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Numerical Solution of the Fredholme-Volterra Integral Equation by the Sinc Function

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ABSTRACT

In this paper, we use the Sinc Function to solve the Fredholme-Volterra Integral Equations. By using collocation method we estimate a solution for Fredholme-Volterra Integral Equations. Finally convergence of this method will be discussed and efficiency of this method is shown by some examples. Numerical examples show that the approximate solutions have a good degree of accuracy.

Keywords: Fredholme-Volterra Integral Equation; Sinc Function; Collocation Method

1. Introduction

In recent years, many different methods have been used to approximate the solution of the Fredholme-Volterra Integral Equations, such as [1,2]. In this paper, we first present the Sinc Function and their properties. Then we consider the Fredholme-Volterra Integral Equation types in the forms

$$u(x) = f(x) + \int_a^b k_1(x,t)u(t)\,dt + \int_a^b k_2(x,t)u(t)\,dt \tag{1.1}$$

where $k_1(x,t)$, $k_2(x,t)$ and $f(x)$ are known functions, but $u(x)$ is an unknown function. Then we use the Sinc Function and convert the problem to a system of linear equations.

2. Sinc Function Properties

The sinc function properties are discussed thoroughly in [3-10]. The sinc function is defined on the real line by

$$\sin c(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0 \end{cases} \tag{2.1}$$

For $h > 0$, and $k = 0, \pm 1, \pm 2, \cdots$, The translated sinc functions with evenly spaced nodes are given by

$$S(j,h)(x) = \sin c\left(\frac{x - jh}{h}\right) = \begin{cases} \dfrac{\sin\left[\dfrac{\pi}{h}(x - jh)\right]}{\dfrac{\pi}{h}(x - jh)}, & x \neq jh, \\ 1, & x = jh, \end{cases} \tag{2.2}$$

The sinc function form for the interpolating point $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{2.3}$$

Let

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t}\,dt, \tag{2.4}$$

If a function $u(x)$ is defined on the real axis, then for $h > 0$ the series

$$c(u,h)(x) = \sum_{j=-\infty}^{+\infty} u(jh)\sin c\left(\frac{x - jh}{h}\right), \tag{2.5}$$

called whittaker cardinal expansion of u, whenever this series converges. The properties of the whittaker cardinal expansion have been extensively studied in [8].

These properties are derived in the infinite stripe D of the complex w- plane, where for $d > 0$,

$$D = \left\{ w \in \mathbb{C}, \left|\mathrm{Im}(w)\right| \langle d, d \rangle 0 \right\}$$

Approximations can be constructed for infinite, semi-infinite and finite intervals. To construct approximations on the interval [a,b], which is used in this paper, the eye-shaped domain in the z-plane.

$$D_E = \left\{ z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2} \right\},$$

Is mapped conformably onto the infinite strip D via

$$w = \varphi(z) = Ln\left(\frac{z-a}{b-z}\right), z = \varphi^{-1}(w) = \frac{a + b\exp(w)}{1 + \exp(w)}$$

The basis functions on [a,b] are taken to be composite translated sinc functions,

$$s(k,h)o\varphi(x)=\sin c\left(\frac{\varphi(x)-jh}{h}\right), \qquad (2.6)$$

Thus we may define the inverse images of the real line and of evenly spaced nodes $\{jh\}_{j=-\infty}^{+\infty}$ as

$$\Gamma=\{\psi(t)\in D_E:-\infty<t<\infty\}=[a,b],$$

and

$$x_k=\varphi^{-1}(kh)=\frac{a+be^{kh}}{1+e^{kh}}, k=0,\pm1,\pm2,\cdots \qquad (2.7)$$

We consider the following definitions and theorems in [8-10].

Definition 2.1:

Let $L_\alpha(D_E)$ be the set of all analytic functions, for which there exists a constant, *C*, such that

$$|u(z)|\le c\frac{|\rho(z)|}{[1+|\rho(z)|]^{2\alpha}}, z\in D_E, 0<\alpha\le1 \qquad (2.8)$$

Theorem 2.1:

Let $u\in L_\alpha(D_E)$, let *N* be appositive integer, and let *h* be

$$h=\left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}} \qquad (2.9)$$

Then there exists positive constant C_1, independent of *N*, such that

$$\sup_{z\in\Gamma}\left|u(z)-\sum_{j=-N}^{N}u(z_j)S(j,h)o\varphi(z)\right|\le C_1e^{(-\pi d\alpha N)^{\frac{1}{2}}} \qquad (2.10)$$

Proof: See [8,9].

