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A Cubic Spline Method for Solving a Unilateral Obstacle Problem

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ABSTRACT

This paper, we develop a numerical method for solving a unilateral obstacle problem by using the cubic spline collocation method and the generalized Newton method. This method converges quadratically if a relation-ship between the penalty parameter ε and the discretization parameter *h* is satisfied. An error estimate between the penalty solution and the discret penalty solution is provided. To validate the theoretical results, some numerical tests on one dimensional obstacle problem are presented.

Keywords: Obstacle Problem; Spline Collocation; Nonsmooth Equation; Generalized Newton Method

1. Introduction

Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and let ψ be an element of $H^1(\Omega)$ with $\psi \le 0$ on $\partial\Omega$. Set

$$K = \left\{ v \in H_0^1(\Omega) \middle| v \ge \psi \ a.e. \text{ in } \Omega \right\}.$$

We consider the following variational inequality problem:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla (v-u) \mathrm{d}x + \int_\Omega f (v-u) \mathrm{d}x \ge 0, \forall v \in K, \end{cases} \quad (1)$$

where *f* is an element of $L^2(\Omega)$. This problem is called a unilateral obstacle problem. It is well known that problem (1) admits a unique solution *u*, and if $\Delta\psi \in L^2(\Omega)$, then *u* is an element of $H^2(\Omega)$ (see [1,2]). There are several alternative solution methods of the obstacle problem; see, e.g., [1,3-5]. Numerical solution by penalty methods have been considered, e.g. by [4,6]. In this paper we develop a numerical method for solving a one dimentional obstacle problem by using the cubic spline collocation method and the generalized Newton method. First, problem (1) is approximated by a sequence of nonlinear equation problems by using the penalty method given in [2,7]. Then we apply the spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the cubic spline coefficients of a nonsmooth system $\varphi(Y) = Y$, where $\varphi : \mathbb{R}^m \to \mathbb{R}^m$. In order to solve the nonsmooth equation we apply the generalized Newton method (see [8-10], for instance). We prove that the cubic spline collocation method converges quadratically provided that a property coupling the penalty parameter ε and the discretization parameter *h* is satisfied.

Numerical methods to approximate the solution of boundary value problems have been considered by several authors. We only mention the papers [11,12] and references therein, which use the spline collocation method for solving the boundary value problems.

The present paper is organized as follows. In Section 2, we present the penalty method to approximate the obstacle problem by a sequence of second order boundary value problems. In Section 3 we construct a cubic spline to approximate the solution of the boundary problem. Section 4 is devoted to the presentation of the generalized Newton method. In Section 5 we show the convergence of the cubic spline to the solution of the boundary problem and provide an error estimate. Finally, some numerical results are given in Section 6 to validate our methodology.

2. Penalty Problem

Let ψ be an element of $H^1(\Omega)$ with $\psi \le 0$ on $\partial\Omega$.

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Assume that $\Delta\psi$ is an element of $L^2(\Omega)$, then the solution u of problem (1) is an element of $H^2(\Omega)$ and can be characterized as (see [1], for instance):

$$\begin{cases} -\Delta u + f \geq 0 & a.e. \text{ on } \Omega, \\ (-\Delta u + f)(u-\psi) = 0 & a.e. \text{ on } \Omega, \\ u - \psi \geq 0 & a.e. \text{ on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

The penalty problem is given by the following boundary value problem (see [10], p. 107, [12]):

$$\begin{cases} -\Delta u_\varepsilon = \max(-\Delta\psi + f, 0)\theta_\varepsilon(u_\varepsilon - \psi) - f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

where θ_ε is a sequence of Lipschitz functions which tend to the function θ defined by

$$\theta(t) = \begin{cases} 1 & t \leq 0 \\ 0 & t > 0, \end{cases} \tag{4}$$

almost everywhere on $\mathbb{R}$, as ε goes to zero. Assume that the function $\theta_\varepsilon(t)$, $-\infty < t < +\infty$, is uniformly Lipschitz, non increasing and satisfy $0 \leq \theta_\varepsilon(t) \leq 1$. Then problem (3) admits a unique solution (see [2] p. 107). We now specify the function

$$\theta_\varepsilon(t) = \begin{cases} 1, & t \leq 0, \\ 1 - t/\varepsilon, & 0 \leq t \leq \varepsilon, \\ 0, & t \geq \varepsilon. \end{cases} \tag{5}$$

We have the interesting properties.

