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假分数阶 Chen 混沌系统同步

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摘要: 提出真分数阶系统和假分数阶系统的概念以及对分数阶系统分段研究的思想, 建立假分数阶系统稳定性理论. 研究分数阶系统中阶次大于 1 (假分数阶) 的分数阶系统同步问题, 并设计控制器实现假分数阶 Chen 混沌系统的同步. 仿真结果证实该理论的正确性.

关键词: 假分数阶; 稳定性; 混沌; 同步

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Synchronizing Improper Fractional Chen Chaotic System

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Abstract: This paper proposes a concept of proper and improper fractional systems. A fractional system can be studied individually based on proper and improper fractional systems. A stable theorem of improper fractional system is given. Synchronization of improper fractional systems is studied, and synchronization of an improper fractional Chen chaotic system realized by designing a controller. Numerical simulation shows effectiveness of the method.

Key words: improper fraction; stability; chaos; synchronization

分数阶微分理论已有近 300 年的历史, 但分数阶微分方程的应用却直到 20 世纪 80 年代 Mandelbort^[1] 发现自然界中存在大量分数维现象才引起人们的研究兴趣. 研究^[2-3] 表明, 分数阶微积分是整数阶微积分的推广, 整数阶微积分是分数阶微积分的特例. 实际上, 自然界中的所有现象几乎都是以分数阶形式存在的, 整数阶数学模型是对实际物理模型的近似. 由于混沌系统具有参数敏感性, 因而, 研究分数阶混沌系统较研究整数阶混沌系统更

具有普遍性和实际意义^[4-6].

自 1990 年 Pecora 等^[7] 实现混沌同步以来, 由于其在保密通信和震荡发射器等领域的潜在应用而得到了广泛的研究, 但这些研究更多地集中于整数阶混沌系统同步^[8-9]. 由于分数阶非线性系统稳定性研究起步较晚, 因此, 尽管分数阶混沌同步近几年也取得了一些成果, 但远不如整数阶混沌同步研究得充分. 分数阶混沌同步方法主要可以分为以下几种: ① 根据分数阶线性系统稳定性理论设计控制器, 使同

步误差为定常的负定矩阵,该方法一方面牺牲了非线性项,另一方面控制代价大;② 基于拉氏变换终值定理合理设计控制器,该方法一方面缺乏灵活性,另一方面很多同步方法难以应用^[10-11]。同时,基于上述方法,很多同步难以实现,如参数未知的分数阶混沌同步等。针对上述问题,Hu 等^[12-13]提出了分数阶非线性系统的稳定性理论,而基于该理论,整数阶混沌系统同步方法几乎都可用于阶次小于1的分数阶混沌系统的同步。但对于阶次大于1的分数阶混沌系统的同步问题还未见相关报道,因此,研究阶次大于1的分数阶系统具有积极意义。

本工作针对上述问题,提出将分数阶系统分为真分数阶系统和假分数阶系统的概念,并建立了对分数阶系统分段研究的思想(对真分数阶系统和假分数阶系统分别研究),研究了假分数阶混沌系统稳定性理论,并设计控制器实现了假分数阶 Chen 混沌系统同步。

1 分数阶系统稳定理论

分数阶微分当前有多种定义,其中常用的有 Riemann-Liouville (R-L) 定义和 Caputo 定义。令 n 为大于 α 的最小整数, $n-1 < \alpha < n$, $\Gamma(\cdot)$ 为伽马函数。

定义 1 Riemann-Liouville (R-L) 定义数学表达式^[11]为

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \times \frac{d^n}{dt^n} \left[\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \right]. \quad (1)$$

定义 2 Caputo 分数阶微分^[11]定义为

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_a^t (t-\tau)^{-\alpha+n-1} f^{(n)}(\tau) d\tau. \quad (2)$$

Caputo 分数阶微分定义的系统初始条件可以借助于位置函数的整数阶导数形式给出,具有可知的物理解释,在实际应用中更有意义。因此,本工作以 Caputo 分数阶微分为基础进行研究。