Theorem 2.2:

Let $\frac{u}{\varphi'}\in L_\alpha(D_E)$, Let *N* be a positive integer and let *h* be selected by the relation (2.9) then there exist positive constant C_2, independent of *N*, such that

$$\left|\int_\Gamma u(z)\mathrm{d}_Z-h\sum_{j=-N}^{N}\frac{u(z_j)}{\varphi(z_j)}\right|\le C_2e^{(-\pi d\alpha N)^{\frac{1}{2}}} \qquad (2.11)$$

and also for $0<d\le\pi, 0<\alpha\le1$, let $\delta_{kj}^{(-1)}$ be defined as in (2.4) then there exists a constant, C_3 which is independent of N, such that

$$\left|\int_a^{z_k}u(t)\mathrm{d}t-h\sum_{J=-N}^{N}\delta_{kj}^{(-1)}\frac{u(zk)}{\varphi'(zk)}\right|\le C_3e^{(-\pi d\alpha N)^{\frac{1}{2}}} \qquad (2.12)$$

Proof: See [8].

3. The Sinc Collocation Method

The solution of linear Fredholm-Volterra integral Equation (1.1) is approximated by the following linear combination of the sinc functions and auxiliary functions:

$$u(x)=\sum_{j=-N}^{N}u(x_j)\gamma_j(x), x\in[a,b] \qquad (3.1)$$

where

$$\gamma_j(x)=\begin{cases}w_a(x), & j=-N\\ s(j,h)o\varphi(x), & j=-N+1,\cdots,N-1,\\ w_b(x), & j=N\end{cases} \qquad (3.2)$$

where the basis functions $w_a(x), w_b(x)$ defined by

$$w_a(x)=\frac{1}{1+\rho(x)}, \qquad (3.3)$$

$$w_b(x)=\frac{\rho(x)}{1+\rho(x)}, \qquad (3.4)$$

$$\rho(x)=e^{\varphi(x)} \qquad (3.5)$$

We denote $\Gamma_a=a$ and $\Gamma_b=b$ then basis function must satisfy the following conditions:

$$\lim_{x\to\Gamma_a}w_a(x)=1, \lim_{x\to\Gamma_a}w_b(x)=0, \qquad (3.6)$$

$$\lim_{x\to\Gamma_b}w_b(x)=0, \lim_{x\to\Gamma b}w_b(x)=1 \qquad (3.7)$$

Obviously by using Equations (2.3) and (3.1) we have

$$u(\mathrm{a})=u_{\mathrm{-N}}, u(\mathrm{b})=u_{\mathrm{N}}.$$

Lemma 3.1:

$u(x)\in L_\alpha(D_E)$, let N be a positive integer and $h=(\pi d/\alpha N)^{\frac{1}{2}}$, Then (see Equation (3.8)), where $\gamma_j(x)$ is defined in (2.14) and C_4 is a positive constant, independent of *N*.

Proof: See [9].

Lemma 3.2:

For $u(x)$ defined in (3.1), let

$$\left(\frac{k_1}{\varphi'}\right)\gamma_j\in L_\alpha(D_E), \left(\frac{k_2}{\varphi'}\right)\gamma_j\in L_\alpha(D_E),$$

and h be selected from (2.9) then (see Equation(3.9))

$$\sup_{x\in\Gamma}\left|u(x)-(u(x_{-N})w_a(x_{-N})+\sum_{j=-N+1}^{N-1}u(x_j)S(j,h)o\varphi(x)+u(x_N)w_b(x_N)\right|\le C_4e^{(-\pi d\alpha N)^{\frac{1}{2}}} \qquad (3.8)$$

$$\int_a^b k_1(x,t)u(t)\mathrm{d}t+\int_a^{x_k} k_2(x,t)u(t)\mathrm{d}t = hu(x_{-N})\sum_{j=-N}^{N}\left[\frac{k_1(x,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]w_a(t_j)$$
$$+h\sum_{j=-N+1}^{N-1}u(x_j)\left[\frac{k_1(x,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]+hu(x_N)\sum_{j=-N}^{N}\left[\frac{k_1(x,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]w_b(t_j) \quad (3.9)$$

Now let $u(x)$ be the exact solution (1.1) that is approximated by following expansion.