Theorem 1 ([2,7]) *Let u denote the solution of the variational inequality problem* (1) *and u_ε, $\varepsilon > 0$, denotes the solution of the penalty problem* (3) *with θ_ε defined by relation* (5). *Then $\{u_\varepsilon\}$ is a nondecreasing sequence and*

$$u(x) \leq u_\varepsilon(x) \leq u(x) + \varepsilon, \ x \in \Omega, \text{ for } \varepsilon > 0.$$

3. Cubic Spline Collocation Method

In this section we construct a cubic spline which approximates the solution u_ε of problem (3), with Ω is the interval $I = (a,b) \subset \mathbb{R}$ and θ_ε is the function given by (5).

Cubic Spline Solution

Let

$$\tau = \{a = x_{-3} = x_{-2} = x_{-1} = x_0 < x_1 < \cdots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = x_{n+3} = b\}$$

be a subdivision of the interval I. Without loss of generality, we put $x_i = a + ih$, where $0 \leq i \leq n$ and $h = \frac{b-a}{n}$.

Denote by $S_4(I,\tau)$ the space of piecewise polynomials of degree 3 over the subdivision τ and of class C^2 everywhere on $[a,b]$. Let B_i, $i = -3, \cdots, n-1$, be the B-splines of degree 3 associated with τ. These B-splines are positives and form a basis of the space $S_4(I,\tau)$. If we put

$$J_\varepsilon(x, u_\varepsilon(x)) = \max(-\Delta\psi(x) + f(x), 0)\theta_\varepsilon(u_\varepsilon(x) - \psi(x)) - f(x), \tag{6}$$

then problem (3) becomes

$$\begin{cases} -\Delta u_\varepsilon = J_\varepsilon(\cdot, u_\varepsilon) & \text{on } I, \\ u_\varepsilon(a) = u_\varepsilon(b) = 0. \end{cases} \tag{7}$$

It is easy to see that J_ε is a nonlinear continuous function on u_ε; and for any two functions u_ε and v_ε, J_ε satisfies the following Lipschitz condition:

$$\left|J_\varepsilon(x, u_\varepsilon(x)) - J_\varepsilon(x, v_\varepsilon(x))\right| \leq L_\varepsilon \left|u_\varepsilon(x) - v_\varepsilon(x)\right| a.e. \text{ on } x \in I, \tag{8}$$

where

$$L_\varepsilon = \frac{1}{\varepsilon}\|-\Delta\psi + f\|_\infty = \frac{1}{\varepsilon}\max_{x\in I}\left|-\Delta\psi(x) + f(x)\right|.$$

Now, we define the following interpolation cubic spline of the solution u_ε of the nonlinear second order boundary value problem (7).

Proposition 2 *Let u_ε be the solution of problem* (7). *Then, there exists a unique cubic spline interpolant $S_\varepsilon \in S_4(I,\tau)$ of u_ε which satisfies*:

$$S_\varepsilon(t_i) = u_\varepsilon(t_i), \quad i = 0, \cdots, n+2,$$

where $t_0 = x_0$, $t_i = \frac{x_{i-1} + x_i}{2}$, $i = 1, \cdots, n$, $t_{n+1} = x_{n-1}$ and $t_{n+2} = x_n$.

Proof Using the Schoenberg-Whitney theorem (see [13]), it is easy to see that there exits a unique cubic spline which interpolates u_ε at the points t_i, $i = 0, \cdots, n+2$.

