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Nonlinear self-adjointness and conservation laws of forced KdV equation

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Abstract: Using the concept of nonlinear self-adjointness and the general theorem on conservation laws developed by Ibragimov, nonlinear self-adjointness and conservation laws for the forced KdV equation are investigated. Its self-adjointness was disscussed firstly, and it's found that the forced KdV equation is nonlinearly self-adjoint. At the same time, formal Lagrangian for the equation is obtained. Having performed Lie symmetry analysis for the equation, lots of nontrivial conservation laws for the equation were derived according to the difference of Lie symmetries.

Key words: nonlinear equation; forced KdV equation; conservation laws; Lie symmetry; formal Lagrangian

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带强迫项 KdV 方程的非线性自伴随性和守恒律

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摘 要： 应用非线性自伴随性的概念和伊布拉基莫夫的一般守恒律定理，研究了带强迫 KdV 方程的非线性自伴随性和守恒律。首先讨论了自伴随性，结果表明这个方程具有非线性自伴随性，同时得到了这个方程的形式拉格朗日量。在对此方程进行李对称分析后，根据李对称的不同得到了此方程的一些非平凡守恒律。

关键词： 非线性方程；带强迫项的 KdV 方程；守恒律；李对称；形式拉格朗日量

1 Introduction

Conservation laws play important roles in the study of differential equations in mathematical physics[1,2]. Explicit forms of conservation laws are also used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness and stability analysis[3]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. To search for explicit conservation laws of nonlinear partial differential equations (PDEs), a number of methods have been presented, such as Noether's Theorem[4], multiplier approach[5,6], partial

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Noether approach[7] and so on[8∼17]. Among those, the new conservation theorem given by Ibragimov[8] is one of the most frequently used methods[18,19].

Based on the concept of adjoint equation for a given differential equation[9], Ibragimov gives a general conservation theorem by which conservation laws for the system consisting of the given equation and its adjoint equation can be obtained. In fact, we are only interested in the conservation laws for the given equation. Therefore one has to eliminate the nonlocal variable which is introduced in the adjoint equation. For a self-adjoint nonlinear equation, its adjoint equation is equivalent with the original equation after replacing the nonlocal variable with the dependent variable in the original equation. However, many equations, which have remarkable symmetry properties and physical significance, are not self-adjoint. Thus the nonlocal variables of these equations cannot be eliminated easily. To solve this problem, Ibragimov has introduced the concept of nonlinear self-adjointness, which extends the self-adjointness to the most generalized meaning.

The KdV equation is the most popular soliton equation and has been extensively investigated. In many geophysical and marine applications it is necessary to include a forcing term. Typical examples are when the waves are generated by moving ships, or by flow over bottom topography. In this paper, we consider the forced KdV equation[20,21]

$$E_1 \equiv u_t + cu_x + \alpha uu_x + \beta u_{xxx} - F(t) = 0, \tag{1}$$

where c, α and β are constants, $F(t)$ is a smooth function. Bilinear form and multiple-soliton solutions for Eq.(1) have been obtained in Ref.[21]. To our best knowledge, nonlinear self-adjointness, Lie symmetries and conservation laws of Eq.(1) have not been discussed up to now.

The rest of the paper is organized as follows. In section 2, we introduce the main notations and theorems used in this paper. In section 3, we first discuss the nonlinear self-adjointness for the forced KdV equation (1) and get its formal Lagrangian. In section 4, after performing Lie symmetry analysis, nontrivial conservation laws of Eq.(1) are derived making use of the obtained formal Lagrangian and Lie symmetries. Some conclusions and discussions are given in the last section.

2 Preliminaries

In this section, we briefly present the main notations and theorems used in this paper. Consider a sth-order nonlinear evolution equation

$$F(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)}) = 0, \tag{2}$$

with n independent variables $x = (x_1, x_2, \cdots, x_n)$ and a dependent variable u, where $u_{(1)}, u_{(2)}, \cdots, u_{(s)}$ denote the collection of all first, second, $\cdots$, sth-order partial derivatives. $u_i = D_i(u), u_{ij} = D_jD_i(u), \cdots$. Here

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \cdots, n,$$

is the total differential operator with respect to x_i.