对于一般的分数阶系统,可以表示为如下形式:

$$\frac{d^\alpha X}{dt^\alpha} = A(X)X, \quad (3)$$

式中, $X = (x_1, x_2, \dots, x_n)$ 为系统状态变量, α 为系统阶次, $A(X)$ 为包含变量的系数矩阵。

引理 1^[12] 对于分数阶系统(3),当阶数 $0 < \alpha < 1$ 时,如果存在实对称正定矩阵 P ,使 $X^T P \frac{d^\alpha X}{dt^\alpha} \leq 0$ 恒成立,则分数阶系统(3)稳定。

当 $X^T P \frac{d^\alpha X}{dt^\alpha} \leq 0$ 时,系数矩阵 $A(X)$ 特征值的实部恒小于 0,依据分数阶系统稳定性理论,当阶数 $\alpha < 1$ 时,分数阶系统(3)稳定。

当状态变量的微分阶次不相等时,即

$$\frac{d^\alpha X}{dt^\alpha} = \begin{bmatrix} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} \\ \vdots \\ \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} \end{bmatrix} = A(X)X, \quad (4)$$

式中, $X = (x_1, x_2, \dots, x_n)$ 为系统状态变量,系统阶次 $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$,其中 $\alpha_1, \alpha_2, \dots, \alpha_n$ 不完全相等, $A(X)$ 为包含变量的系数矩阵。当 $\max(\alpha_1, \alpha_2, \dots, \alpha_n) \leq 1$ 且 $\min(\alpha_1, \alpha_2, \dots, \alpha_n) > 0$ 时,如果 $X^T P \frac{d^\alpha X}{dt^\alpha} \leq 0$,则系数矩阵 $A(X)$ 特征值的实部恒小于 0,分数阶系统(4)也稳定。该稳定性判据同样适用于阶次不等的分数阶系统(4)。

该引理虽然给出了阶次 $\alpha_1, \alpha_2, \dots, \alpha_n$ 都小于 1 时分数阶系统的稳定性判据,但当存在任意 $\alpha_i > 1$ 时,还不能直接使用该理论来判断系统的稳定性。

定义 3 当阶数 $\alpha_1, \alpha_2, \dots, \alpha_n$ 都小于 1 时, $\alpha_1, \alpha_2, \dots, \alpha_n$ 都为真分数,形如式(4)的系统称为“真分数阶”系统;当阶数 $\alpha_1, \alpha_2, \dots, \alpha_n$ 都等于 1 时,形如式(4)的系统称为“整数阶”系统;当至少有一个 $\alpha_i > 1 (i=1, 2, \dots, n)$ 时, α_i 为假分数,形如式(4)的系统称为“假分数阶”系统。

引理 1 给出了真分数阶系统的稳定性理论,本工作进一步给出假分数阶系统的稳定性理论。

定理 1 对于分数阶系统(4),当阶数 $1 < \max(\alpha_1, \alpha_2, \dots, \alpha_n) < 2$ 时,如果存在实对称正定矩阵 $P, Q, m = (m_1, m_2, \dots, m_n), 0 < m_i < 1, 0 < \alpha_i - m_i < 1, i=1, 2, \dots, n$,使 $\left(\frac{d^m X}{dt^m}\right)^T P \frac{d^\alpha X}{dt^\alpha} + (X)^T Q \frac{d^{\alpha-m} X}{dt^{\alpha-m}} \leq 0$ 恒成立,则假分数阶系统(4)稳定。

证明

令
$$Y = \frac{d^m X}{dt^m}, \quad (5)$$

则有
$$\frac{d^\alpha X}{dt^\alpha} = \left(\frac{d^{\alpha-m} \left(\frac{d^m X}{dt^m} \right)}{dt^{\alpha-m}} \right) = \frac{d^{\alpha-m} Y}{dt^{\alpha-m}}, \quad (6)$$