If we put $S_\varepsilon = \sum_{i=-3}^{n-1} c_{i,\varepsilon} B_i$, then by using the boundary conditions of problem (7) we obtain

$$c_{-3,\varepsilon} = S_\varepsilon(a) = u_\varepsilon(a) = 0, \text{ and}$$

$$c_{n-1,\varepsilon} = S_\varepsilon(b) = u_\varepsilon(b) = 0$$

Hence $S_\varepsilon = \sum_{i=-2}^{n-2} c_{i,\varepsilon} B_i$.

Furthermore, since the interpolation with splines of degree d gives uniform norm errors of order $O(h^{d+1})$ for the interpolant, and of order $O(h^{d+1-r})$ for the rth derivative of the interpolant (see [13], for instance), then for any $u_\varepsilon \in C^4([a,b])$ we have

$$-\Delta S_\varepsilon(t_i) = J_\varepsilon(t_i, u_\varepsilon) + O(h^2), i = 1, \cdots, n+1. \tag{9}$$

The cubic spline collocation method, that we present in this paper, constructs numerically a cubic spline $\tilde{S}_\varepsilon = \sum_{i=-3}^{n-1} \tilde{c}_{i,\varepsilon} B_i$ which satisfies the Equation (7) at the points t_i, $i = 0,\cdots,n+2$. It is easy to see that

$$\tilde{c}_{-3,\varepsilon} = \tilde{c}_{n-1,\varepsilon} = 0,$$

and the coefficients $\tilde{c}_{i,\varepsilon}$, $i = -2,\cdots,n-2$, satisfy the following nonlinear system with $n + 1$ equations:

$$-\sum_{i=-2}^{n-2} \tilde{c}_{i,\varepsilon} \Delta B_i(t_j) = J_\varepsilon\left(t_j, \sum_{i=-2}^{n-2} \tilde{c}_{i,\varepsilon} B_i(t_j)\right), \quad j = 1,\cdots,n+1. \tag{10}$$

Relations (9) and (10) can be written in the matrix form, respectively, as follows

$$\hat{A}C_\varepsilon = -F_\varepsilon - \hat{E}_\varepsilon, \quad \hat{A}\tilde{C}_\varepsilon = -F_{\tilde{C}_\varepsilon}, \tag{11}$$

where

$$F_\varepsilon = \left[J_\varepsilon(t_1, u_\varepsilon(t_1)),\cdots,J_\varepsilon(t_{n+1}, u_\varepsilon(t_{n+1}))\right]^T,$$

$$F_{\tilde{C}_\varepsilon} = [J_\varepsilon(t_1, \tilde{S}(t_1)),\cdots,J_\varepsilon(t_{n+1}, \tilde{S}(t_{n+1}))]^T,$$

and $\hat{E}_\varepsilon$ is a vector where each component is of order $O(h^2)$. It is well known that $\hat{A} = \frac{1}{h^2}A$, where A is a matrix independent of h given as follows:

$$A = \begin{bmatrix} \frac{-15}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \cdots & & & 0 \\ \frac{3}{4} & \frac{-3}{4} & \frac{-1}{2} & \frac{1}{2} & 0 & \cdots & & 0 \\ 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & \cdots & & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{4} & \frac{3}{4} \\ 0 & \cdots & & & 0 & \frac{1}{2} & \frac{1}{4} & \frac{-15}{4} \\ 0 & \cdots & & & 0 & 1 & \frac{-5}{2} & \frac{3}{2} \end{bmatrix}.$$

Then, relation (11) becomes

$$AC_\varepsilon = -h^2 F_\varepsilon - E_\varepsilon, \quad A\tilde{C}_\varepsilon = -h^2 F_{\tilde{C}_\varepsilon}, \tag{12}$$

with E_ε is a vector where each one of its components is of order $O(h^4)$.

