二粒子 Boltzmann 方程组的奇异扰动解法：初始层解 *

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摘　要　本文讨论了二粒子 Boltzmann 方程组的初始层解. 为此先对未知变量进行了 Fourier 变换，然后运用奇异扰动解法得到了二粒子 Boltzmann 方程组的正规解和初始层解以及其初始层解的初级和高级近似，并且得到了初始层解和正规解的连接.

关键词　二粒子 Boltzmann 方程组；正规解；初始层解

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1　引言

近几十年来， Tsugé S, Lewis M B, 陈天权和其他作者对气体的湍流运动用新的方法进行了一系列研究. 较早前， Grad H[1] 在讨论 Boltzmann 方程链与湍流之间可能的关系时认为：对于稀薄气体，非线性非稳定流和湍流必须用 Boltzmann 方程链描述，而 Boltzmann 方程和 Navier-Stokes 方程都不再适用. Tsugé S, Lewis M B[2−4] 利用 Grad-13 矩方法从 Boltzmann 方程链得到了相关气体的宏观运动方程组. 陈天权 [5] 应用推广的 Hilbert-Enskog-Chapman 展开方法和 Kenberry-Truesdell I 的 Maxwell 迭代法也从 Boltzmann 方程链得到了相关气体的宏观运动方程组. 陈建宁 [6] 对 Boltzmann 方程链在三点混乱水平上考虑问题，即只考虑二点相关 (即 $h = 0$), 并假定两点相关函数可以变量分离. 在这种情况下，由 Boltzmann 方程的两个解可以构造二粒子 Boltzmann 方程组的一类解. 在此基础上，杨梅荣 [7] 用奇异扰动法求解了二粒子 Boltzmann 方程组的边界层解. 在 [8] 中 Grad 介绍了 Boltzmann 方程的渐近方法. 在 [9] 中 Mudan 讨论了混合气体 Boltzmann 方程组的奇异扰动解法，并得到了其正规解. 本文将以上结论推广到二粒子

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Boltzmann 方程组的情形．我们从正规解出发，主要应用 [10] 的方法得到方程组的初始层解并给出了其初级和高级近似，然后进一步得到正规解和初始层解的连接．

本文将在三点混乱水平上讨论二粒子 Boltzmann 方程组，即假定 $h=0$[5−7]，并且只限于平面对称情况．此时 Boltzmann 方程组的前两个方程可以独立求解．在这种假定下，经过 Fourier 变换后方程组可以写为 [7]

$$\begin{cases} \dfrac{\partial\varphi}{\partial t}+i\mu\dfrac{\partial^2\varphi}{\partial\kappa\partial x}+\dfrac{i(1-\mu^2)}{\kappa}\dfrac{\partial^2\varphi}{\partial\mu\partial x}=\dfrac{1}{\varepsilon}J(\varphi,\varphi)+\dfrac{1}{\varepsilon}J(\varphi_*,\varphi_*), \\ \dfrac{\partial\varphi_*}{\partial t}+i\mu\dfrac{\partial^2\varphi_*}{\partial\kappa\partial x}+\dfrac{i(1-\mu^2)}{\kappa}\dfrac{\partial^2\varphi_*}{\partial\mu\partial x}=\dfrac{1}{\varepsilon}J(\varphi,\varphi_*)+\dfrac{1}{\varepsilon}J(\varphi_*,\varphi), \end{cases} \tag{1}$$

其中

$$\varphi=\varphi(x,k,\mu,t)=\int e^{-i\mathbf{k}\cdot\mathbf{v}}fd\mathbf{v},\varphi_*=\varphi_*(x,k,\mu,t)=\int e^{-i\mathbf{k}\cdot\mathbf{v}}g_1d\mathbf{v}, \tag{2}$$

$$\mu=\widehat{\mathbf{k}}\cdot\widehat{\mathbf{e}}_x=k_x/k. \tag{3}$$

(2) 中 $f(x,v,\nu,t)$，$g(x,v,\nu,\widehat{x},\widehat{v},\widehat{\nu},t)=g_1(x,v,\nu,t)g_1(\widehat{x},\widehat{v},\widehat{\nu},t)$ 为二粒子累积分布函数，并与单粒子分布函数的关系为 [6]

$$f(x,v,\nu,t)=\frac{1}{2}\big[f_1(x,v,\nu,t)+f_2(x,v,\nu,t)\big],$$

$$g(x,v,\nu,\widehat{x},\widehat{v},\widehat{\nu},t)=\frac{1}{4}\big[f_1(x,v,\nu,t)-f_2(x,v,\nu,t)\big]\big[f_1(\widehat{x},\widehat{v},\widehat{\nu},t)-f_2(\widehat{x},\widehat{v},\widehat{\nu},t)\big],$$

其中 f_j $(j=1,2)$ 为单粒子分布函数．由此可见

命题 1.1　设 $\varphi_j=\int e^{-i\mathbf{k}\cdot\mathbf{v}}f_j\,\mathrm{d}\mathbf{v}$ $(j=1,2)$，且令

$$\varphi_j(t=0)=\rho_{j0}e^{-\frac{k^2}{2}\theta_{j0}-ik\mu c_{j0}}(1+\eta_{j0}),\qquad j=1,2,$$

则方程组 (1) 的初条件为

$$\begin{cases} \varphi(t=0)=\frac{1}{2}\big[\rho_{10}e^{-\frac{k^2}{2}\theta_{10}-ik\mu c_{10}}(1+\eta_{10})+\rho_{20}e^{-\frac{k^2}{2}\theta_{20}-ik\mu c_{20}}(1+\eta_{20})\big], \\ \varphi_*(t=0)=\frac{1}{2}\big[\rho_{10}e^{-\frac{k^2}{2}\theta_{10}-ik\mu c_{10}}(1+\eta_{10})-\rho_{20}e^{-\frac{k^2}{2}\theta_{20}-ik\mu c_{20}}(1+\eta_{20})\big], \end{cases} \tag{4}$$

其中

$$\begin{aligned} &\rho_{j0}=\rho_j(x,0),\qquad c_{j0}=c_j(x,0),\qquad \theta_{j0}=\theta_j(x,0), \\ &\eta_{j0}=\sum_{nl}b^{(0)}_{jnl}(x,0)e_{nl},\qquad (n,l)=(2,2),(3,1),\cdots,\qquad j=1,2. \end{aligned} \tag{5}$$

那么方程组 (1) 及初条件 (4) 构成了我们要求解的 Cauchy 问题．

已知正规解 φ_n,φ_{*n} 满足 (1) 式 [10]，但一般不能满足初条件 (4)．现在假定式 (1),(4) 的解由两部分组成：

$$\varphi=\varphi_n+\varphi_b,\qquad \varphi_*=\varphi_{*n}+\varphi_{*b}, \tag{6}$$