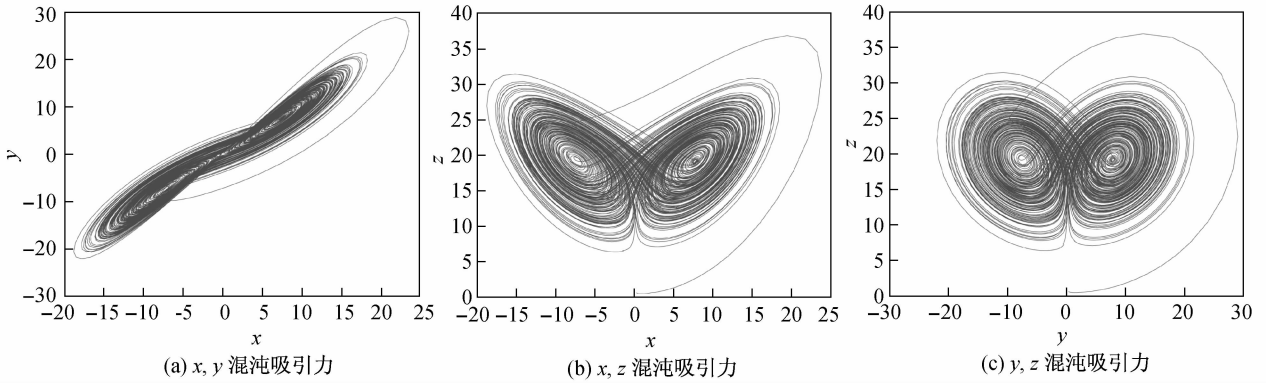


图 1 $\alpha_1, \alpha_2, \alpha_3 = 0.86$ 时, 真分数阶 Chen 混沌系统吸引子

Fig. 1 When $\alpha_1, \alpha_2, \alpha_3 = 0.86$, the chaotic attractor of proper fraction at Chen system

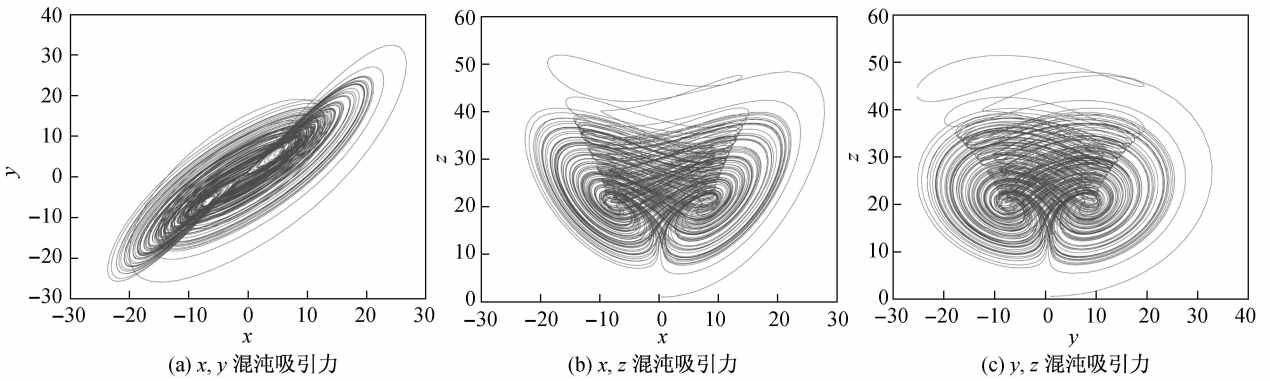


图 2 $\alpha_1, \alpha_2, \alpha_3 = 1$ 时, 整数阶 Chen 混沌系统吸引子

Fig. 2 When $\alpha_1, \alpha_2, \alpha_3 = 1$, the chaotic attractor of integer Chen system