Definition 1(see Ref.[9])

The adjoint equation of Eq.(2) is defined by

$$F^*(x,u,v,u_{(1)},v_{(1)},u_{(2)},v_{(2)},\cdots,u_{(s)}v_{(s)})=0, \tag{3}$$

with $\quad F^*(x,u,v,u_{(1)},v_{(1)},u_{(2)},v_{(2)},\cdots,u_{(s)}v_{(s)})=\dfrac{\delta(vF)}{\delta u},$

where $\quad \dfrac{\delta}{\delta u}=\dfrac{\partial}{\partial u}+\sum\limits_{m=1}^{\infty}(-1)^m D_{i_1}\cdots D_{i_m}\dfrac{\partial}{\partial u_{i_1 i_2\cdots i_m}}$

denotes the Euler-Lagrange operator, v is a new dependent variable, $v=v(x)$.

Definition 2(see Ref.[8])

Eq.(2) is said to be self-adjoint if the equation obtained from the adjoint equation(3) by the substitution $v=u$:

$$F^*(x,u,u,u_{(1)},u_{(1)},u_{(2)},u_{(2)},\cdots,u_{(s)}u_{(s)})=0$$

is identical with the original equation (2). In other words, Eq.(2) is self-adjoint if and only if

$$F^*(x,u,u,u_{(1)},u_{(1)},u_{(2)},u_{(2)},\cdots,u_{(s)}u_{(s)})=\lambda(x,u,u_{(1)},u_{(2)},\cdots)F(x,u,u_{(1)},u_{(2)},\cdots,u_{(s)}),$$

where λ is an undetermined coefficient.

Definition 3 (see Ref.[16])

Eq.(2) is said to be nonlinear self-adjoint if its adjoint equation (3) is satisfied for all solutions of Eq.(2) upon a substitution

$$v=\phi(x,u),\qquad \phi(x,u)\neq 0.$$

Theorem 1 (see Ref.[9])

The system consisting of Eq.(2) and its adjoint equation (3)

$$\begin{cases} F(x,u,u_{(1)},u_{(2)},\cdots,u_{(s)})=0\\ F^*(x,u,v,u_{(1)},v_{(1)},u_{(2)},v_{(2)},\cdots,u_{(s)}v_{(s)})=0\end{cases} \tag{4}$$

has a formal Lagrangian, namely

$$L=vF(x,u,u_{(1)},u_{(2)},\cdots,u_{(s)}). \tag{5}$$

In the following we recall the “ new conservation theorem ” given by Ibragimov in Ref.[8].

Theorem 2

Any Lie point, Lie-Backlund and non-local symmetry

$$V=\xi^i\frac{\partial}{\partial x_i}+\eta\frac{\partial}{\partial u} \tag{6}$$

of Eq.(2) provides a conservation law $D_i(T^i)=0$ for the system (4). The conserved vector is given by

$$\begin{aligned}T^i=\ &\xi^i L+W\Big[\frac{\partial L}{\partial u_i}-D_j\Big(\frac{\partial L}{\partial u_{ij}}\Big)+D_jD_k\Big(\frac{\partial L}{\partial u_{ijk}}\Big)-D_jD_kD_r\Big(\frac{\partial L}{\partial u_{ijk_r}}\Big)+\cdots\Big]+\\ &D_jW\Big[\frac{\partial L}{\partial u_{ij}}-D_k\Big(\frac{\partial L}{\partial u_{ijk}}\Big)+D_kD_r\Big(\frac{\partial L}{\partial u_{ijk_r}}\Big)-\cdots\Big]+\\ &D_jD_kW\Big[\frac{\partial L}{\partial u_{ijk}}-D_r\Big(\frac{\partial L}{\partial u_{ijk_r}}\Big)+\cdots\Big],\end{aligned} \tag{7}$$