那么 φ_b,φ_{*b} 应当满足方程

$$\begin{aligned}&\frac{\partial\varphi_b}{\partial t}+i\mu\frac{\partial^2\varphi_b}{\partial\kappa\partial x}+\frac{i(1-\mu^2)}{\kappa}\frac{\partial^2\varphi_b}{\partial\mu\partial x}\\=&\frac{1}{\varepsilon}\big[J(\varphi_n,\varphi_b)+J(\varphi_b,\varphi_b)+J(\varphi_b,\varphi_n)+J(\varphi_{*n},\varphi_{*b})+J(\varphi_{*b},\varphi_{*b})+J(\varphi_{*b},\varphi_{*n})\big],\\&\frac{\partial\varphi_{*b}}{\partial t}+i\mu\frac{\partial^2\varphi_{*b}}{\partial\kappa\partial x}+\frac{i(1-\mu^2)}{\kappa}\frac{\partial^2\varphi_{*b}}{\partial\mu\partial x}\\=&\frac{1}{\varepsilon}[J(\varphi_n,\varphi_{*b})+J(\varphi_b,\varphi_{*b})+J(\varphi_{*b},\varphi_n)+J(\varphi_{*n},\varphi_b)+J(\varphi_{*b},\varphi_b)+J(\varphi_b,\varphi_{*n})],\end{aligned}\tag{7}$$

$$\begin{cases}\varphi_b(t=0)=\varphi(t=0)-\varphi_n(t=0),\\\varphi_{*b}(t=0)=\varphi_*(t=0)-\varphi_{*n}(t=0).\end{cases}\tag{8}$$

而且在初始层之外为了保证 φ,φ_* 能归结为 φ_n,φ_{*n}, 从而 φ_b,φ_{*b} 应满足

$$\varphi_b(t\to\infty)=0,\qquad\varphi_{*b}(t\to\infty)=0.\tag{9}$$

命题 1.2 假设

$$\begin{cases}\varphi_b=\dfrac{1}{2}[\rho_1e^{-\frac{k^2}{2}\theta_1-ik\mu c_1}\eta_1+\rho_2e^{-\frac{k^2}{2}\theta_2-ik\mu c_2}\eta_2],\\\varphi_{*b}=\dfrac{1}{2}[\rho_1e^{-\frac{k^2}{2}\theta_1-ik\mu c_1}\eta_1-\rho_2e^{-\frac{k^2}{2}\theta_2-ik\mu c_2}\eta_2],\end{cases}\tag{10}$$

且令 $\varphi_{1b}=\rho_1e^{-\frac{k^2}{2}\theta_1-ik\mu c_1}\eta_1$, $\varphi_{2b}=\rho_2e^{-\frac{k^2}{2}\theta_2-ik\mu c_2}\eta_2$, 则有 $\varphi_1=\varphi_{1n}+\varphi_{1b},\varphi_2=\varphi_{2n}+\varphi_{2b}$; 若假定 $\varphi_{1b}(t\to\infty)=0$, $\varphi_{2b}(t\to\infty)=0$, 则 (9) 式也得以满足.

下面将 (10) 式代入 (7) 中可得

$$\begin{cases}\dfrac{\partial(\varphi_{1b}+\varphi_{2b})}{\partial t}+i\mu\dfrac{\partial^2(\varphi_{1b}+\varphi_{2b})}{\partial\kappa\partial x}+\dfrac{i(1-\mu^2)}{\kappa}\dfrac{\partial^2(\varphi_{1b}+\varphi_{2b})}{\partial\mu\partial x}\\=\dfrac{1}{\varepsilon}\big[J(\varphi_{1n},\varphi_{1b})+J(\varphi_{2n},\varphi_{2b})+J(\varphi_{1b},\varphi_{1n})\\\quad+J(\varphi_{2b},\varphi_{2n})+J(\varphi_{1b},\varphi_{1b})+J(\varphi_{2b},\varphi_{2b})\big],\\\dfrac{\partial(\varphi_{1b}-\varphi_{2b})}{\partial t}+i\mu\dfrac{\partial^2(\varphi_{1b}-\varphi_{2b})}{\partial\kappa\partial x}+\dfrac{i(1-\mu^2)}{\kappa}\dfrac{\partial^2(\varphi_{1b}-\varphi_{2b})}{\partial\mu\partial x}\\=\dfrac{1}{\varepsilon}\big[J(\varphi_{1n},\varphi_{1b})+J(\varphi_{1b},\varphi_{1n})+J(\varphi_{1b},\varphi_{1b})\\\quad-J(\varphi_{2n},\varphi_{2b})-J(\varphi_{2b},\varphi_{2n})-J(\varphi_{2b},\varphi_{2b})\big],\end{cases}\tag{11}$$

将 (11) 中两个式子相加减得出

$$\begin{cases}\dfrac{\partial\varphi_{1b}}{\partial t}+i\mu\dfrac{\partial^2\varphi_{1b}}{\partial\kappa\partial x}+\dfrac{i(1-\mu^2)}{\kappa}\dfrac{\partial^2\varphi_{1b}}{\partial\mu\partial x}=\dfrac{1}{\varepsilon}\big[J(\varphi_{1n},\varphi_{1b})+J(\varphi_{1b},\varphi_{1n})+J(\varphi_{1b},\varphi_{1b})\big],\\\dfrac{\partial\varphi_{2b}}{\partial t}+i\mu\dfrac{\partial^2\varphi_{2b}}{\partial\kappa\partial x}+\dfrac{i(1-\mu^2)}{\kappa}\dfrac{\partial^2\varphi_{2b}}{\partial\mu\partial x}=\dfrac{1}{\varepsilon}\big[J(\varphi_{2n},\varphi_{2b})+J(\varphi_{2b},\varphi_{2n})+J(\varphi_{2b},\varphi_{2b})\big].\end{cases}\tag{12}$$

由于 (12) 中两个方程形式相同，所以下面只考虑其第一个方程. 将 φ_{1b} 代入到 (12) 式

的第一个方程，则经过计算可得 η_1 所满足的方程

$$\varepsilon\frac{\partial\eta_1}{\partial t}-\rho_1 I(\eta_1)=\rho_1 J(\eta_1,\eta_1)+\rho_1\sum_{j=1}^{\infty}\varepsilon^j\big[J(\eta_1,\xi_1^{(j)})+J(\xi_1^{(j)},\eta_1)\big]-\varepsilon\big[D_0\eta_1+D_1(\eta_1)+D_2(\eta_1)\big],\tag{13}$$

其中 $I(\eta_1)=J(1,\eta_1)+J(\eta_1,1)$,

$\xi_1^{(j)}$ 由正规解给出 [10]:

$$\varphi_{1n}=\rho_1 e^{-\frac{k^2}{2}\theta_1-ik\mu c_1}\Big[1+\sum_{j=1}^{\infty}\varepsilon^j\xi_1^{(j)}\Big],\tag{14}$$

