}\right)^p\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,dt,\tag{14}$$

$$L_\lambda=\int_0^\infty x^{\frac{q\lambda}{r}-1}\left(\int_x^\infty\frac{1}{y^\lambda}g(y)\,dy\right)^q dx<\left(\frac{r}{\lambda}\right)^q\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}g^q(t)\,dt,\tag{15}$$

式中, 常数因子 $\left(\frac{r}{\lambda}\right)^\rho\ (\rho=p,q)$ 均为最佳值, 且式(14), (15)与式(9)等价.

(1) 当 $r=p$ 时, 有如下不等式与式(10)等价:

$$\int_0^\infty\frac{1}{y^{\lambda+1}}\left(\int_0^y f(x)\,dx\right)^p dy<\left(\frac{p}{\lambda}\right)^p\int_0^\infty t^{p-\lambda-1}f^p(t)\,dt,\tag{16}$$

$$\int_0^\infty x^{(q-1)\lambda-1}\left(\int_x^\infty\frac{1}{y^\lambda}g(y)\,dy\right)^q dx<\left(\frac{p}{\lambda}\right)^q\int_0^\infty t^{q-\lambda-1}g^q(t)\,dt.\tag{17}$$

(2) 当 $r=q$ 时, 有如下不等式与式(11)等价:

$$\int_0^\infty\frac{1}{y^{(p-1)\lambda+1}}\left(\int_0^y f(x)\,dx\right)^p dy<\left(\frac{q}{\lambda}\right)^p\int_0^\infty t^{(p-1)(1-\lambda)}f^p(t)\,dt,\tag{18}$$

$$\int_0^\infty x^{\lambda-1}\left(\int_x^\infty\frac{1}{y^\lambda}g(y)\,dy\right)^q dx<\left(\frac{q}{\lambda}\right)^q\int_0^\infty t^{(q-1)(1-\lambda)}g^q(t)\,dt.\tag{19}$$

证明 取 $I_\lambda=\int_0^\infty\frac{1}{y^\lambda}\left(\int_0^y f(x)\,dx\right)g(y)\,dy$, 在 $(0,\infty)$ 上定义有界可测函数 $[f(x)]_n$,

$$[f(x)]_n=\min\{f(x),n\}=\begin{cases}f(x), & f(x)\leqslant n,\\ n, & f(x)>n.\end{cases}$$

由 $0<\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\,dt<\infty$, 存在 $n_0\in\mathbf{N}$, 使当 $n\geqslant n_0$ 时, $0<\int_{\frac{1}{n}}^n x^{p\left(1-\frac{\lambda}{r}\right)-1}[f(x)]_n^p\,dx<\infty$. 令

$$g_n(y)=\frac{1}{y^{\frac{p\lambda}{r}+1}}\left(\int_{\frac{1}{n}}^y[f(x)]_n\,dx\right)^{p-1}>0,$$
$$\frac{1}{n}\leqslant y\leqslant n, n\geqslant n_0,$$

必存在 $a>0$, 使 $[f(x)]_n\leqslant n\leqslant ax^{\frac{\lambda}{r}-1}, x\in\left[\frac{1}{n},n\right]$. 当 $n\geqslant n_0$ 时, 有

$$0<\int_{\frac{1}{n}}^n y^{q\left(1-\frac{\lambda}{s}\right)-1}g_n^q(y)\,dy=\int_{\frac{1}{n}}^n\frac{1}{y^{\frac{p\lambda}{r}+1}}\left(\int_{\frac{1}{n}}^y[f(x)]_n\,dx\right)^p dy\leqslant$$
$$a^p\int_{\frac{1}{n}}^n\frac{1}{y^{\frac{p\lambda}{r}+1}}\left(\int_0^y x^{\frac{\lambda}{r}-1}dx\right)^p dy=a^p\left(\frac{r}{\lambda}\right)^p\int_{\frac{1}{n}}^n y^{-1}dy=$$
$$2a^p\left(\frac{r}{\lambda}\right)^p\ln n<\infty.$$