The results of this work are basically based on the invertibility of the matrix *A*. Then, in order to prove that *A* is invertible we give the flowing lemma.

Lemma 3 (*de Boor* [13]) *Let* $S \in \mathsf{S}_{k+1}$ *such that* $S = 0$ *on* $[x_{p-1}, x_p] \cup [x_q, x_{q+1}]$ *where* $p < q$. *If S admits r zeros in* $[x_p, x_q]$ *then* $r \le p - q - (k+1)$.

Proposition 4 *The matrix A is invertible.*

Proof Let $D = [d_1,\cdots,d_{n+1}]^T$ be a vector of R^{n+1} such that $AD = 0$. If we put $S(x) = \sum_{j=-2}^{n-2} d_j B_j$, then we have $S(a) = S(b) = 0$ and $\Delta S(t_i) = 0$ for any $i = 1,\cdots,n+1$. Since $S \in S_4(I,\tau)$ then $\Delta S \in S_2(I,\tau)$. If we assume that $\Delta S \ne 0$ in $[x_0, x_n]$, then using the above lemma and the fact that ΔS has $n+1$ zeros in $[x_0, x_n]$, we conclude that $n+1 \le n-2$, which is impossible. Therefore $\Delta S = 0$ for each $x \in I$. This means that the function *S* is a piecewise linear polynomial in *I*. Since $S(a) = S(b) = 0$, then we obtain $S(x) = 0$ for any $x \in I$. Consequently $D = 0$ and the matrix *A* is invertible.

Proposition 5 *Assume that the penalty parameter* ε *and the discretization parameter h satisfy the following relation*:

$$h^2 \left\|-\Delta\psi + f\right\|_\infty \left\|A^{-1}\right\|_\infty < \varepsilon. \tag{13}$$

Then there exists a unique cubic spline which approximates the exact solution u_ε of problem (7).

Proof From relation (12), we have $\tilde{C}_\varepsilon = -h^2 A^{-1} F_{\tilde{C}_\varepsilon}$. Let $\varphi : \mathsf{R}^{n+1} \to \mathsf{R}^{n+1}$ be a function defined by

$$\varphi(Y) = -h^2 A^{-1} F_{\tilde{Y}}. \tag{14}$$

To prove the existence of cubic spline collocation it suffices to prove that φ admits a unique fixed point. Indeed, let Y_1 and Y_2 be two vectors of R^{n+1}. Then we have

$$\left\|\varphi(Y_1) - \varphi(Y_2)\right\| \le h^2 \left\|A\right\|_\infty \left\|F_{Y_1} - F_{Y_2}\right\|_\infty. \tag{15}$$

Using relation (8) and the fact that $\sum_{j=-2}^{n-2} B_j \le 1$, we get

$$\left|J_\varepsilon(t_i, S_{Y_1}(t_i)) - J_\varepsilon(t_i, S_{Y_2}(t_i))\right| \le L_\varepsilon \left|S_{Y_1}(t_i) - S_{Y_2}(t_i)\right| \le L_\varepsilon \left\|Y_1 - Y_2\right\|_\infty,$$

where $L_\varepsilon = \frac{1}{\varepsilon}\left\|-\Delta\psi + f\right\|_\infty$. Then we obtain

$$\left\|F_{Y_1} - F_{Y_2}\right\|_\infty \le L_\varepsilon \left\|Y_1 - Y_2\right\|_\infty.$$

From relation (15), we conclude that

$$\left\|\varphi(Y_1) - \varphi(Y_2)\right\| \le L_\varepsilon h^2 \left\|A^{-1}\right\|_\infty \left\|Y_1 - Y_2\right\|_\infty.$$

Then we have

$$\left\|\varphi(Y_1) - \varphi(Y_2)\right\| \le k \left\|Y_1 - Y_2\right\|_\infty,$$

With $k = L_\varepsilon h^2 \left\|A^{-1}\right\|_\infty$.

by relation (13). Hence the function φ admits a unique fixed point.