而

$$\begin{aligned}
&D_0=D_{00}+\varepsilon D_{01},\qquad D_{00}=\frac{4}{3}\theta_1\frac{\partial c_1}{\partial x}e_{22}+3\theta_1\frac{\partial\theta_1}{\partial x}e_{31},\\
&D_{01}=-\frac{1}{\rho_1}\frac{\partial}{\partial x}(\rho_1 a_{22}^{(1)})e_{11}-\Big[\frac{2}{3}a_{22}^{(1)}\frac{\partial c_1}{\partial x}+\frac{5}{9\rho_1}\frac{\partial}{\partial x}(\rho_1 a_{31}^{(1)})\Big]e_{20},\\
&D_1(\eta_1)=(c_1+\theta_1 e_{11})\frac{\partial\eta_1}{\partial x}+i\mu\frac{\partial^2\eta_1}{\partial\kappa\partial x}+\frac{i(1-\mu^2)}{\kappa}\frac{\partial^2\eta_1}{\partial\mu\partial x},\\
&D_2(\eta_1)=\Big(\frac{1}{\rho_1}\frac{\partial\rho_1}{\partial x}-\frac{k^2}{2}\frac{\partial\theta_1}{\partial x}-ik\mu\frac{\partial c_1}{\partial x}\Big)\Big[i\mu\frac{\partial\eta_1}{\partial\kappa}+\frac{i(1-\mu^2)}{\kappa}\frac{\partial\eta_1}{\partial\mu}\Big].
\end{aligned}\tag{15}$$

由 (4),(8),(10) 和 (14) 式可知， η_1 应满足初条件

$$\eta_1(t=0)=\eta_{10}-\sum_{j=1}^{\infty}\varepsilon^j\xi_1^{(j)}(t=0)=\sum_{nl}\Big[b_{1nl}^{(0)}(x,0)-\sum_{j=1}^{\infty}\varepsilon^j a_{1nl}^{(j)}(x,0)\Big]e_{nl}.\tag{16}$$

同样地， η_2 满足初条件

$$\eta_2(t=0)=\sum_{nl}\Big[b_{2nl}^{(0)}(x,0)-\sum_{j=1}^{\infty}\varepsilon^j a_{2nl}^{(j)}(x,0)\Big]e_{nl}.$$

一般来说， $b_{1nl}^{(0)}(x,0)$ 不都是零，因此 $\eta_1(t=0)\sim O(1)$, 从 (13) 可看出 $\frac{\partial\eta_1}{\partial t}|_{t=0}\sim O(1/\varepsilon)$, 就是说，在开始的一段时间内 η_1 随 t 的变化是十分剧烈的．为反映这种剧烈的变化，引进新的时间标度 [10], $\tau=t/\varepsilon$, 于是 (13) 式改写为

$$\frac{\partial\eta_1}{\partial\tau}-\rho_1 I(\eta_1)=\rho_1 J(\eta_1,\eta_1)+\rho_1\sum_{j=1}^{\infty}\varepsilon^j\big[J(\eta_1,\xi_1^{(j)})+J(\xi_1^{(j)},\eta_1)\big]-\varepsilon[D_0\eta_1+D_1(\eta_1)+D_2(\eta_1)],\tag{17}$$

这就是讨论初始层解的基本方程式.

令

$$\eta_1=\sum_{j=0}^{\infty}\varepsilon^j\eta_1^{(j)}=\sum_{j=0}^{\infty}\varepsilon^j\sum_{nl}b_{1nl}^{(j)}(x,\tau)e_{nl},\tag{18}$$

与 (5) 式比较, $\eta_1^{(0)}(\tau=0)=\eta_{10}$. 再把 $\frac{\partial\eta_1}{\partial\tau}$ 展开为

$$\frac{\partial\eta_1}{\partial\tau}=\sum_{j=0}^{\infty}\varepsilon^j\Big(\frac{\partial\eta_1}{\partial\tau}\Big)_j. \tag{19}$$

与正规解类似, 假定

$$\frac{\partial b_{1nl}^{(j)}(x,\tau)}{\partial\tau}=\sum_{i=0}^{\infty}\varepsilon^i\beta_{1nl}^{(j,i)}, \tag{20}$$

其中 $\beta_{1nl}^{(j,i)}$ 只含 $\rho_1,c_1,\theta_1,a_{1n'l'}^{(1)},\cdots,a_{1n'l'}^{(j)},b_{1n'l'}^{(0)},\cdots,b_{1n'l'}^{(j)}$ 及其对 x 的导数, 以至于保证 (20) 式成为封闭的方程组. 由 (18)–(20) 式可确定

$$\Big(\frac{\partial\eta_1}{\partial\tau}\Big)_j=\sum_{nl}\Big[\sum_{i=0}^{j}\beta_{1nl}^{(i,j-i)}\Big]e_{nl}. \tag{21}$$

同样地,

$$\Big(\frac{\partial\eta_2}{\partial\tau}\Big)_j=\sum_{nl}\Big[\sum_{i=0}^{j}\beta_{2nl}^{(i,j-i)}\Big]e_{nl}.$$

2 初级近似

将 (18) 和 (19) 式代入 (17) 式, 得 ε^0 级方程

$$\Big(\frac{\partial\eta_1}{\partial t}\Big)_0-\rho_1 I(\eta_1^{(0)})=\rho_1 J(\eta_1^{(0)},\eta_1^{(0)}), \tag{22}$$

它的 e_{nl} 分量为

$$\beta_{1nl}^{(0,0)}+\lambda_{nl}\rho_1 b_{1nl}^{(0)}=R_{1nl}^{(0)}, \tag{23}$$

其中

$$I(e_{nl})=-\lambda_{nl}e_{nl},\qquad R_{1nl}^{(0)}=\rho_1\sum_{n'l'n''l''}b_{1n'l'}^{(0)}b_{1n''l''}^{(0)}h_{nl}^{n'l'n''l''}, \tag{24}$$

$h_{nl}^{n'l'n''l''}$ 由下式定义:

$$\frac{1}{2}\big[J(e_{n'l'},e_{n''l''})+J(e_{n''l''},e_{n'l'})\big]=\sum_{nl}h_{nl}^{n'l'n''l''}e_{nl}. \tag{25}$$

可以看出, 在除 $n=n'+n'',|l'-l''|\le l\le l'+l''$ 之外的情况下, $h_{nl}^{n'l'n''l''}=0$. 如同正规解中 ε^0 级方程不能确定 ρ_1,c_1,θ_1 一样, 在初始层解中, ε^0 级方程 (22) 也不能确定 $b_{1nl}^{(0)}$.