由式(9), 当 $x\notin\left[\frac{1}{n},n\right]$ 时, $g_n(x)=[f(x)]_n=0$, 有

$$0<\int_{\frac{1}{n}}^n y^{q\left(1-\frac{\lambda}{s}\right)-1}g_n^q(y)\,dy=\int_{\frac{1}{n}}^n\frac{1}{y^{\frac{p\lambda}{r}+1}}\left(\int_{\frac{1}{n}}^y[f(x)]_n\,dx\right)^p dy=$$
$$\int_{\frac{1}{n}}^n\int_{\frac{1}{n}}^y\frac{1}{y^\lambda}[f(x)]_n g_n(y)\,dx\,dy<$$
$$\frac{r}{\lambda}\left\{\int_{\frac{1}{n}}^n x^{p\left(1-\frac{\lambda}{r}\right)-1}[f(x)]_n^p\,dx\right\}^{\frac{1}{p}}\cdot$$
$$\left\{\int_{\frac{1}{n}}^n y^{q\left(1-\frac{\lambda}{s}\right)-1}g_n^q(y)\,dy\right\}^{\frac{1}{q}}<\infty,\tag{20}$$

$$0 < \int_{\frac{1}{n}}^{n} y^{q\left(1-\frac{\lambda}{s}\right)-1} g_n^q(y)\,\mathrm{d}y < \left(\frac{r}{\lambda}\right)^p \cdot \int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(x)\,\mathrm{d}x < \infty. \tag{21}$$

因而 $0 < \int_0^\infty y^{q\left(1-\frac{\lambda}{s}\right)-1} g_\infty^q(y)\,\mathrm{d}y = J_\lambda < \infty$. 由条件 $0 < \int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(t)\,\mathrm{d}t < \infty$，当 $n\to\infty$ 时，式(9)，(20)，(21)保持严格不等号，故式(14)成立.

反之，设式(14)成立，由 Hölder 不等式[9]，有

$$I_\lambda = \int_0^\infty \left(y^{\frac{\lambda}{s}-\frac{1}{p}} \int_0^y \frac{1}{y^\lambda} f(x)\,\mathrm{d}x\right)\left(y^{\frac{1}{p}-\frac{\lambda}{s}} g(y)\right)\mathrm{d}y \leqslant J_\lambda^{\frac{1}{p}} \left\{\int_0^\infty y^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{22}$$

再由式(14)，得式(9)，故式(14)与式(9)等价.

式(14)的常数因子必是最佳值. 不然，由式(22)，易得式(9)的常数因子也不是最佳值，矛盾.

同法，取 $I_\lambda = \int_0^\infty \left(\int_x^\infty \frac{1}{y^\lambda} g(y)\,\mathrm{d}y\right) f(x)\,\mathrm{d}x$，定义 $[g(y)]_n$，$f_n(x) = x^{\lambda-1}\left(\int_x^n \frac{1}{y^\lambda}[g(y)]_n \mathrm{d}y\right)^{q-1}$，建立不等式 $I_\lambda \leqslant \left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}} L_\lambda^{\frac{1}{q}}$，可证式(15)成立及其等价性、最佳值等结论. 证毕.

若在定理 1，定理 2 的证明中，取 $r=1$，并默认 $\frac{1}{s}=0$，则有

推论 1 设 $\lambda>0$，$p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$f(t)$，$g(t)$ 为 $(0,\infty)$ 上的非负可测函数，且 $0 < \int_0^\infty t^{p(1-\lambda)-1} f^p(t)\,\mathrm{d}t < \infty$，$0 < \int_0^\infty t^{q-1} g^q(t)\,\mathrm{d}t < \infty$，则有如下等价不等式:

$$I_\lambda < \frac{1}{\lambda}\left\{\int_0^\infty t^{p(1-\lambda)-1} f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}} \left\{\int_0^\infty t^{q-1} g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{23}$$

$$\int_0^\infty \frac{1}{y^{p\lambda+1}}\left(\int_0^y f(x)\,\mathrm{d}x\right)^p \mathrm{d}y < \left(\frac{1}{\lambda}\right)^p \int_0^\infty t^{p(1-\lambda)-1} f^p(t)\,\mathrm{d}t, \tag{24}$$

$$\int_0^\infty x^{q\lambda-1}\left(\int_x^\infty \frac{1}{y^\lambda} g(y)\,\mathrm{d}y\right)^q \mathrm{d}x < \left(\frac{1}{\lambda}\right)^q \int_0^\infty t^{q-1} g^q(t)\,\mathrm{d}t, \tag{25}$$

式中，常数因子 $\frac{1}{\lambda}$，$\left(\frac{1}{\lambda}\right)^\rho$ $(\rho=p,q)$ 均为最佳值.