where L is determined by Eq.(5), W is the Lie characteristic function and

$$W = \eta - \xi^j u_j.$$

3 Nonlinear self-adjointness and conservation laws

To search for conservation laws of the forced KdV equation (1) by Theorem 2, Lie symmetry and formal Lagrangian of Eq.(1) must be known. We first construct its adjoint equation. According to Definition 1, the adjoint equation of Eq.(1) is

$$E_1^* \equiv v_t + cv_x + \alpha u v_x + \beta v_{xxx} = 0, \tag{8}$$

where v is a new dependent variable with respect to x and t.

According to Theorem 1, the formal Lagrangian for the system consisting of Eq.(1) and its adjoint equation (8) is

$$L = v[u_t + cu_x + \alpha u u_x + \beta u_{xxx} - F(t)]. \tag{9}$$

According to Definition 3, we recall that the forced KdV equation (1) is nonlinearly self-adjoint if its adjoint equation (8) coincides with Eq.(1) after the following substitution

$$v = \phi(x, t, u), \tag{10}$$

where $\phi(x, t, u)$ is a nonzero and smooth function. In other words, Eq.(1) is nonlinearly self-adjoint if and only if

$$E_1^*|_{v=\phi(x,t,u)} = \lambda(x, t, u, u_x, u_t, u_{xx}, \cdots)E_1, \tag{11}$$

where λ is an undetermined and smooth function.

The substitution of the expressions of E_1 and E_1^* into (11) results in the following equation

$$\begin{aligned}(\phi_u - \lambda)u_t \ &+\beta(\phi_u - \lambda)u_{xxx} + \phi_t + c\phi_x + c\phi_u u_x + \alpha u\phi_x + \alpha u\phi_u u_x + \\ &\beta\phi_{xxx} + 3\beta\phi_{xxu}u_x + 3\beta\phi_{xuu}u_x^2 + 3\beta\phi_{xu}u_{xx} + \beta\phi_{uuu}u_x^3 + \\ &3\beta\phi_{uu}u_x u_{xx} - \lambda c u_x - \lambda\alpha u u_x + \lambda F(t) = 0.\end{aligned} \tag{12}$$

Solving the above system with the aid of Maple, the final results read as

$$\lambda = C_1 t + C_2,$$

$$\phi = C_1\Big[tu - \frac{x}{\alpha} + \frac{ct}{\alpha} - \int tF(t)\mathrm{d}t\Big] + C_2\Big[u - \int F(t)\mathrm{d}t\Big] + C_3,$$

where C_1, C_2 and C_3 are integral constants, and

$$C_1^2 + C_2^2 + C_3^3 \neq 0.$$

Obviously, $\phi \neq 0$. In summary, we have the following statements.

Theorem 3

The forced KdV equation (1) is nonlinearly self-adjoint.

Corollary 1

The formal Lagrangian of Eq.(1) reads as

$$L=\Big\{C_1\Big[tu-\frac{x}{\alpha}+\frac{ct}{\alpha}-\int tF(t)\mathrm{d}t\Big]+C_2\Big[u-\int F(t)\mathrm{d}t\Big]+C_3\Big\}[u_t+cu_x+\alpha uu_x+\beta u_{xxx}-F(t)]. \quad (13)$$

Remark 1

When the formal Lagrangian has the form of (13), the adjoint equation of Eq.(1) expressed by Eq.(8) is equivalent with Eq.(1).