3 一级近似

(17) 式的 ε 级方程为

$$\Big(\frac{\partial\eta_1}{\partial\tau}\Big)_1-\rho_1 I(\eta_1^{(1)})=\rho_1\big[J(\eta_1^{(0)},\eta_1^{(1)})+J(\eta_1^{(1)},\eta_1^{(0)})\big]+\rho_1\big[J(\eta_1^{(0)},\xi_1^{(1)})+J(\xi_1^{(1)},\eta_1^{(0)})\big]$$

$$-\big[D_{00}\eta_1^{(0)}+D_1(\eta_1^{(0)})+D_2(\eta_1^{(0)})\big], \tag{26}$$

它的 e_{nl} 分量为

$$\begin{aligned}\beta_{1nl}^{(1,0)}+\beta_{1nl}^{(0,1)}+\lambda_{nl}\rho_1 b_{1nl}^{(1)}=&-\frac{n(l+1)}{2l+3}\theta_1\frac{\partial b_{1,n-1,l+1}^{(0)}}{\partial x}-c_1\frac{\partial b_{1nl}^{(0)}}{\partial x}\\&-\frac{(n-l+2)l}{(n+1)(2l-1)}\frac{\partial b_{1,n+1,l-1}^{(0)}}{\partial x}+\widetilde{\beta}_{1nl}^{(1)},\end{aligned} \tag{27}$$

其中

$$\begin{aligned}\widetilde{\beta}_{1nl}^{(1)}=&\,2\rho_1\sum_{n'l'n''l''}b_{1n'l'}^{(0)}\big[b_{1n''l''}^{(1)}+a_{1n''l''}^{(1)}\big]h_{nl}^{n'l'n''l''}\\&-\frac{1}{2}n(n-1)(n-2)\theta_1\frac{\partial\theta_1}{\partial x}\Big[\frac{l}{2l-1}b_{1,n-3,l-1}^{(0)}+\frac{l+1}{2l+3}b_{1,n-3,l+1}^{(0)}\Big]\\&-n(n-1)\theta_1\frac{\partial c_1}{\partial x}\Big[\frac{l(l-1)}{(2l-1)(2l-3)}b_{1,n-2,l-2}^{(0)}+\frac{2l(l+1)}{3(2l-1)(2l+3)}b_{1,n-2,l}^{(0)}\\&+\frac{(l+1)(l+2)}{(2l+3)(2l+5)}b_{1,n-2,l+2}^{(0)}\Big]-\frac{nl}{(2l-1)}\theta_1\frac{\partial b_{1,n-1,l-1}^{(0)}}{\partial x}\\&-\frac{1}{2}n\frac{\partial\theta_1}{\partial x}\Big[\frac{l(n-l)}{2l-1}b_{1,n-1,l-1}^{(0)}+\frac{(l+1)(n+l+1)}{2l+3}b_{1,n-1,l+1}^{(0)}\Big]\\&-\frac{\partial c_1}{\partial x}\Big\{\frac{(n-l+2)l(l-1)}{(2l-1)(2l-3)}b_{1,n,l-2}^{(0)}+\Big[\frac{(n-1)(l+1)^2}{(2l+1)(2l+3)}+\frac{(n+l+1)l^2}{(2l-1)(2l+1)}\Big]b_{1nl}^{(0)}\\&+\frac{(l+1)(l+2)(n+l+3)}{(2l+3)(2l+5)}b_{1,n,l+2}^{(0)}\Big\}-\frac{(n-l+2)l}{(n+1)(2l-1)}\frac{1}{\rho_1}\frac{\partial\rho_1}{\partial x}b_{1,n+1,l-1}^{(0)}\\&-\frac{(n+l+3)(l+1)}{(n+1)(2l+3)}\frac{1}{\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{1,n+1,l+1}^{(0)}).\end{aligned} \tag{28}$$

如果在 (27) 式中取 $\beta_{1nl}^{(0,1)}=0$, 那么 (27) 式的右边就是关于 $b_{1nl}^{(1)}$ 的微分方程的非齐次项，而 $b_{1nl}^{(0)}\sim e^{-\lambda_{nl}\rho_1\tau}$, 所以 (27) 式右边有一项 $-c_1\frac{\partial b_{1nl}^{(0)}}{\partial x}\sim\tau e^{-\lambda_{nl}\rho_1\tau}$, 这使得 $b_{1nl}^{(1)}$ 含有一项 $-\frac{1}{2}\cdot\frac{1}{4}\tau^4 e^{-\lambda_{nl}\rho_1\tau}$. 以此类推，在 $b_{1nl}^{(j)}$ 中有项 $\frac{1}{(2j)!!}\tau^{2j}e^{-\lambda_{nl}\rho_1\tau}$, 它是久期项，因为随着 j 的增大，这一项可以任意大．为了消去久期项，必须把 (27) 式右边的 $-c_1\frac{\partial b_{1nl}^{(0)}}{\partial x}$ 归入 $\beta_{1nl}^{(0,1)}$ 之中．注意到 $\lambda_{n,0}=\lambda_{n-1,1}$, 这时还有新的久期项出现，所以必须区别简并与非简并两种情况．

若 $l\neq 0$ 及 1, 那么可以令 $\beta_{1nl}^{(0,1)}=-c_1\frac{\partial b_{1nl}^{(0)}}{\partial x}$, $\beta_{1nl}^{(1,0)}+\lambda_{nl}\rho_1 b_{1nl}^{(1)}=R_{1nl}^{(1)}$, 其中

$$R_{1nl}^{(1)}=-\frac{n(l+1)}{2l+3}\theta_1\frac{\partial b_{1,n-1,l+1}^{(0)}}{\partial x}-\frac{(n-l+2)l}{(n+1)(2l-1)}\frac{\partial b_{1,n+1,l-1}^{(0)}}{\partial x}+\widetilde{\beta}_{1nl}^{(1)}. \tag{29}$$

若 $l=0$ 或 1, $n\geq 4$, 那么令

$$\beta_{1,n,0}^{(0,1)}=-c_1\frac{\partial b_{1,n,0}^{(0)}}{\partial x}-\frac{n}{3}\theta_1\frac{\partial b_{1,n-1,1}^{(0)}}{\partial x}-\frac{n}{6}\frac{\partial\theta_1}{\partial x}b_{1,n-1,1}^{(0)}, \tag{30}$$

$$\beta_{1,n-1,1}^{(0,1)}=-c_1\frac{\partial b_{1,n-1,1}^{(0)}}{\partial x}-\frac{\partial b_{1,n,0}^{(0)}}{\partial x}-\frac{1}{2\theta_1}\Big(\frac{\partial\theta_1}{\partial t}+c_1\frac{\partial\theta_1}{\partial x}\Big)b_{1,n-1,1}^{(0)}. \tag{31}$$