若在定理 1，定理 2 的证明中，改设 $k_\lambda(x,y)=\frac{1}{y^\lambda}$，$x\geqslant y$；$k_\lambda(x,y)=0$，$x<y$，$r<0$，则有如下定理.

定理 3 设 $\lambda>0$，$p>1$，$r<0$，$\frac{1}{p}+\frac{1}{q}=1$，$\frac{1}{r}+\frac{1}{s}=1$，$f(t)$，$g(t)$ 为 $(0,\infty)$ 上的非负可测函数，且 $0 < \int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(t)\,\mathrm{d}t < \infty$，$0 < \int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(t)\,\mathrm{d}t < \infty$，则有如下等价不等式:

$$\tilde{I}_\lambda = \int_0^\infty \int_y^\infty \frac{1}{y^\lambda} f(x)g(y)\,\mathrm{d}x\mathrm{d}y = \int_0^\infty \int_0^x \frac{1}{y^\lambda} f(x)g(y)\,\mathrm{d}y\mathrm{d}x < \left(\frac{-r}{\lambda}\right)\left\{\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}} \left\{\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{26}$$

$$\int_0^\infty \frac{1}{y^{\frac{p\lambda}{r}+1}}\left(\int_y^\infty f(x)\,\mathrm{d}x\right)^p \mathrm{d}y < \left(\frac{-r}{\lambda}\right)^p \int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(t)\,\mathrm{d}t, \tag{27}$$

$$\int_0^\infty x^{\frac{q\lambda}{r}-1}\left(\int_0^x \frac{1}{y^\lambda} g(y)\,\mathrm{d}y\right)^q \mathrm{d}x < \left(\frac{-r}{\lambda}\right)^q \int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(t)\,\mathrm{d}t, \tag{28}$$

式中，常数因子 $\left(\frac{-r}{\lambda}\right)$，$\left(\frac{-r}{\lambda}\right)^\rho$ $(\rho=p,q)$ 均为最佳值.

特别地，当 $r=-1$，$s=\frac{1}{2}$ 时，有如下等价式:

$$\tilde{I}_\lambda < \frac{1}{\lambda}\left\{\int_0^\infty t^{p(1+\lambda)-1} f^p(t)\,\mathrm{d}t\right\}^{\frac{1}{p}} \left\{\int_0^\infty t^{q(1-2\lambda)-1} g^q(t)\,\mathrm{d}t\right\}^{\frac{1}{q}}, \tag{29}$$

$$\int_0^\infty \frac{1}{y^{1-p\lambda}}\left(\int_y^\infty f(x)\,\mathrm{d}x\right)^p \mathrm{d}y < \left(\frac{1}{\lambda}\right)^p \int_0^\infty t^{p(1+\lambda)-1} f^p(t)\,\mathrm{d}t, \tag{30}$$

$$\int_0^\infty \frac{1}{x^{1+q\lambda}}\left(\int_0^x \frac{1}{y^\lambda} g(y)\,\mathrm{d}y\right)^q \mathrm{d}x < \left(\frac{1}{\lambda}\right)^q \int_0^\infty t^{q(1-2\lambda)-1} g^q(t)\,\mathrm{d}t. \tag{31}$$

(1) 当 $\lambda=1$ 时，式(18)和(19)分别变为式(4)和(5)；式(16)和(17)分别变为如下等价不等式:

$$\int_0^\infty \frac{1}{y^2}\left(\int_0^y f(x)\,\mathrm{d}x\right)^p \mathrm{d}y < p^p \int_0^\infty t^{p-2} f^p(t)\,\mathrm{d}t, \tag{32}$$

$$\int_0^\infty x^{q-2}\left(\int_x^\infty \frac{1}{y} g(y)\,\mathrm{d}y\right)^q \mathrm{d}x < p^q \int_0^\infty t^{q-2} g^q(t)\,\mathrm{d}t; \tag{33}$$

