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Sine-Gordon 方程的极限对称及应用

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摘要: 研究 sine-Gordon 方程的极限对称及应用. 由对称引出相似约化,求得 sine-Gordon 方程的极限解;利用对称与孤子方程的自相容源之间的联系,得到带新自相容源的 sine-Gordon 方程,并求出该方程的解.

关键词: 对称;sine-Gordon 方程;相似约化;自相容源;Hirota 方法

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Limit Symmetry of Sine-Gordon Equation and Applications

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Abstract: Limit symmetry of the sine-Gordon equation and its applications are considered. The similarity reduction leads to limit solutions of the sine-Gordon equation. Besides, based on the relationship between symmetries and sources of soliton equations, a sine-Gordon equation with new self-consistent sources is obtained and its solutions are derived.

Key words: symmetries; sine-Gordon equation; similarity reduction; self-consistent sources; Hirota method

众所周知,孤立子解可以由许多不同的方法求得,例如反散射变换、Darboux 变换、代数几何方法、Hirota 方法等. 在反散射变换中,N-孤子解是由谱问题中 N 个不同的特征值,即透射系数$\frac{1}{a(k)}$的 N 个不同的简单极点$\{k_j\}$决定的. 而在对称理论[1]中,经典的 N-孤子解与平方本征函数的对称约束有着密切联系[2]. 对称方法为微分方程的求解提供了强有力的工具[3-4],基于对称的扰动方法已被成功应用于许多扰动方程的求解[3-5]. 另外,利用平方本征函数对称的极限形式,Zhang 等[6]得到了 Korteweg-de Vries (KdV)方程的极限解以及带极限源的 KdV 系统.

本研究将讨论 sine-Gordon 方程的一个新对称. 该对称可以由已知的平方本征函数对称通过一个极限过程得到,而且由相应的对称约束得到的新解是一个二重极点解[7-9],可以看作是方程的极限解. 另外,平方本征函数与孤立子方程的自相容源之间有着紧密关系[10-11]. 新的极限对称引出一个带新自相容源的 sine-Gordon 方程,本研究将利用双线性方法求解这个带源的方程.

1 Sine-Gordon 方程的一个极限对称

Sine-Gordon 方程为

$$u_{xt} = \sin u. \tag{1}$$

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该方程最早来自于负常曲率曲面,可用于描述 Josephson 传输线中的磁通量子[12-13]、共振介质中的超短脉冲传播[14]等,具有丰富的物理与几何背景. Sine-Gordon 方程是可积的,其 Lax 对为

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}-\lambda & \frac{u_x}{2}\\ -\frac{u_x}{2} & \lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{2}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t = \frac{1}{4\lambda}\begin{pmatrix}-\cos u & \sin u\\ \sin u & \cos u\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{3}$$

式中,λ 为谱参数. 可以验证,当 ϕ_1 和 ϕ_2 满足式(2)和(3)时,有

$$\sigma = (\phi_1^2 + \phi_2^2). \tag{4}$$

式(4)为 sine-Gordon 方程的一个对称,即满足 $\sigma_{xt} = \sigma\cos u$. 利用对称所满足的线性方程的线性性质,由式(4)以及方程的另一个对称 u_x,可以得到方程的一个对称约束为

$$u_x = \sum_{j=1}^{N}(\phi_{1j}^2 + \phi_{2j}^2), \tag{5}$$

式中,ϕ_{ij}为 Lax 对当 $\lambda = \lambda_j$ 时的解. 由式(5)可以引出 sine-Gordon 方程的 N-孤子解[15-16].

引入

$$\tilde{\sigma} = \sum_{j=1}^{N}(\phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}), \tag{6}$$

式中,ϕ_{1j}和 ϕ_{2j},ψ_{1j}和 ψ_{2j}满足如下关系:

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}_x = \begin{pmatrix}-\lambda_j & \frac{u_x}{2}\\ -\frac{u_x}{2} & \lambda_j\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}, \tag{7}$$

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}_t = \frac{1}{4\lambda_j}\begin{pmatrix}-\cos u & \sin u\\ \sin u & \cos u\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}, \tag{8}$$