(30) 和 (31) 式右边的末项是为了后面求解 $b^{(0)}_{1,n,0}$ 和 $b^{(0)}_{1,n-1,1}$ 的需要而引进的. 由 (27), (30) 及 (31) 式，有

$$\beta^{(1,0)}_{1,n,0}+\lambda_{n,0}\rho_1 b^{(1)}_{1,n,0}=R^{(1)}_{1,n,0},\qquad \beta^{(1,0)}_{1,n-1,1}+\lambda_{n,0}\rho_1 b^{(1)}_{1,n-1,1}=R^{(1)}_{1,n-1,1},$$

其中

$$R^{(1)}_{1,n,0}=\frac{n}{6}\frac{\partial\theta_1}{\partial x}b^{(0)}_{1,n-1,1}+\widetilde{\beta}^{(1)}_{1,n,0},$$

$$R^{(1)}_{1,n-1,1}=-\frac{2(n-1)}{5}\theta_1\frac{\partial b^{(0)}_{1,n-2,2}}{\partial x}+\frac{1}{2\theta_1}\Big(\frac{\partial\theta_1}{\partial t}+c_1\frac{\partial\theta_1}{\partial x}\Big)b^{(0)}_{1,n-1,1}+\widetilde{\beta}^{(1)}_{1,n-1,1}.$$

4 高级近似

(17) 式的 ε^j 级 ($j\geq 2$) 方程为

$$\begin{aligned}\Big(\frac{\partial\eta_1}{\partial\tau}\Big)_j-\rho_1 I(\eta_1^{(j)})=&\rho_1\sum_{j'=0}^{j-1}\big[J(\eta_1^{(j')},\xi_1^{(j-j')})+J(\xi_1^{(j-j')},\eta_1^{(j')})\big]+\rho_1\sum_{j'=0}^{j}J(\eta_1^{(j')},\eta_1^{(j-j')})\\&-\big[D_{01}\eta_1^{(j-2)}+D_{00}\eta_1^{(j-2)}+D_1(\eta_1^{(j-1)})+D_2(\eta_1^{(j-1)})\big].\end{aligned}\tag{32}$$

它的 e_{nl} 分量为

$$\begin{aligned}\sum_{i=0}^{j}\beta^{(i,j-i)}_{1nl}+\lambda_{nl}\rho_1 b^{(j)}_{1nl}=&-\frac{n(l+1)}{2l+3}\theta_1\frac{\partial b^{(j-1)}_{1,n-1,l+1}}{\partial x}-c_1\frac{\partial b^{(j-1)}_{1nl}}{\partial x}\\&-\frac{(n-l+2)l}{(n+1)(2l-1)}\frac{\partial b^{(j-1)}_{1,n+1,l-1}}{\partial x}+\widetilde{\beta}^{(j)}_{1nl}.\end{aligned}\tag{33}$$

如果认为 $a^{(0)}_{1nl}=0$, 那么 (33) 式之中的 $\widetilde{\beta}^{(j)}_{1nl}$ 可以写成

$$\begin{aligned}\widetilde{\beta}^{(j)}_{1nl}=&2\rho_1\sum_{i=0}^{j}\sum_{n'l'n''l''}b^{(i)}_{1n'l'}\Big[a^{(j-i)}_{1n''l''}+\frac{1}{2}b^{(j-i)}_{1n''l''}\Big]h_{nl}^{n'l'n''l''}\\&-\Big\{nq_1\Big[\frac{l}{2l-1}b^{(j-2)}_{1,n-1,l-1}+\frac{l+1}{2l+3}b^{(j-2)}_{1,n-1,l+1}\Big]+\frac{1}{2}n(n-1)s_1 b^{(j-2)}_{1,n-2,l}\Big\}\\&-\frac{1}{2}n(n-1)(n-2)\theta_1\frac{\partial\theta_1}{\partial x}\Big[\frac{l}{2l-1}b^{(j-1)}_{1,n-3,l-1}+\frac{l+1}{2l+3}b^{(j-1)}_{1,n-3,l+1}\Big]\\&-n(n-1)\theta_1\frac{\partial c_1}{\partial x}\Big[\frac{l(l-1)}{(2l-1)(2l-3)}b^{(j-1)}_{1,n-2,l-2}+\frac{2l(l+1)}{3(2l-1)(2l+3)}b^{(j-1)}_{1,n-2,l}\\&+\frac{(l+1)(l+2)}{(2l+3)(2l+5)}b^{(j-1)}_{1,n-2,l+2}\Big]-\frac{nl}{2l-1}\theta_1\frac{\partial b^{(j-1)}_{1,n-1,l-1}}{\partial x}\\&-\frac{1}{2}n\frac{\partial\theta_1}{\partial x}\Big[\frac{l(n-l)}{2l-1}b^{(j-1)}_{1,n-1,l-1}+\frac{(l+1)(n+l+1)}{2l+3}b^{(j-1)}_{1,n-1,l+1}\Big]\\&-\frac{\partial c_1}{\partial x}\Big\{\frac{(n-l+2)l(l-1)}{(2l-1)(2l-3)}b^{(j-1)}_{1,n,l-2}+\Big[\frac{(n-1)(l+1)^2}{(2l+1)(2l+3)}+\frac{(n+l+1)l^2}{(2l-1)(2l+1)}\Big]b^{(j-1)}_{1nl}\end{aligned}$$

$$+\frac{(l+1)(l+2)(n+l+3)}{(2l+3)(2l+5)}b_{1,n,l+2}^{(j-1)}\Big\}-\frac{(n-l+2)l}{(n+1)(2l-1)}\frac{1}{\rho_1}\frac{\partial\rho_1}{\partial x}b_{1,n+1,l-1}^{(j-1)}$$
$$-\frac{(n+l+3)(l+1)}{(n+1)(2l+3)}\frac{1}{\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{1,n+1,l+1}^{(j-1)}). \tag{34}$$

与上节的讨论相似，为消去久期项，只须选择 $\beta_{1nl}^{(0,j)}=\beta_{1nl}^{(1,j-1)}=\cdots=\beta_{1nl}^{(j-2,2)}=0$. 当 $l\neq 0$ 及 1 时，令

$$\beta_{1nl}^{(j-1,1)}=-c_1\frac{\partial b_{1nl}^{(j-1)}}{\partial x},\qquad \beta_{1nl}^{(j,0)}+\lambda_{nl}\rho_1 b_{1nl}^{(j)}=R_{1nl}^{(j)},$$
$$R_{1nl}^{(j)}=-\frac{n(l+1)}{2l+3}\theta_1\frac{\partial b_{1,n-1,l+1}^{(j-1)}}{\partial x}-\frac{(n-l+2)l}{(n+1)(2l-1)}\frac{\partial b_{1,n+1,l-1}^{(j-1)}}{\partial x}+\widetilde{\beta}_{1nl}^{(j)}.$$