式(24)和(25)分别变为如下等价不等式:

$$\int_0^\infty \frac{1}{y^{p+1}}\left(\int_0^y f(x)\,\mathrm{d}x\right)^p \mathrm{d}y < \int_0^\infty \frac{1}{t} f^p(t)\,\mathrm{d}t, \tag{34}$$

$$\int_0^\infty x^{q-1}\left(\int_x^\infty \frac{1}{y}g(y)\mathrm{d}y\right)^q \mathrm{d}x < \int_0^\infty t^{q-1}g^q(t)\mathrm{d}t;\tag{35}$$

式(30)和(31)分别变为如下等价不等式：

$$\int_0^\infty y^{p-1}\left(\int_y^\infty f(x)\mathrm{d}x\right)^p \mathrm{d}y < \int_0^\infty t^{2p-1}f^p(t)\mathrm{d}t,\tag{36}$$

$$\int_0^\infty \frac{1}{x^{q+1}}\left(\int_0^x \frac{1}{y}g(y)\mathrm{d}y\right)^q \mathrm{d}x < \int_0^\infty \frac{1}{t^{q+1}}g^q(t)\mathrm{d}t.\tag{37}$$

（2）当 $p=q=2,\lambda=1$ 时，从以上结果可导出如下3组等价的、基本的 Hardy 型积分不等式：

$$I_1 < 2\left(\int_0^\infty f^2(t)\mathrm{d}t\int_0^\infty g^2(t)\mathrm{d}t\right)^{\frac{1}{2}},\tag{38}$$

$$\int_0^\infty \left(\frac{1}{y}\int_0^y f(x)\mathrm{d}x\right)^2 \mathrm{d}y < 4\int_0^\infty f^2(t)\mathrm{d}t,\tag{39}$$

$$\int_0^\infty \left(\int_x^\infty \frac{1}{y}g(y)\mathrm{d}y\right)^2 \mathrm{d}x < 4\int_0^\infty g^2(t)\mathrm{d}t;\tag{40}$$

$$I_1 < \left(\int_0^\infty \frac{1}{t}f^2(t)\mathrm{d}t\int_0^\infty tg^2(t)\mathrm{d}t\right)^{\frac{1}{2}},\tag{41}$$

$$\int_0^\infty \frac{1}{y^3}\left(\int_0^y f(x)\mathrm{d}x\right)^2 \mathrm{d}y < \int_0^\infty \frac{1}{t}f^2(t)\mathrm{d}t,\tag{42}$$

$$\int_0^\infty x\left(\int_x^\infty \frac{1}{y}g(y)\mathrm{d}y\right)^2 \mathrm{d}x < \int_0^\infty tg^2(t)\mathrm{d}t;\tag{43}$$

$$\widetilde{I}_1 < \left(\int_0^\infty t^3f^2(t)\mathrm{d}t\int_0^\infty \frac{1}{t^3}g^2(t)\mathrm{d}t\right)^{\frac{1}{2}},\tag{44}$$

$$\int_0^\infty y\left(\int_y^\infty f(x)\mathrm{d}x\right)^2 \mathrm{d}y < \int_0^\infty t^3f^2(t)\mathrm{d}t,\tag{45}$$

$$\int_0^\infty \frac{1}{x^3}\left(\int_0^x \frac{1}{y}g(y)\mathrm{d}y\right)^2 \mathrm{d}x < \int_0^\infty \frac{1}{t^3}g^2(t)\mathrm{d}t.\tag{46}$$

不等式(38)～(46)右边的积分都收敛于正数，且不含参数，常数因子皆为最佳值，其中

$$I_1 = \int_0^\infty\int_0^y \frac{1}{y}f(x)g(y)\mathrm{d}x\mathrm{d}y = \int_0^\infty\int_x^\infty \frac{1}{y}f(x)g(y)\mathrm{d}y\mathrm{d}x,$$

$$\widetilde{I}_1 = \int_0^\infty\int_y^\infty \frac{1}{y}f(x)g(y)\mathrm{d}x\mathrm{d}y = \int_0^\infty\int_0^x \frac{1}{y}f(x)g(y)\mathrm{d}y\mathrm{d}x.$$

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