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_x = \begin{pmatrix}-\lambda_j & \frac{u_x}{2}\\ -\frac{u_x}{2} & \lambda_j\end{pmatrix}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}, \tag{9}$$

$$\begin{aligned}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_t = {}& \frac{1}{4\lambda_j}\begin{pmatrix}-\cos u & \sin u\\ \sin u & \cos u\end{pmatrix}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix} - \\ & \frac{1}{4\lambda_j^2}\begin{pmatrix}-\cos u & \sin u\\ \sin u & \cos u\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}.\end{aligned} \tag{10}$$

容易验证,$\tilde{\sigma}$ 满足等式

$$\tilde{\sigma}_{xt} = \tilde{\sigma}\cos u. \tag{11}$$

这表明 $\tilde{\sigma}$ 也是 sine-Gordon 方程的一个对称,式(9)和(10)可以视为式(7)和(8)关于 λ_j 的微分,因此,称 $\tilde{\sigma}$ 为"极限对称"(相当于式(2)和(3)关于 $\lambda = \lambda_j + \varepsilon$ 的扰动展开,Taylor 展开式中的领头项即为式(9)和(10)).

2　相似约化

2.1　相似约化与精确解

考虑式(1)的对称的组合

$$\hat{\sigma} = u_x - \sum_{j=1}^{N}(\phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}), \tag{12}$$

式中,ϕ_{1j},ϕ_{2j}满足式(7)和(8),ψ_{1j},ψ_{2j}满足式(9)和(10). 令 $\hat{\sigma} = 0$,有

$$u_x = \sum_{j=1}^{N}(\phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}). \tag{13}$$

这是一个新对称约束. 整个系统由式(1),(7)~(10),(13)组成,其中 $j = 1,2,\cdots,N$. 直接代入验证发现,当 ϕ_{kj},ψ_{kj}($k = 1,2$)满足式(7)~(10)时,由式(13)定义的 u 自动满足 sine-Gordon 方程. 所以,此约束系统可以简化为

$$u_x = \sum_{j=1}^{N}(\phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}), \tag{14}$$

$$\begin{cases}\phi_{1j,x} = -\lambda_j\phi_{1j} + \frac{u_x}{2}\phi_{2j},\\ \phi_{2j,x} = -\frac{u_x}{2}\phi_{1j} + \lambda_j\phi_{2j},\end{cases} \tag{15}$$

$$\begin{cases}\phi_{1j,t} = \frac{1}{4\lambda_j}(-\phi_{1j}\cos u + \phi_{2j}\sin u),\\ \phi_{2j,t} = \frac{1}{4\lambda_j}(\phi_{1j}\sin u + \phi_{2j}\cos u),\end{cases} \tag{16}$$

$$\begin{cases}\psi_{1j,x} = -\lambda_j\psi_{1j} + \frac{u_x}{2}\psi_{2j} - \phi_{1j},\\ \psi_{2j,x} = -\frac{u_x}{2}\psi_{1j} + \lambda_j\psi_{2j} + \phi_{2j},\end{cases} \tag{17}$$

$$\begin{aligned}\psi_{1j,t} = {}& \frac{1}{4\lambda_j}(-\psi_{1j}\cos u + \psi_{2j}\sin u) - \\ & \frac{1}{4\lambda_j^2}(-\phi_{1j}\cos u + \phi_{2j}\sin u),\end{aligned} \tag{18}$$

$$\begin{aligned}\psi_{2j,t} = {}& \frac{1}{4\lambda_j}(\psi_{1j}\sin u + \psi_{2j}\cos u) - \\ & \frac{1}{4\lambda_j^2}(\phi_{1j}\sin u + \phi_{2j}\cos u).\end{aligned} \tag{19}$$

引入如下变换:

$$u = 2\mathrm{i}\ln\frac{\bar{f}}{f}, \tag{20}$$

$$\begin{cases}\phi_{1j}=\dfrac{\bar g_j}{\bar f}+\dfrac{g_j}{f}, \quad \phi_{2j}=\mathrm{i}\left(\dfrac{\bar g_j}{\bar f}-\dfrac{g_j}{f}\right),\\ \psi_{1j}=\dfrac{\bar h_j}{\bar f}+\dfrac{h_j}{f}, \quad \psi_{2j}=\mathrm{i}\left(\dfrac{\bar h_j}{\bar f}-\dfrac{h_j}{f}\right),\end{cases} \tag{21}$$