4 Lie symmetries and conservation laws

4.1 Lie symmetry analysis of Eq.(1)

In this section, we first perform Lie symmetry analysis for the force KdV equation 1) using the classical Lie group approach. Suppose that the Lie symmetry of Eq.(1) is as follows

$$V=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}, \quad (14)$$

where ξ,τ and η are undetermined functions with respect to x,t and u. According to the procedures of Lie group method, the undetermined functions ξ,τ and η must satisfy the following invariant condition

$$\eta^t+c\eta^x+\alpha u_x\eta+\alpha u\eta^x+\beta\eta^{xxx}-\tau F'(t)=0, \quad (15)$$

where

$$\eta^x=D_x(\eta-\xi u_x-\tau u_t)+\xi u_{xx}+\tau u_{xt},$$

$$\eta^t=D_t(\eta-\xi u_x-\tau u_t)+\xi u_{xt}+\tau u_{tt},$$

$$\eta^{xxx}=D_{xxx}(\eta-\xi u_x-\tau u_t)+\xi u_{xxxx}+\tau u_{xxxt}.$$

Substituting the expressions of η^x,η^t and η^{xxx} into (15) with u be a solution of Eq.(1), i.e.

$$u_t=-cu_x-\alpha uu_x-\beta u_{xxx}+F(t).$$

We obtain a system of over-determined differential equations with respect to ξ,τ and η. Solving the system of over-determined differential equations with the aid of Maple, we get the following cases.

Case I $F(t)$ is arbitrary,

$$V=V_1+V_2,\quad V_1=\frac{\partial}{\partial x},\quad V_2=t\frac{\partial}{\partial x}+\frac{1}{\alpha}\frac{\partial}{\partial u}.$$

Case II $F(t)=H,H$ is a constant,

$$V_3=\frac{\partial}{\partial t}.$$

Case III $F(t)=t^{-\frac{5}{3}}$,

$$V_4=\frac{x}{3}\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\Big(\frac{2u}{3}+\frac{2c}{3\alpha}\Big)\frac{\partial}{\partial u}.$$

The above Lie symmetries can be used to get the similarity reduction solutions of Eq.(1). For example, making use of V_2, an exact solution of Eq.(1) is

$$u=\frac{x}{\alpha t}+\theta(t),$$

where $\theta(t)$ satisfies

$$\alpha t\theta'(t)+c+\theta(t)\alpha-\alpha tF(t)=0. \tag{16}$$

Solving (16), we can obtain the following similarity reduction solution of Eq.(1)

$$u=\frac{x}{\alpha t}+\frac{1}{t}\Big(\int\frac{-c+\alpha tF(t)}{\alpha}\mathrm{d}t+C_4\Big),$$

here C_4 is an integral constant. Other similarity reduction solutions of Eq.(1) are omitted here, since searching for exact solutions of Eq.(1) is not the main purpose of this paper.

4.2 Conservation laws of Eq.(1)

Through analysis of self-adjointness, the adjoint equation (8) of the forced KdV equation (1) has become equivalent with the original equation (1). Using the formal Lagrangian and Lie symmetries of Eq.(1), conservation laws for Eq.(1) can be obtained by Theorem 2 and they are listed as follows.

Case I For the first Lie symmetry V_1, its Lie characteristic function has the form $W=-u_x$, the conservation laws for Eq.(1) associated with is

$$\begin{aligned}X=&-C_2\int F(t)\mathrm{d}tu_t-c_2uF(t)+C_2uu_t+C_2\int F(t)\mathrm{d}tF(t)-C_1\int tF(t)\mathrm{d}t(u_t-F(t))-\\&\frac{cC_1tF(t)}{\alpha}+\frac{cC_1tu_t}{\alpha}+C_1tuu_t-C_1tuF(t)-\frac{C_1xu_t}{\alpha}+\frac{C_1xF(t)}{\alpha}+C_3(u_t-F(t))-\frac{C_1u_{xx}\beta}{\alpha},\\T=&-u_xC_1tu+\frac{C_1xu_x}{\alpha}-\frac{cC_1tu_x}{\alpha}+u_xC_1\int tF(t)\mathrm{d}t-u_xC_2u+u_xC_2\int F(t)\mathrm{d}t-C_3u_x.\end{aligned}$$