In order to calculate the coefficients of the cubic spline collocation given by the nonsmooth system

$$\tilde{C}_\varepsilon = \varphi\left(\tilde{C}_\varepsilon\right), \tag{16}$$

we propose the generalized Newton method defined by

$$\tilde{C}_\varepsilon^{(k+1)} = \tilde{C}_\varepsilon^{(k)} - \left(I_{n+1} - V_k\right)^{-1}\left(\tilde{C}_\varepsilon^{(k)} - \varphi\left(\tilde{C}_\varepsilon^{(k)}\right)\right), \tag{17}$$

where I_{n+1} is the unit matrix of order $n+1$ and V_k is the generalized Jacobian of the function $\tilde{C}_\varepsilon \mapsto \varphi\left(\tilde{C}_\varepsilon\right)$, (see [8-10], for instance).

4. Generalized Newton Method

Let $F:\mathbb{R}^m \to \mathbb{R}^m$ be a function. Consider the equation

$$F(x) = 0.$$

The Newton method assumes that *F* is Fréchet differentiable, and is defined by

$$x_{k+1} = x_k - \left(F'(x_k)\right)^{-1} F'(x_k), \tag{18}$$

where $\left(F'(x_k)\right)^{-1}$ is the inverse of the Jacobian of the function *F*. However, in nonsmooth case $F'(x_k)$ may not exists. The generalized Jacobian of the function *F* may play the role of F' in the relation (18). Rademacher's theorem states that a locally Lipschitz function is almost everywhere differentiable (see [14], for instance). Assume that *F* is a locally Lipschitz function and let D_F be the set where *F* is differentiable. We denote

$$\partial_B F(x) = \left\{\lim_{x_i \to x} F'(x_i), x_i \in D_F\right\}.$$

The generalized Jacobian of *F* at $x \in \mathbb{R}^m$, $\partial F(x)$, in the sense of Clarke [15] is the convex hull of $\partial_B F(x)$:

$$\partial F(x) = conv\partial_B F(x). \tag{19}$$

For nonsmooth equations with a locally Lipschitz function *F*, the generalized Newton method is defined by

$$x_{k+1} = x_k - V_k^{-1} F(x_k), \tag{20}$$

where V_k is an element of $\partial F(x_k)$. If the function *F* is semismooth and BD-regular at *x*, then the sequence x_k in (20) superlinearly converges to a solution *x* (see [8,9, 16,17]). A Function *F* is said to be BD-regular at a point *x* if all the elements of $\partial_B F(x)$ are nonsingular, and it is said to be semismooth at *x* if it is locally Lipshitz at *x* and the limit

$$\lim_{V \in \partial F(x+th'), h' \to h, t \downarrow 0} Vh',$$

exists for any $h \in \mathbb{R}^m$. The class of semismooth functions includes, obviously smooth functions, convex functions, the piecewise-smooth functions, and others (see [10,18], for instance). Since the function J_ε defined by (6) is a Lipshitz and piecewise smooth function on u_ε, then the function $\varphi:\mathbb{R}^m \to \mathbb{R}^m$ given by (14) is also a Lipshitz and piecewise smooth function on $\mathbb{R}^m$. Hence we may apply the generalized Newton method to solve the problem (16).

5. Convergence of the Method

Theorem 6 *If we assume that the penalty parameter* ε *and the discretization parameter h satisfy the following relation*

$$2h^2 \left\|-\Delta\psi + f\right\|_\infty \left\|A^{-1}\right\|_\infty < \varepsilon. \tag{21}$$

then the cubic spline $\tilde{S}_\varepsilon$ converges to the solution u_ε. Moreover the error estimate $\left\|u_\varepsilon - \tilde{S}_\varepsilon\right\|_\infty$ is of order $O\left(h^2\right)$.