式中,i 为虚数单位,“ - ”表示复共轭. 将式(14)两边对 x 微分,利用式(20)和(21),可以将式(14)~(19)写成如下双线性形式:

$$D_x^2 f\cdot f=2\mathrm{i}\sum_{j=1}^{N}(2k_jg_jh_j-g_j^2), \tag{22}$$

$$D_x\bar g_j\cdot f=k_jg_j\bar f, \tag{23}$$

$$D_x\bar h_j\cdot f=k_jh_j\bar f-g_j\bar f, \tag{24}$$

$$D_t\, g_j\cdot f=\frac{1}{4k_j}\bar g_j\bar f, \tag{25}$$

$$D_t\, h_j\cdot f=\frac{1}{4k_j}\bar h_j\bar f+\frac{1}{4k_j^2}\bar g_j\bar f. \tag{26}$$

为了方便,在式(22)~(26)中已将 λ_j 记为 $-k_j$,算子 D 即为所熟悉的 Hirota 双线性算子[17],定义为

$$D_t^mD_x^na(t,x)\cdot b(t,x)=\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}a(t+s,x+y)\cdot b(t-s,x-y)\mid_{s=0,y=0}, m,n=0,1,2,\cdots.$$

为了精确地求解式(22)~(26),将 f,g_j,h_j 分别按 ε 级数展开,有

$$\begin{cases}f=1+\sum\limits_{l=1}^{\infty}f^{(2l)}\varepsilon^{2l},\\ g_j=\sum\limits_{l=1}^{\infty}g_j^{(2l-1)}\varepsilon^{2l-1},\\ h_j=\sum\limits_{l=1}^{\infty}h_j^{(2l-1)}\varepsilon^{2l-1}.\end{cases} \tag{27}$$

将式(27)代入式(22)~(26). 当 $N=1$ 时,经过计算发现,式(22)~(26)的解可以由截断的级数展开式(27)给出,其中

$$f^{(2)}=\mathrm{i}\left(-x+\frac{t}{4k_1^2}+\frac{1}{2k_1}\right)\mathrm{e}^{2\xi_1}, f^{(4)}=\frac{1}{16k_1^2}\mathrm{e}^{4\xi_1}, \tag{28}$$

$$g_1^{(1)}=\sqrt{2k_1}\mathrm{e}^{\xi_1}, \quad g_1^{(3)}=\mathrm{i}\frac{\sqrt{2}}{4\sqrt{k_1}}\mathrm{e}^{3\xi_1}, \tag{29}$$

$$\begin{cases}h_1^{(1)}=-\sqrt{2k_1}\left(x-\dfrac{t}{4k_1^2}\right)\mathrm{e}^{\xi_1},\\ h_1^{(3)}=\mathrm{i}\dfrac{\sqrt{2}}{4\sqrt{k_1}}\left(x-\dfrac{t}{4k_1^2}-\dfrac{1}{k_1}\right)\mathrm{e}^{3\xi_1},\end{cases} \tag{30}$$

$$f^{(2l)}=g_1^{(2l-1)}=h_1^{(2l-1)}=0, \quad l\geqslant 3, \tag{31}$$

式中,k_1,$\mathrm{e}^{\xi_1^{(0)}}$ 都为实参数,且

$$\xi_1=k_1x+\frac{t}{4k_1}+\xi_1^{(0)}. \tag{32}$$

在式(27)中,取 $\varepsilon=1$,由式(20)和(21),可求得 sine-Gordon 方程的解为

$$\begin{cases}u=2\mathrm{i}\ \ln\dfrac{\bar f}{f},\\ f=1+\mathrm{i}\left(-x+\dfrac{t}{4k_1^2}+\dfrac{1}{2k_1}\right)\mathrm{e}^{2\xi_1}+\dfrac{1}{16k_1^2}\mathrm{e}^{4\xi_1},\end{cases} \tag{33}$$

或表示为

$$u=4\arctan\frac{-x+\dfrac{t}{4k_1^2}+\dfrac{1}{2k_1}}{\mathrm{e}^{-2\xi_1}+\dfrac{1}{16k_1^2}\mathrm{e}^{2\xi_1}}. \tag{34}$$