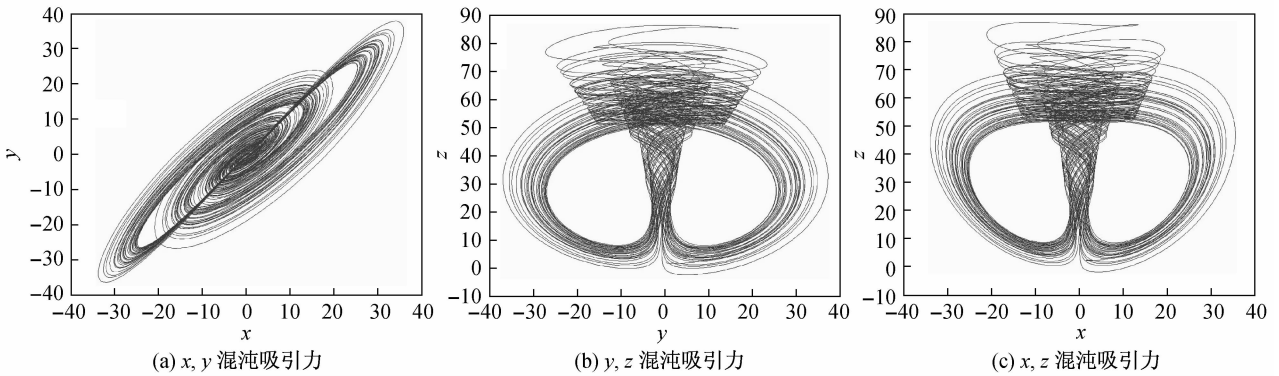


图 3 $\alpha_1, \alpha_2, \alpha_3 = [1, 1.24, 1.24]'$ 时, 假分数阶 Chen 混沌系统吸引子

Fig. 3 When $\alpha_1, \alpha_2, \alpha_3 = [1, 1.24, 1.24]'$, the chaotic attractor of improper fraction at Chen system

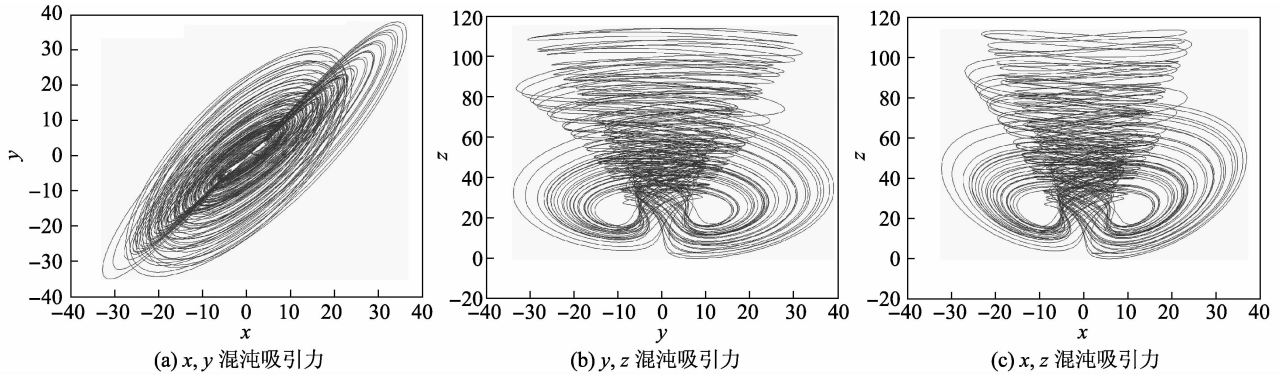


图4 $\alpha_1, \alpha_2, \alpha_3 = [1.05, 1.28, 1.28]'$ 时, 假分数阶 Chen 混沌系统吸引子

Fig. 4 When $\alpha_1, \alpha_2, \alpha_3 = [1.05, 1.28, 1.28]'$, the chaotic attractor of improper fraction at Chen system

$$\begin{bmatrix} \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} \\ \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}} \end{bmatrix} = \begin{bmatrix} -a & a & 0 \\ c-a-x_3 & c & -y_1 \\ x_2 & y_1 & -b \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} - U(t). \quad (13)$$

定理2 如果设计的控制器满足

$$\begin{cases} u_1 = (1-a)e_1 + ae_2 + \frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}}, \\ u_2 = (c-a-x_3)e_1 + (c+1)e_2 - y_1 e_3 + \frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}}, \\ u_3 = x_2 e_1 + y_1 e_2 + (1-b)e_3 + \frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}}, \end{cases} \quad (14)$$

则同步误差系统(13)渐近稳定.

证明

令 P, Q 为单位阵, $m = \alpha/2$, 根据定理1 构造如下函数:

$$\begin{aligned} & \left(\frac{d^m E}{dt^m} \right)^T P \frac{d^\alpha E}{dt^\alpha} + (E)^T Q \frac{d^{\alpha-m} E}{dt^{\alpha-m}} = \\ & \left(\frac{d^{\alpha/2} E}{dt^{\alpha/2}} \right)^T \frac{d^\alpha E}{dt^\alpha} + (E)^T \frac{d^{\alpha/2} E}{dt^{\alpha/2}}. \end{aligned} \quad (15)$$