当 $l=0$ 或 1，$n\geq 4$, 令

$$\beta_{1,n,0}^{(j-1,1)}=-c_1\frac{\partial b_{1,n,0}^{(j-1)}}{\partial x}-\frac{n}{3}\theta_1\frac{\partial b_{1,n-1,1}^{(j-1)}}{\partial x}-\frac{n}{6}\frac{\partial\theta_1}{\partial x}b_{1,n-1,1}^{(j-1)},$$
$$\beta_{1,n-1,1}^{(j-1,1)}=-c_1\frac{\partial b_{1,n-1,1}^{(j-1)}}{\partial x}-\frac{\partial b_{1,n,0}^{(j-1)}}{\partial x}-\frac{1}{2\theta_1}\Big(\frac{\partial\theta_1}{\partial t}+c_1\frac{\partial\theta_1}{\partial x}\Big)b_{1,n-1,1}^{(j-1)},$$

而 $\beta_{1,n,0}^{(j,0)}+\lambda_{n,0}\rho_1 b_{1,n,0}^{(j)}=R_{1,n,0}^{(j)}$, $\beta_{1,n-1,1}^{(j,0)}+\lambda_{n,0}\rho_1 b_{1,n-1,1}^{(j)}=R_{1,n-1,1}^{(j)}$, 其中

$$R_{1,n,0}^{(j)}=\frac{n}{6}\frac{\partial\theta_1}{\partial x}b_{1,n-1,1}^{(j-1)}+\widetilde{\beta}_{1,n,0}^{(j)},$$
$$R_{1,n-1,1}^{(j)}=-\frac{2(n-1)}{5}\theta_1\frac{\partial b_{1,n-2,2}^{(j-1)}}{\partial x}+\frac{1}{2\theta_1}\Big(\frac{\partial\theta_1}{\partial t}+c_1\frac{\partial\theta_1}{\partial x}\Big)b_{1,n-1,1}^{(j-1)}+\widetilde{\beta}_{1,n-1,1}^{(j)}.$$

于是 (20) 式成为

$$\frac{\partial b_{1nl}^{(j)}}{\partial\tau}=\beta_{1nl}^{(j,0)}+\varepsilon\beta_{1nl}^{(j,1)},\qquad j=0,1,2,\cdots. \tag{35}$$

它是关于 $b_{1nl}^{(j)}$ 的精确方程. 在初始层讨论中，(35) 式起基本作用. 下面讨论这个方程的解法.

当 $l\neq 0$ 及 1 时，(35) 式就是

$$\frac{\partial b_{1nl}^{(j)}(x,\tau)}{\partial\tau}+\varepsilon c_1(x,\tau)\frac{\partial b_{1nl}^{(j)}(x,\tau)}{\partial x}+\lambda_{nl}\rho_1(x,\tau)b_{1nl}^{(j)}(x,\tau)=R_{1nl}^{(j)}(x,\tau), \tag{36}$$

它的特征线为

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=\varepsilon c_1(x,\tau). \tag{37}$$

记 $x(\tau=0)=x_0$, 那么给定 x_0 之后由 (37) 式可以求得 $x=x(x_0,\tau)$, 由此反解 $x_0=x_0(x,\tau)$, 于是 (36) 式的解可以写成

$$b_{1nl}^{(j)}(x,\tau)=b_{1nl}^{(j)}(x_0,0)e^{-\gamma_{nl}}+e^{-\gamma_{nl}}\int_0^\tau e^{\gamma_{nl'}}R_{1nl}^{(j)}(x(x_0,\tau'),\tau')\,\mathrm{d}\tau', \tag{38}$$

其中

$$\gamma_{nl}(x_0,\tau)=\lambda_{nl}\int_0^{\tau}\rho_1(x(x_0,\tau^{'}),\tau^{'})\,\mathrm{d}\tau^{'},\quad \gamma_{nl^{'}}=\gamma_{nl}(x_0,\tau^{'})=\lambda_{nl}\int_0^{\tau^{'}}\rho_1(x(x_0,\tau^{''}),\tau^{''})\,\mathrm{d}\tau^{''}.$$

当 $l=0$ 或 1，$n\geq 4$ 时，(35) 式就是

$$\begin{aligned}&\frac{\partial b_{1,n,0}^{(j)}}{\partial\tau}+\varepsilon c_1\frac{\partial b_{1,n,0}^{(j)}}{\partial x}+\frac{\varepsilon n}{3}\theta_1\frac{\partial b_{1,n-1,1}^{(j)}}{\partial x}+\frac{\varepsilon n}{6}\frac{\partial\theta_1}{\partial x}b_{1,n-1,1}^{(j)}+\lambda_{n,0}\rho_1 b_{1,n,0}^{(j)}=R_{1,n,0}^{(j)},\\&\frac{\partial b_{1,n-1,1}^{(j)}}{\partial\tau}+\varepsilon c_1\frac{\partial b_{1,n-1,1}^{(j)}}{\partial x}+\varepsilon\frac{\partial b_{1,n,0}^{(j)}}{\partial x}+\frac{\varepsilon}{2\theta_1}\Big(\frac{\partial\theta_1}{\partial t}+c_1\frac{\partial\theta_1}{\partial x}\Big)b_{1,n-1,1}^{(j)}\\&+\lambda_{n,0}\rho_1 b_{1,n-1,1}^{(j)}=R_{1,n-1,1}^{(j)}.\end{aligned}\tag{39}$$

这是互相耦合的两个方程．令

$$u_{1,n,1}^{(j)}=b_{1,n,0}^{(j)}-\sqrt{\frac{n\theta_1}{3}}\,b_{1,n-1,1}^{(j)},\qquad u_{1,n,2}^{(j)}=b_{1,n,0}^{(j)}+\sqrt{\frac{n\theta_1}{3}}\,b_{1,n-1,1}^{(j)},$$

那么 (39) 式可以改写成两个独立的方程

$$\frac{\partial u_{1,n,1}^{(j)}}{\partial\tau}+\varepsilon\Big(c_1-\sqrt{\frac{n\theta_1}{3}}\Big)\frac{\partial u_{1,n,1}^{(j)}}{\partial x}+\lambda_{n,0}\rho_1 u_{1,n,1}^{(j)}=R_{1,n-}^{(j)},\tag{40}$$

$$\frac{\partial u_{1,n,2}^{(j)}}{\partial\tau}+\varepsilon\Big(c_1+\sqrt{\frac{n\theta_1}{3}}\Big)\frac{\partial u_{1,n,2}^{(j)}}{\partial x}+\lambda_{n,0}\rho_1 u_{1,n,2}^{(j)}=R_{1,n+}^{(j)},\tag{41}$$

其中

$$R_{1,n-}^{(j)}=R_{1,n,0}^{(j)}-\sqrt{\frac{n\theta_1}{3}}R_{1,n-1,1}^{(j)},\qquad R_{1,n+}^{(j)}=R_{1,n,0}^{(j)}+\sqrt{\frac{n\theta_1}{3}}R_{1,n-1,1}^{(j)}.\tag{42}$$