Proof From (12) and Lemma 4, we have

$$C_\varepsilon - \tilde{C}_\varepsilon = -h^2 A^{-1}\left(F_\varepsilon - F_{\tilde{C}_\varepsilon}\right) - A^{-1}E_\varepsilon.$$

Since E_ε is of order $O\left(h^4\right)$, then there exists a constant K_1 such that $\left\|E_\varepsilon\right\|_\infty \le k_1 h^4$. Hence we have

$$\left\|C_\varepsilon - \tilde{C}_\varepsilon\right\|_\infty \le h^2 \left\|A^{-1}\right\|_\infty \left\|F_\varepsilon - F_{\tilde{C}_\varepsilon}\right\|_\infty + K_1 \left\|A^{-1}\right\|_\infty h^4. \tag{22}$$

On the other hand we have

$$\begin{aligned} &\left|J_\varepsilon\left(t_i, u_\varepsilon(t_i)\right) - J_\varepsilon\left(t_i, \tilde{S}(t_i)\right)\right| \\ &\le L_\varepsilon \left|u_\varepsilon(t_i) - \tilde{S}_\varepsilon(t_i)\right| \\ &\le L_\varepsilon \left|u_\varepsilon(t_i) - S_\varepsilon(t_i)\right| + L_\varepsilon \left|S_\varepsilon(t_i) - \tilde{S}_\varepsilon(t_i)\right|. \end{aligned}$$

Since S_ε is the cubic spline interpolation of u_ε, then there exists a constant K_2 such that

$$\left\|u_\varepsilon - S_\varepsilon\right\|_\infty \le K_2 h^4 \left\|u_\varepsilon\right\|_\infty. \tag{23}$$

Using the fact that

$$\left|S_\varepsilon - \tilde{S}_\varepsilon\right| \le \left\|C_\varepsilon - \tilde{C}_\varepsilon\right\|_\infty \sum_{j=-2}^{n-2} B_j \le \left\|C_\varepsilon - \tilde{C}_\varepsilon\right\|_\infty, \tag{24}$$

then, we obtain

$$\left|F_\varepsilon - F_{\tilde{C}_\varepsilon}\right| \le L_\varepsilon \left\|C_\varepsilon - \tilde{C}_\varepsilon\right\|_\infty + L_\varepsilon K_2 h^4 \left\|u_\varepsilon^4\right\|_\infty.$$

By using relation (22) and assumption (21) it is easy to see that

$$\begin{aligned} \left\|C_\varepsilon - \tilde{C}_\varepsilon\right\|_\infty &\le \frac{h^2 \left\|A^{-1}\right\|_\infty}{1 - L_\varepsilon h^2 \left\|A^{-1}\right\|_\infty}\left(K_2 L_\varepsilon h^4 \left\|u_\varepsilon^{(4)}\right\|_\infty + K_1 h^2\right) \\ &\le \frac{K_2 L_\varepsilon h^2 \left\|u_\varepsilon^{(4)}\right\|_\infty + K_1}{L_\varepsilon} h^2 \end{aligned} \tag{25}$$

We have

$$\left\|u_\varepsilon - \tilde{S}_\varepsilon\right\|_\infty \le \left\|u_\varepsilon - S_\varepsilon\right\|_\infty + \left\|S_\varepsilon - \tilde{S}_\varepsilon\right\|_\infty .$$

Then from relations (23), (24) and (25), we deduce that $\left\|u_\varepsilon - \tilde{S}_\varepsilon\right\|_\infty$ is of order $O\left(h^2\right)$. Hence the proof is complete.