将设计的控制器代入上式, 可得

$$\begin{aligned} & \left(\frac{d^m E}{dt^m} \right)^T P \frac{d^\alpha E}{dt^\alpha} + (E)^T Q \frac{d^{\alpha-m} E}{dt^{\alpha-m}} = \\ & \left(\frac{d^{\alpha/2} E}{dt^{\alpha/2}} \right)^T \frac{d^\alpha E}{dt^\alpha} + (E)^T \frac{d^{\alpha/2} E}{dt^{\alpha/2}} = \\ & \frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}} \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} + e_1 \frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}} + \frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}} \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} + \\ & e_2 \frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}} + \frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}} \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}} + e_3 \frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}} = \end{aligned}$$

$$\begin{aligned} & \left(\frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} + e_1 \right) \frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}} + \left(\frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} + e_2 \right) \frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}} + \\ & \left(\frac{d^{\alpha_3} e_3}{dt^{\alpha_3}} + e_3 \right) \frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}} = \\ & \left(-e_1 - \frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}} + e_1 \right) \frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}} + \left(-e_2 - \frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}} + e_2 \right) \cdot \\ & \frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}} + \left(-e_3 - \frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}} + e_3 \right) \frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}} = \\ & - \left(\frac{d^{\alpha_1/2} e_1}{dt^{\alpha_1/2}} \right)^2 - \left(\frac{d^{\alpha_2/2} e_2}{dt^{\alpha_2/2}} \right)^2 - \left(\frac{d^{\alpha_3/2} e_3}{dt^{\alpha_3/2}} \right)^2 \leq 0. \end{aligned} \quad (16)$$

根据定理1 可知, 结论成立. 定理2 得证.

3 数值仿真

基于改进的 Adams-Bashforth-Moulton 理论^[15], 文献[16]提出了分数阶混沌系统仿真算法. 采用该算法进行仿真, 仿真时选择系统参数 $a = 35, b = 3, c = 27, x_1 = 3.123, x_2 = 1.145, x_3 = 2.453, y_1 = 0.423, y_2 = 0.451, y_3 = 2.453$ 为初始值. 分数阶系统阶次取 $\alpha = [1, 1.24, 1.24]$ 和 $\alpha = [1.05, 1.28, 1.28]'$, 仿真结果分别如图5和图6所示. 仿真结果表明, 同步误差渐近稳定, 所设计的控制器有效, 这也证实了定理1 的正确性.

4 结束语

本工作提出了真分数阶系统和假分数阶系统的概念以及对分数阶系统分段研究的思想, 建立了假分数阶系统稳定性理论, 该理论和真分数阶系统稳定性理论一起构成了分数阶系统稳定性理论. 该成果不仅可用于阶次小于2的分数阶混沌系统同步,

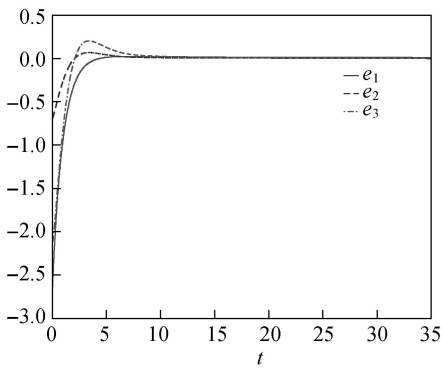


图5 $\alpha = [1, 1.24, 1.24]'$ 时,同步误差随时间的演化

Fig. 5 When $\alpha = [1, 1.24, 1.24]'$, the synchronization error with time

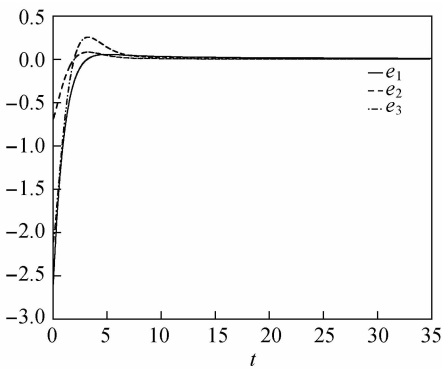


图6 $\alpha = [1.05, 1.28, 1.28]'$ 时,同步误差随时间的演化

Fig. 6 When $\alpha = [1.05, 1.28, 1.28]'$, the synchronization error with time

对于阶次大于2的分数阶系统,按照相似的方法同样可降阶为真分数阶系统,因而,对于阶次大于2的分数阶系统同步依然具有借鉴意义。

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