For the second Lie symmetry V_2, its Lie characteristic function has the form $W=\frac{1}{\alpha}-tu_x$, the conservation laws for Eq.(1) associated with V_2 is

$$\begin{aligned}X=&-C_3tF(t)+\frac{cC_3}{\alpha}-C_2u\int F(t)\mathrm{d}t-c_1u\int tF(t)\mathrm{d}t+C_1u^2t+C_3tu_t+\frac{cC_1t^2u_t}{\alpha}-\frac{C_1txu_t}{\alpha}+\\&\frac{2cC_1tu}{\alpha}+C_3u+C_2u^2+\frac{\beta C_2u_{xx}}{\alpha}-\frac{C_1xu}{\alpha}+\frac{cC_2u}{\alpha}+C_2tuu_t-C_1t\int tF(t)\mathrm{d}tu_t-\\&C_2tuF(t)-C_2t\int F(t)\mathrm{d}tu_t-C_1t^2uF(t)+C_1t^2uu_t+C_2t\int F(t)\mathrm{d}tF(t)+\frac{C_1txF(t)}{\alpha}+\\&C_1t\int tF(t)\mathrm{d}tF(t)-\frac{cC_1x}{\alpha^2}+\frac{c^2C_1t}{\alpha^2}-\frac{cC_1\int tF(t)\mathrm{d}t}{\alpha}-\frac{cC_2\int tF(t)\mathrm{d}t}{\alpha}-\frac{cC_1t^2F(t)}{\alpha},\\T=&\frac{c_1tu}{\alpha}-\frac{c_1x}{\alpha^2}+\frac{cC_1t}{\alpha^2}-\frac{C_1\int tF(t)\mathrm{d}t}{\alpha}+\frac{C_2u}{\alpha}-\frac{C_2\int F(t)\mathrm{d}t}{\alpha}+\frac{C_3}{\alpha}-C_1t^2u_xu+\\&\frac{C_1txu_x}{\alpha}-\frac{cC_1t^2u_x}{\alpha}+C_1tu_x\int tF(t)\mathrm{d}t-C_2tuu_x+C_2tu_x\int F(t)\mathrm{d}t-C_3tu_x.\end{aligned}$$

Case II For the Lie symmetry V_3, its Lie characteristic function has the form $W=-u_t$, the conservation laws for Eq.(1) associated with V_3 is

$$X=-cC_3u_t-\beta C_3u_{txx}+\frac{cC_1H_0}{2}t^2u_t-cC_2uu_t+C_1xuu_t-C_2\alpha u^2u_t-C_3\alpha uu_t-\beta C_2u_tu_{xx}-\frac{\beta C_1u_{tx}}{\alpha}+\beta C_2u_{tx}u_x-\beta C_2uu_{txx}-2cC_1tuu_t+\frac{cC_1xu_t}{\alpha}-\frac{C_1c^2tu_t}{\alpha}+cC_2H_0tu_t-\alpha C_1tu^2u_t+\frac{\alpha C_1H_0}{2}t^2u_tu+C_2H_0\alpha tuu_t-\beta C_1tu_tu_{xx}+\beta C_1tu_{tx}u_x-\beta C_1tuu_{txx}+\frac{\beta C_1xu_{txx}}{\alpha}-\frac{\beta C_1ctu_{txx}}{\alpha}+\frac{\beta C_1H_0t^2}{2}u_{txx}+\beta C_2H_0tu_{txx},$$

$$T=C_2H_0^2t+\tfrac{1}{2}C_1H_0^2t^2+C_3\beta u_{xxx}-C_2H_0u+C_3cu_x+\frac{C_1c\beta tu_{xxx}}{\alpha}+C_1\alpha tu^2u_x-\frac{C_1cH_0t}{\alpha}+\frac{C_1H_0x}{\alpha}-C_1H_0tu+C_2cuu_x+C_2\alpha u^2u_x+C_2\beta uu_{xxx}+C_3\alpha uu_x-C_1xuu_x-C_3H_0+2C_1ctuu_x+C_1\beta tuu_{xxx}-\frac{C_1cxu_x}{\alpha}-\frac{C_1\beta xu_{xxx}}{\alpha}+\frac{C_1c^2tu_x}{\alpha}-\frac{1}{2}C_1H_0ct^2u_x-\frac{1}{2}C_1H_0\alpha t^2uu_x-\frac{1}{2}C_1H_0\beta t^2u_{xxx}-C_2H_0ctu_x-C_2H_0\alpha tuu_x-C_2H_0\beta tu_{xxx}.$$