(40) 和 (41) 式也可以精确求解．(40) 式的特征线为

$$\frac{\partial x(x_0^-,\tau)}{\partial\tau}=\varepsilon\Big(c_1-\sqrt{\frac{n\theta_1}{3}}\Big),\qquad x(x_0^-,0)=x_0^-.\tag{43}$$

如果由 (41) 式确定 $x_0^-=x_0^-(x,\tau)$, 那么 (40) 式的解为

$$u_{1,n,1}^{(j)}=u_{1,n,1}^{(j)}(x_0^-,0)e^{-\gamma_{n,0}^-}+e^{-\gamma_{n,0}^-}\int_0^{\tau}e^{\gamma_{n,0}^{-'}}R_{1,n-}^{(j)}(x(x_0^-,\tau^{'}),\tau^{'})\,\mathrm{d}\tau^{'},\tag{44}$$

其中

$$\begin{aligned}&\gamma_{n,0}^-=\lambda_{n,0}\int_0^{\tau}\rho_1(x(x_0^-,\tau^{'}),\tau^{'})\,\mathrm{d}\tau^{'},\\&\gamma_{n,0}^{-'}=\gamma_{n,0}^-(x_0^-,\tau^{'})=\lambda_{n,0}\int_0^{\tau^{'}}\rho_1(x(x_0^-,\tau^{''}),\tau^{''})\,\mathrm{d}\tau^{''}.\end{aligned}\tag{45}$$

(41) 式的特征线为

$$\frac{\partial x(x_0^+,\tau)}{\partial\tau}=\varepsilon\Big(c_1+\sqrt{\frac{n\theta_1}{3}}\Big),\qquad x(x_0^+,0)=x_0^+.\tag{46}$$

如果由 (46) 式确定 $x_0^+ = x_0^+(x,\tau)$, 那么 (41) 式的解为

$$u_{1,n,2}^{(j)} = u_{1,n,2}^{(j)}(x_0^+,0)e^{-\gamma_{n,0}^+} + e^{-\gamma_{n,0}^+}\int_0^\tau e^{\gamma_{n,0}^{+'}} R_{1,n+}^{(j)}(x(x_0^+,\tau'),\tau')\,\mathrm{d}\tau', \tag{47}$$

其中

$$\begin{aligned} \gamma_{n,0}^+ &= \lambda_{n,0}\int_0^\tau \rho_1(x(x_0^+,\tau'),\tau')\,\mathrm{d}\tau', \\ \gamma_{n,0}^{+'} &= \gamma_{n,0}^+(x_0^+,\tau') = \lambda_{n,0}\int_0^{\tau'} \rho_1(x(x_0^+,\tau''),\tau'')\,\mathrm{d}\tau''. \end{aligned} \tag{48}$$

在求得 $u_{1,n,1}^{(j)}, u_{1,n,2}^{(j)}$ 之后，同样也可求得

$$b_{1,n,0}^{(j)} = \frac{1}{2}(u_{1,n,1}^{(j)} + u_{1,n,2}^{(j)}), \qquad b_{1,n-1,1}^{(j)} = \frac{1}{2}\sqrt{\frac{3}{n\theta_1}}\,(u_{1,n,2}^{(j)} - u_{1,n,1}^{(j)}). \tag{49}$$

由 (37), (43) 和 (46) 式看出，非简并矩传播的速度就是流体速度 c_1, 而简并的矩传播的速度是 $c_1 \pm \sqrt{\frac{n\theta_1}{3}}$. 这说明存在着高阶矩声波，在静止的气体之中传播的速度是 $\pm\sqrt{\frac{n\theta_1}{3}}$. 当然，在传播的同时，它的衰减是很快的.

5 初始层解与正规解的连接

由 (9) 式可知， $b_{1nl}^{(j)}$ 应当满足

$$\lim_{\tau\to\infty} b_{1nl}^{(j)}(x,\tau) = 0. \tag{50}$$

当 $(n,l) = (2,2),(3,1),\cdots$ 时， (50) 式已经得到保证，因为它们都有 $e^{-\lambda\int_0^\tau \rho_1\,\mathrm{d}\tau'}$ 这样的因子. 由 (16) 及 (18) 式可知， $b_{1nl}^{(j)}$ 的初条件应当写成

$$b_{1nl}^{(j)}(x,0) = -a_{1nl}^{(j)}(x,0), \qquad j=1,2,\cdots; \quad (n,l) = (2,2),(3,1),\cdots. \tag{51}$$

由于 ρ_1, c_1, θ_1 的初条件给定之后 $a_{1nl}^{(j)}(x,0)\big[(n,l) = (2,2),(3,1),\cdots\big]$ 也随着给定，因此相应地由 (51) 式也给定了 $b_{1nl}^{(j)}(x,0)[(n,l) = (2,2),(3,1),\cdots]$. 对于 $(n,l) = (0,0),(1,1),(2,0)$ 三个矩来说，情况完全不同. 由于 $\lambda_{00} = \lambda_{11} = \lambda_{20} = 0$, 所以，以 (27) 式为例，只有当 $\beta_{1nl}^{(j,0)}$ 之中有不带如 $e^{-\lambda\int_0^\tau \rho_1\,\mathrm{d}\tau'}$ 这样指数因子的项时才会出现久期项，但 (27) 式的右边不存在这样的项，因此在保持 $\beta_{1nl}^{(0,j)} = \beta_{1nl}^{(1,j-1)} = \cdots = \beta_{1nl}^{(j-2,2)} = 0$ 的同时还可以令

$$\beta_{100}^{(0,1)} = \beta_{111}^{(0,1)} = \beta_{120}^{(0,1)} = 0. \tag{52}$$

所以， (20) 式成为

$$\frac{\partial b_{1nl}^{(0)}}{\partial\tau} = \beta_{1nl}^{(0,0)}, \qquad (n,l) = (0,0),(1,1),(2,0),\cdots. \tag{53}$$

考虑到 (23) 式及 $b_{100}^{(0)}(x,0)=b_{111}^{(0)}(x,0)=b_{120}^{(0)}(x,0)=0$, 故

$$b_{1nl}^{(0)}(x,\tau)=0, \qquad (n,l)=(0,0),(1,1),(2,0). \tag{54}$$

类似地可以知道，取

$$\beta_{1nl}^{(j-1,1)}=0, \qquad (n,l)=(0,0),(1,1),(2,0);\quad j=1,2,\cdots. \tag{55}$$