2.2 动力学分析

为了更好地分析式(34)的动力学特征,先来看 sine-Gordon 方程的 2-孤子解,它可以写为[18-20]

$$\begin{cases}u=2\mathrm{i}\ \ln\dfrac{\bar f}{f},\\ f=1+\mathrm{i}(\mathrm{e}^{2\xi_1}+\mathrm{e}^{2\xi_2})-\left(\dfrac{k_1-k_2}{k_1+k_2}\right)^2\mathrm{e}^{2\xi_1+2\xi_2},\end{cases} \tag{35}$$

$$\xi_j=k_jx+\frac{t}{4k_j}+\xi_j^{(0)}, \quad j=1,2. \tag{36}$$

众所周知,sine-Gordon 方程的单孤子解具有 kink 和反-kink 两种类型,因此,2-孤子的相互作用也自然较 KdV 方程更丰富.

为了与式(33)建立联系,先将式(35)中的 $\mathrm{e}^{2\xi_1^{(0)}}$, $\mathrm{e}^{2\xi_2^{(0)}}$ 分别替换为 $\dfrac{\alpha_1\mathrm{e}^{2(\xi_1^{(0)}+\beta_1(k_1))}}{k_1-k_2}$ 和 $\dfrac{\alpha_1\mathrm{e}^{2(\xi_1^{(0)}+\beta_1(k_2))}}{k_2-k_1}$,其中 α_1 为实参数,$\beta_1(k_j)$ 为关于 k_j 的可微函数. 则式(35)可以写成

$$\begin{cases}u=2\mathrm{i}\ \ln\dfrac{\bar f}{f},\\ f=1+\mathrm{i}\alpha_1\dfrac{\mathrm{e}^{2\xi_1}-\mathrm{e}^{2\xi_2}}{k_1-k_2}+\left(\dfrac{\alpha_1}{k_1+k_2}\right)^2\mathrm{e}^{2\xi_1+2\xi_2},\end{cases} \tag{37}$$

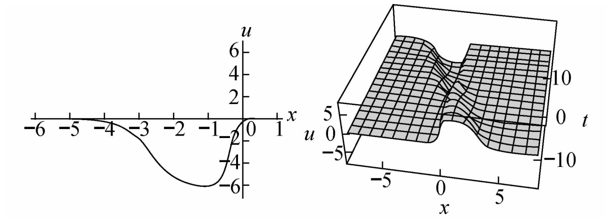
式中,

$$\xi_j=k_jx+\frac{t}{4k_j}+\beta_1(k_j)+\xi_1^{(0)}, \quad j=1,2. \tag{38}$$

式(37)的图像如图 1 所示,其中 $k_1=1$,$k_2=3$,$\alpha_1=-\dfrac{1}{2}$,$\xi_1^{(0)}=0$,$\beta_1(k_1)=-\dfrac{1}{2}\ln k_1$.

从图 1(a)中看出,波形是非对称的. 事实上,在波的两侧各有一个拐点,拐点处的斜率分别为 $4k_2$

和 $-4k_1$ ($k_2 > k_1 > 0$),而拐点移动的速度分别为 $-\frac{1}{4k_1^2}$ 和 $-\frac{1}{4k_2^2}$,它们分别代表相互作用的 2 个孤子. 通过渐进分析发现,这些特征在相互作用以后并不改变.



(a) 当 $t=5$ 时,二维平面图　(b) u 的三维图

图 1　Sine-Gordon 方程的解(37)的图像

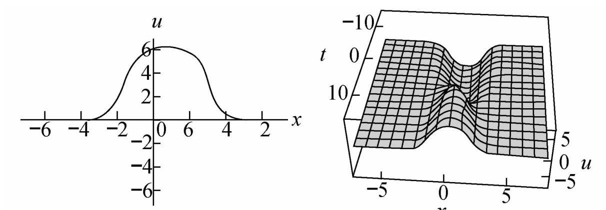
Fig. 1　Plots for solution of sine-Gordon equation given by (37)

在式(37)中,令 $k_2 \to k_1$,并利用L' Hospital法则,可得

$$f \to 1 + 2i\alpha_1\left(x - \frac{t}{4k_1^2} + \partial_{k_1}\beta_1(k_1)\right)e^{2\xi_1} + \frac{\alpha_1^2}{4k_1^2}e^{4\xi_1}. \tag{39}$$

当 $\alpha_1 = -\frac{1}{2}, \beta_1(k_1) = -\frac{1}{2}\ln k_1$ 时,式(39)与式(33)是一致的,这就意味着由式(13)得到的解即是 sine-Gordon 方程 2-孤子解的极限解.