Remark 7 *Theorem* 6 *provides a relation coupling the penalty parameter* ε *and the discretization parameter h, which guarantees the quadratic convergence of the cubic spline collocation* $\tilde{S}_\varepsilon$ *to the solution* u_ε *of the penalty problem.*

6. Numerical Examples

In this section we give numerical experiments in order to validate the theoretical results presented in this paper. We report numerical results for solving a one dimensional obstacle problem by using the cubic spline method to approximate the solution of the penalty problem (7), and the generalized Newton method (20) to determine the coefficients of the cubic spline collocation. Consider the obstacle problem (1) with the following data: $\Omega = \left]0,2\right[$, $\psi = 0$ and

$$f = \begin{cases} -1 & \text{on } \left]0,1\right], \\ 1 & \text{on } \left]1,2\right[. \end{cases}$$

The true solution $u(x)$ of this problem is given by

$$u(x) = \begin{cases} -\frac{1}{2}x^2 + \left(2-\sqrt{2}\right)x & \text{if } x \in \left]0,1\right], \\ \frac{1}{2}x^2 - \sqrt{2}x + 1 & \text{if } x \in \left[1,\sqrt{2}\right], \\ 0 & \text{if } x \in \left[\sqrt{2},2\right[. \end{cases}$$

As a stopping criteria for the generalized Newton's iterations, we have considered that the absolute value of the difference between the input coefficients and the output coefficients is less than 10^{-9}.

Tables 1-4 show, for different values of the discretization parameter *h*, the error between the cubic spline collocation $\tilde{S}_\varepsilon$ and the true solution *u*. We note the convergence of the solution $\tilde{S}_\varepsilon$ to the function *u* depends on the discretization parameter *h* and the penalty parameter ε. Theorem 6 implies that for a fixed *h*, this convergence is guaranteed only if there exists $\varepsilon_h > 0$ such that $\varepsilon \ge \varepsilon_h$. Some experimental values of ε_h are given in **Tables 1-4**.

Theorems 1 and 6 imply that we have the error estimate between the exact solution and the discret penalty solution is given by $\left|u - \tilde{S}_\varepsilon\right|_\infty \le \varepsilon + kh^2$. The obtained results show the convergence of the discret penalty solution to the solution of the original obstacle problem as the parameters *h* and ε get smaller provided they satisfy the relation (21). Moreover, the numerical error estimates behave like $\varepsilon + kh^2$ which confirms what we were expecting.

Table 1. Results for $h = \frac{1}{20}$.

ε	e−2	e−3	5e−4	2e−4 = ε_h
$\left\|u - \tilde{S}_\varepsilon\right\|_\infty$	4.7e−3	7.61e−4	7.12e−4	6.84e−4
Number of iterations	5	7	9	10

Table 2. Results for $h = \frac{1}{50}$.

ε	e−2	e−3	e−6	2e−5 = ε_h
$\left\|u - \tilde{S}_\varepsilon\right\|_\infty$	4.5e−3	4.94e−4	1.75e−4	1.59e−4
Number of iterations	6	9	15	22

Table 3. Results for $h = \frac{1}{100}$.

ε	e−3	e−4	e−5	5e−6 = ε_h
$\left\|u - \tilde{S}_\varepsilon\right\|_\infty$	4.87e−4	4.41e−5	4.12e−6	2.74e−6
Number of iterations	9	16	31	43

Table 4. Results for $h = \frac{1}{200}$.

ε	e−3	e−4	e−5	2e−6 = ε_h
$\left\|u - \tilde{S}_\varepsilon\right\|_\infty$	4.86e−4	4.92e−5	5.25e−6	8.26e−7
Number of iterations	9	18	35	56

7. Concluding Remarks

In this paper, we have consider an approximation of a unilateral obstacle problem by a sequence of penalty problems, which are nonsmooth equation problems, presented in [2,7]. Then we have developed a numerical method for solving each nonsmooth equation, based on a cubic collocation spline method and the generalized Newton method. We have shown the convergence of the method provided that the penalty and discret parameters satisfy the relation (21). Moreover we have provided an error estimate of order $O\left(h^2\right)$ with respect to the norm

$\|\cdot\|_\infty$. The obtained numerical results show the convergence of the approximate penalty solutions to the exact one and confirm the error estimates provided in this paper.

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