Case III For the Lie symmetry V_4, its Lie characteristic function has the form $W=-\left(\frac{2u}{3}+\frac{2c}{3\alpha}\right)-\frac{x}{3}u_x-tu_t$, the conservation laws for Eq.(1) associated with V_4 is

$$X=-\frac{4}{3}cC_2u^2-\frac{2}{3}\alpha C_3u^2-\frac{4}{3}\beta C_3u_{xx}-\frac{C_3x}{3t^{5/3}}-\frac{2c^2C_3}{3\alpha}+\frac{1}{3}C_3xu_t-\frac{4cC_3}{3}u-\frac{2}{3}C_2\alpha u^3-\frac{C_2x}{2t^{7/3}}+\beta C_2u_x^2-\beta C_2tu_tu_{xx}-\frac{2c\beta C_2u_{xx}}{3\alpha}-\frac{c^2C_2}{\alpha t^{2/3}}-\frac{C_2xu}{3t^{5/3}}+\frac{1}{3}C_2xuu_t+\frac{C_2xu_t}{2t^{2/3}}-\frac{2cC_2u}{t^{2/3}}-\frac{\alpha C_2u^2}{t^{2/3}}-2\beta C_2uu_{xx}-\frac{2c^2C_2u}{3\alpha}-\frac{3}{2}cC_2t^{1/3}u_t-cC_3tu_t-\frac{2\beta C_2u_{xx}}{t^{2/3}}-\frac{3}{2}\beta C_2t^{1/3}u_{txx}-\beta C_3tu_{txx}-\frac{3}{2}\alpha C_2t^{1/3}u_tu-cC_2tu_tu-\alpha C_2tu_tu^2-\alpha C_3tu_tu+\beta C_2tu_xu_{tx}-\beta C_2tuu_{txx},$$

$$T=-\frac{2cC_3}{3\alpha}-\frac{2C_2u}{t^{2/3}}-\frac{1}{3}C_3xu_x+C_2\beta tuu_{xxx}+cC_2tuu_x+C_2\alpha tu^2u_x+\frac{3}{2}cC_2t^{1/3}u_x+\frac{3}{2}C_2\beta t^{1/3}u_{xxx}+C_3ctu_x+C_3\beta tu_{xxx}-\frac{2cC_2u}{3\alpha}-\frac{cC_2}{\alpha t^{2/3}}-\frac{1}{3}C_2xu_xu-\frac{C_2xu_x}{2t^{2/3}}-\frac{3C_2}{2t^{4/3}}-\frac{C_3}{t^{2/3}}-\frac{2}{3}C_2u^2-\frac{2}{3}C_3u+\frac{3}{2}C_2\alpha t^{1/3}uu_x+C_3\alpha tuu_x.$$

Remark 2 All the conservation laws listed here are nontrivial. The correctness of them has been checked by Maple software.

5 Conclusions

Recently, the concept of nonlinear self-adjointness, which extends the self-adjointness to the most generalized meaning, has been introduced in order to find conservation laws of non-self-adjoint differential equations. Through analysis of self-adjointness, it's show that Eq.(1) possesses nonlinear self-adjointness. Making use of the Lie symmetries obtained by Lie symmetry analysis, many nontrivial conservation laws for Eq.(1) are derived. These conservation laws may be useful for the explanation of some practical physical problems. As far as we know, self-adjointness, formal Lagrangian, Lie symmetries and conservation laws for Eq.(1) have not been reported in the existent literature, so they are completely new.

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