极限解(34)的图像如图 2 所示,其中 $k_1 = 1$, $\xi_1^{(0)} = 0$.



(a) 当 $t=50$ 时,二维平面图　(b) u 在移动坐标系中的三维图

图 2　Sine-Gordon 方程的解(34)的图像

Fig. 2　Plots for solution of sine-Gordon equation given by (34)

显然,图 2(a)中的波形是对称的,这正是 2-孤子解(37)中 $k_2 \to k_1$ 的体现. 为了更好地研究解(33)的渐进性,将其放入如下移动坐标系内(见图 2(b)):

$$\left(X = x + \frac{t}{4k_1^2}, t\right). \tag{40}$$

通过渐进分析发现,图 2(b)中 4 个拐点的轨迹可以用下述 4 条曲线来描述.

定理 1　设式(34)中,$k_1 > 0$,则当 $t \to -\infty$ 时,有 2 条移动的拐点轨迹,分别为

$$X_{BR} = \frac{1}{2k_1}[\ln(-t) + \ln 8], \quad X_{BR} \to +\infty,$$

$$X_{BL} = \frac{1}{2k_1}[-\ln(-t) + \ln k_1 + \ln 2], \quad X_{BL} \to -\infty.$$

在拐点处,u 的斜率分别为 $4k_1$ 和 $-4k_1$,u 的值为 $u|_{X_{BR}} = u|_{X_{BL}} = -\pi$. 当 $t \to +\infty$ 时,有 2 条移动的拐点轨迹,分别为

$$X_{TR} = \frac{1}{2k_1}(\ln t + \ln 8), \quad X_{TR} \to +\infty,$$

$$X_{TL} = \frac{1}{2k_1}(-\ln t + \ln k_1 + \ln 2), \quad X_{TL} \to -\infty.$$

在拐点处 u 的斜率分别为 $4k_1$ 和 $-4k_1$,u 的值为 $u|_{X_{TL}} = u|_{X_{TR}} = \pi$.

3　带新自相容源的 sine-Gordon 方程

在文献[21]中,带自相容源的 sine-Gordon 方程定义为

$$\begin{cases} u_{xt} = \sin u + 2\sum_{j=1}^{N}(\phi_{1j}^2 + \phi_{2j}^2)_x, \\ \phi_{1j,x} = -\lambda_j\phi_{1j} + \frac{u_x}{2}\phi_{2j}, \\ \phi_{2j,x} = -\frac{u_x}{2}\phi_{1j} + \lambda_j\phi_{2j}. \end{cases} \tag{41}$$

类似地,引入如下带极限源的 sine-Gordon 方程:

$$u_{xt} = \sin u + 2\sum_{j=1}^{N}(\phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j})_x, \tag{42}$$

$$\phi_{1j,x} = -\lambda_j\phi_{1j} + \frac{u_x}{2}\phi_{2j}, \quad \phi_{2j,x} = -\frac{u_x}{2}\phi_{1j} + \lambda_j\phi_{2j}, \tag{43}$$

$$\begin{cases} \psi_{1j,x} = -\lambda_j\psi_{1j} + \frac{u_x}{2}\psi_{2j} - \phi_{1j}, \\ \psi_{2j,x} = -\frac{u_x}{2}\psi_{1j} + \lambda_j\psi_{2j} + \phi_{2j}, \end{cases} \tag{44}$$

式中,$\{\lambda_j\}_{j=1}^N$ 互不相同,$j = 1, 2, \cdots, N$. 式(42)~(44)为 Lax 可积系,Lax 对为

$$\begin{cases} \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}_x = \begin{pmatrix} -\lambda & \frac{u_x}{2} \\ -\frac{u_x}{2} & \lambda \end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \\ \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \end{cases} \tag{45}$$