不会造成久期项，因此关于 $b_{1nl}^{(j)}[(n,l)=(0,0),(1,1),(2,0)]$ 的精确方程为

$$\frac{\partial b_{1nl}^{(j)}}{\partial\tau}=\beta_{1nl}^{(j,0)}=0, \qquad (n,l)=(0,0),(1,1),(2,0);\quad j=1,2,\cdots. \tag{56}$$

当 $j=1$ 时，由 (27) 及 (54) 式可以得到

$$\begin{aligned}&\frac{\partial b_{100}^{(1)}}{\partial\tau}=0, \qquad \frac{\partial b_{111}^{(1)}}{\partial\tau}=-\frac{1}{\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{122}^{(0)}),\\ &\frac{\partial b_{120}^{(1)}}{\partial\tau}=-\frac{5}{9\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{131}^{(0)})-\frac{2}{3}\frac{\partial c_1}{\partial x}b_{122}^{(0)}.\end{aligned} \tag{57}$$

考虑到 (16) 式，有

$$b_{1nl}^{(1)}(x,0)=-a_{1nl}^{(1)}(x,0), \qquad (n,l)=(0,0),(1,1),(2,0). \tag{58}$$

从而得到

$$\begin{aligned}b_{100}^{(1)}(x,\tau)=&-a_{100}^{(1)}(x,0),\\ b_{111}^{(1)}(x,\tau)=&-a_{111}^{(1)}(x,0)-\int_0^\tau\frac{1}{\rho_1(\tau')}\frac{\partial}{\partial x}\left[\rho_1(\tau')b_{122}^{(0)}(x,\tau')\right]\mathrm{d}\tau',\\ b_{120}^{(1)}(x,\tau)=&-a_{120}^{(1)}(x,0)-\int_0^\tau\frac{5}{9\rho_1(\tau')}\frac{\partial}{\partial x}\left[\rho_1(\tau')b_{131}^{(0)}(x,\tau')\right]\mathrm{d}\tau'\\ &-\int_0^\tau\frac{2}{3}\frac{\partial c_1(x,\tau')}{\partial x}b_{122}^{(0)}(x,\tau')\,\mathrm{d}\tau'.\end{aligned} \tag{59}$$

考虑到式 (50) 得知

$$\begin{aligned}&a_{100}^{(1)}(x,0)=0, \qquad a_{111}^{(1)}(x,0)=-\int_0^\infty\frac{1}{\rho_1(\tau')}\frac{\partial}{\partial x}\left[\rho_1(\tau')b_{122}^{(0)}(x,\tau')\right]\mathrm{d}\tau',\\ &a_{120}^{(1)}(x,0)=-\int_0^\infty\frac{5}{9\rho_1(\tau')}\frac{\partial}{\partial x}\left[\rho_1(\tau')b_{131}^{(0)}(x,\tau')\right]\mathrm{d}\tau'\\ &\qquad\qquad-\int_0^\infty\frac{2}{3}\frac{\partial c_1(x,\tau')}{\partial x}b_{122}^{(0)}(x,\tau')\,\mathrm{d}\tau'.\end{aligned} \tag{60}$$

一般地，

$$\begin{aligned}\beta_{100}^{(j,0)}=&-c_1\frac{\partial b_{100}^{(j-1)}}{\partial x}-\frac{1}{\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{111}^{(j-1)}),\\ \beta_{111}^{(j,0)}=&-c_1\frac{\partial b_{111}^{(j-1)}}{\partial x}-\frac{\partial b_{120}^{(j-1)}}{\partial x}-q_1 b_{100}^{(j-2)}-\theta_1\frac{\partial b_{100}^{(j-1)}}{\partial x}\end{aligned}$$

$$-\frac{\partial c_1}{\partial x}b_{111}^{(j-1)}-\frac{1}{\rho_1}\frac{\partial \rho_1}{\partial x}b_{120}^{(j-1)}-\frac{1}{\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{122}^{(j-1)}), \tag{61}$$

$$\beta_{120}^{(j,0)}=-\frac{2\theta_1}{3}\frac{\partial b_{111}^{(j-1)}}{\partial x}-c_1\frac{\partial b_{120}^{(j-1)}}{\partial x}-\frac{2}{3}q_1 b_{111}^{(j-2)}-s_1 b_{100}^{(j-2)}-\frac{\partial \theta_1}{\partial x}b_{111}^{(j-1)}$$
$$-\frac{2}{3}\frac{\partial c_1}{\partial x}(b_{120}^{(j-1)}+b_{122}^{(j-1)})-\frac{5}{9\rho_1}\frac{\partial}{\partial x}(\rho_1 b_{131}^{(j-1)}).$$

所以 (56) 式的解为

$$b_{1nl}^{(j)}(x,\tau)=-\int_\tau^\infty \beta_{1nl}^{(j,0)}d\tau', \qquad (n,l)=(0,0),(1,1),(2,0);\quad j=1,2,3,\cdots. \tag{62}$$

而正规解中待定的初条件也被确定

$$a_{1nl}^{(j)}(x,0)=\int_0^\infty \beta_{1nl}^{(j,0)}\,\mathrm{d}\tau', \qquad (n,l)=(0,0),(1,1),(2,0);\quad j=1,2,3,\cdots. \tag{63}$$

从 (51) 及 (63) 式看出，当 $(n,l)=(2,2),(3,1),\cdots$ 时，是由正规解的需要决定初始层解的初条件 (除 $j=0$ 之外), 而当 $(n,l)=(0,0),(1,1),(2,0)$ 时，恰好相反，是由初始层的需要 (50) 式来决定正规解的初条件.

6 注记

在讨论二粒子 Boltzmann 方程组的初始层解时，使用了双时标度法．它相当于用分子的平均自由飞行时间作为单位来描述系统分布函数的变化．在初始层的讨论中，我们看到线性化的碰撞算子 I 的谱具有决定的意义．它决定了各阶矩衰减的快慢程度，这一点从初级近似的讨论中可以看出．在空间均匀的情况下也是如此．在空间不均匀的情况下， I 的谱是否有简并这一点显得十分重要．当发生简并时，相应的矩就会以一定的速度向外传播，类似于声波那样，只是衰减的速度很快．我们知道 Maxwell 分子的情况下 I 的本征值发生的简并是 “偶然简并”，所以这种高阶矩声波也许只是 Maxwell 分子所特有的现象．由于对非 Maxwell 分子的谱没有详细地了解，因此还不知道在它们中间是否也有可能发生这种波.

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On the Singular Perturbation Solution of Two-particle Boltzmann Equations: Initial Layer Solution

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Abstract The initial layer solution of the Boltzmann Hierarchy for two-particles is discussed in this article. By using the method of The Singular Perturbation Solution, we formulate the Boltzmann Hierarchy with Fourier transform, and then get the normal solution and initial solution. In addition,the primary and high-order approximation of the initial layer solution is obtained and the connection between the normal solution and initial layer solution is given.

Key words Boltzmann Hierarchy; normal solution; initial layer solution

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