式中,

$$A = -\cos u + \partial^{-1}u_x \sum_{j=1}^{N}\left[\frac{2\lambda_j(\phi_{2j}\psi_{2j}-\phi_{1j}\psi_{1j})}{\lambda+\lambda_j} + \frac{\lambda(\phi_{2j}^2-\phi_{1j}^2)}{(\lambda+\lambda_j)^2}\right] + \partial^{-1}u_x \sum_{j=1}^{N}\left[\frac{2\lambda_j(\phi_{2j}\psi_{2j}-\phi_{1j}\psi_{1j})}{\lambda-\lambda_j} + \frac{\lambda(\phi_{2j}^2-\phi_{1j}^2)}{(\lambda-\lambda_j)^2}\right],$$

$$B = \sin u - \sum_{j=1}^{N}\left[-\frac{2\lambda_j\phi_{2j}\psi_{2j}}{\lambda+\lambda_j} - \frac{\phi_{2j}^2}{\lambda+\lambda_j} + \frac{\lambda_j\phi_{2j}^2}{(\lambda+\lambda_j)^2}\right] - \sum_{j=1}^{N}\left[\frac{2\lambda_j\phi_{1j}\psi_{1j}}{\lambda-\lambda_j} + \frac{\phi_{1j}^2}{\lambda-\lambda_j} + \frac{\lambda_j\phi_{1j}^2}{(\lambda-\lambda_j)^2}\right],$$

$$C = \sin u + \sum_{j=1}^{N}\left[-\frac{2\lambda_j\phi_{1j}\psi_{1j}}{\lambda+\lambda_j} - \frac{\phi_{1j}^2}{\lambda+\lambda_j} + \frac{\lambda_j\phi_{1j}^2}{(\lambda+\lambda_j)^2}\right] + \sum_{j=1}^{N}\left[\frac{2\lambda_j\phi_{2j}\psi_{2j}}{\lambda-\lambda_j} + \frac{\phi_{2j}^2}{\lambda-\lambda_j} + \frac{\lambda_j\phi_{2j}^2}{(\lambda-\lambda_j)^2}\right].$$

由式(45)的相容性条件,可导出式(42),其中需利用如下关系:

$$\begin{cases} L\begin{pmatrix}\phi_{1j}^2\\ \phi_{2j}^2\end{pmatrix} = 2\lambda_j\begin{pmatrix}\phi_{1j}^2\\ \phi_{2j}^2\end{pmatrix}, \\ L\begin{pmatrix}\phi_{1j}\psi_{1j}\\ \phi_{2j}\psi_{2j}\end{pmatrix} = 2\lambda_j\begin{pmatrix}\phi_{1j}\psi_{1j}\\ \phi_{2j}\psi_{2j}\end{pmatrix} + \begin{pmatrix}\phi_{1j}^2\\ \phi_{2j}^2\end{pmatrix}, \\ L\begin{pmatrix}\phi_{2j}^2\\ \phi_{1j}^2\end{pmatrix} = -2\lambda_j\begin{pmatrix}\phi_{2j}^2\\ \phi_{1j}^2\end{pmatrix}, \\ L\begin{pmatrix}\phi_{2j}\psi_{2j}\\ \phi_{1j}\psi_{1j}\end{pmatrix} = -2\lambda_j\begin{pmatrix}\phi_{2j}\psi_{2j}\\ \phi_{1j}\psi_{1j}\end{pmatrix} - \begin{pmatrix}\phi_{2j}^2\\ \phi_{1j}^2\end{pmatrix}, \end{cases} \tag{46}$$

$$F^{-1}u_{xx} = \cos u\partial^{-1}(\sin u) - \sin\partial^{-1}(\cos u - 1) = \sin u, \tag{47}$$

式中, $L = \begin{pmatrix} -\partial - \frac{1}{2}u_x\partial^{-1}u_x & \frac{1}{2}u_x\partial^{-1}u_x \\ -\frac{1}{2}u_x\partial^{-1}u_x & \partial + \frac{1}{2}u_x\partial^{-1}u_x \end{pmatrix}$, $F = \partial^2 + \partial u_x\partial^{-1}u_x$, $F^{-1} = \cos u\partial^{-1}\cos u\partial^{-1} + \sin u\partial^{-1}\sin u\partial^{-1}$,且 ϕ_{kj}, ψ_{kj} 满足式(43)和(44).

式(42)~(44)能够被精确求解.采用变换式(20)~(21),则式(42)~(44)转化为如下双线性形式($\lambda_j = -k_j$):

$$D_xD_t f\cdot f = \frac{1}{2}(f^2 - \bar{f}^2) + 4\mathrm{i}\sum_{j=1}^{N}(2k_jg_jh_j + g_j^2), \tag{48}$$

$$D_x\bar{g}_j\cdot f = k_jg_j\bar{f}, \tag{49}$$

$$D_x\bar{h}_j\cdot f = k_jh_j\bar{f} - g_j\bar{f}. \tag{50}$$

类似第2节中的求解过程,如式(27)将 f, g_j, h_j 展开,并代入到式(48)~(50)中.当 $N=1$ 时,可得

$$f^{(2)} = \mathrm{i}\left(-x + \frac{t}{4k_1^2} + \frac{1}{k_1}\right)\mathrm{e}^{2\xi_1},\ f^{(4)} = \frac{1}{16k_1^2}\mathrm{e}^{4\xi_1}, \tag{51}$$

$$g_1^{(1)} = \sqrt{\beta_1(t)}\,\mathrm{e}^{\xi_1},\ g_1^{(3)} = \frac{\mathrm{i}}{4k_1}\sqrt{\beta_1(t)}\,\mathrm{e}^{3\xi_1}, \tag{52}$$

$$\begin{cases} h_1^{(1)} = \sqrt{\beta_1(t)}\left(-x + \frac{t}{4k_1^2}\right)\mathrm{e}^{\xi_1}, \\ h_1^{(3)} = \frac{\mathrm{i}}{4k_1}\sqrt{\beta_1(t)}\left(x - \frac{t}{4k_1^2} - \frac{2}{k_1}\right)\mathrm{e}^{3\xi_1}, \end{cases} \tag{53}$$

$$f^{(2l)} = g_1^{(2l-1)} = h_1^{(2l-1)} = 0,\quad l \geqslant 3,$$

式中,

$$\xi_1 = k_1x + \frac{t}{4k_1} + \int_0^t \beta_1(z)\,\mathrm{d}z + \xi_1^{(0)}, \tag{54}$$

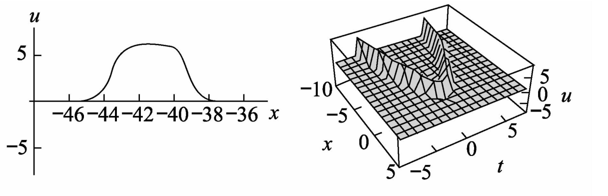
式中,k_1,$\mathrm{e}^{\xi_1^{(0)}}$ 为实参数,$\beta_1(z)$ 为 z 的任意连续函数.在式(27)中,若取 $\varepsilon = 1$,可得式(42)~(44)的一个解为

$$\begin{cases} u = 2\mathrm{i}\ln\frac{\bar{f}}{f}, \\ f = 1 + \mathrm{i}\left(-x + \frac{t}{4k_1^2} + \frac{1}{k_1}\right)\mathrm{e}^{2\xi_1} + \frac{1}{16k_1^2}\mathrm{e}^{4\xi_1}. \end{cases} \tag{55}$$

或写为

$$u = 4\arctan\frac{-x + \frac{t}{4k_1^2} + \frac{1}{k_1}}{\mathrm{e}^{-2\xi_1} + \frac{1}{16k_1^2}\mathrm{e}^{2\xi_1}}. \tag{56}$$

解(56)的图像如图3所示,其中 $k_1 = 1, 2, \xi_1^{(0)} = 0$, $\beta_1(z) = 3z^2$.



(a) 当t=7时, 二维平面图　　(b) u的三维图

图3　带极限源的 sine-Gordon 方程的解(56)的图像

Fig. 3　Plots for the solution of sine-Gordon equation with new self-consistent sources given by (56)

4　结 束 语

本研究给出了与本征函数有关的 sine-Gordon 方程的新对称,这个对称与原有的平方本征函数对称之间存在极限关系,因此,称之为极限对称.由该

对称引出的相似约化,可以得到 sine-Gordon方程 2-孤子解的极限解. 本研究讨论了这个解与 sine-Gordon 方程 2-孤子解之间的极限关系,并分析了解的动力学特征. 此外,本研究还利用极限对称给出了一个新的带源的 sine-Gordon 方程,该方程是 Lax 可积的,可以被双线性化,并且得到的解具有极限解的特征. 本研究所讨论的极限对称与相应的方法可同样应用于其他可积方程.

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