$$u_n(x)=\sum_{j=-N}^{N}u_j\gamma_j(x), x\in\Gamma=[a,b], n=2N+1 \quad (3.10)$$

Upon replacing $u(x)$ in the Fredholm-Volterra integral Equation (1.1) $u_n(x)$, applying Lemma 3.1 and Lemma 3.2, setting sinc collocation points x_k and Then, considering $\delta_{kj}^{(0)}=\delta_{jk}^{(0)}$ we obtain the following system

$$u_{-N}\left\{w_a(x_k)-\lambda h\sum_{j=-N}^{N}\left[\frac{k_1(x_k,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]w_a(t_j)\right\}+\sum_{j=-N+1}^{N-1}\left\{\delta_{kj}^{(0)}-\lambda h\left[\frac{k_1(x_k,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]\right\}u_j$$
$$+u_N\left\{w_b(x_k)-\lambda h\sum_{j=-N}^{N}\left[\frac{k_1(x_k,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]w_b(t_j)\right\} \quad (3.11)$$
$$=f(x_k), k=-N,\cdots,N$$

We write the above system of equations in the matrix forms:

$$TU=P, \text{ where } T=\left[A_{n\times 1}\middle|B_{n(n-2)}\middle|C\right]U \quad n=2N+1 \quad (3.12)$$

where

$$A=\left[w_a(x_{-N})-\lambda h\sum_{j=-N}^{N}\left[\frac{k_1(x_{-N},t_j)}{\varphi'(t_j)}+\delta_{-Nj}^{(-1)}\frac{k_2(x_{-N},t_j)}{\varphi'(t_j)}\right]w_a(t_j),\cdots,w_a(x_N)\right.$$
$$\left.-\lambda h\sum_{j=-N}^{N}\left[\frac{k_1(x_N,t_j)}{\varphi'(t_j)}+\delta_{Nj}^{(-1)}\frac{k_2(x_N,t_j)}{\varphi'(t_j)}\right]w_a(t_j)\right]^T, \quad (3.13)$$

$$B=\left[\delta_{kj}^{(0)}-\lambda h\left[\frac{k_1(x_k,t_j)}{\varphi'(t_j)}+\delta_{kj}^{(-1)}\frac{k_2(x_k,t_j)}{\varphi'(t_j)}\right]\right]^T,$$
$$k=-N,\cdots N, j=-N+1,\cdots,N-1 \quad (3.14)$$

$$C=\left[w_b(x_{-N})-\lambda h\sum_{j=-N}^{N}\left[\frac{k_1(x_{-N},t_j)}{\varphi'(t_j)}+\delta_{-Nj}^{(-1)}\frac{k_2(x_{-N},t_j)}{\varphi'(t_j)}\right]w_a(t_j),\cdots,w_b(x_N)\right.$$
$$\left.-\lambda h\sum_{j=-N}^{N}\left[\frac{k_1(x_N,t_j)}{\varphi'(t_j)}+\delta_{N_j}^{(-1)}\frac{k_2(x_N,t_j)}{\varphi'(t_j)}\right]w_b(t_j)\right]^T \quad (3.15)$$

$$p=\left[f(x_{-N}),f(x_{-N+1}),\cdots,f(x_{N-1}))f(x_N)\right]^T, \quad (3.16)$$

$$U=\left[u_{-N},u_{-N+1},\cdots,u_{N-1},u_N\right]^T, \quad (3.17)$$

By solving the above system we obtain, u_j, then, by using such solution we can obtain the approximate solution un as

$$U_n = I_u.U, I_u = \begin{bmatrix} w_a(x_{-N}) & 0 & \dots & 0 & w_b(x_{-N}) \\ w_a(x_{-N+1}) & 1 & \dots & 0 & w_b(x_{-N+1}) \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ w_a(x_{N-1}) & 0 & \dots & 1 & w_b(x_{N-1}) \\ w_a(x_N) & 0 & \dots & 0 & w_b(x_N) \end{bmatrix} \quad (3.18)$$

4. Convergence Analysis

Now we discuss the convergence each of sinc collocation method. Suppose that $u(x)$ is the exact solution of the Fredholme-Volterra integral Equation (1.1). For each N, we can find u_j which is our solution of the liner system (3.12), also by using u_j we obtain the approximate solution $u_n(x)$, In order to derive a bound for $|u(x) - u_n(x)|$ we need to estimate the norm of the vector $T\tilde{u} - p$, where $\tilde{u}$ is a vector defined by

$$\tilde{u} = \left(u(x_{-N}), \cdots, u(x_N)\right)^T$$

where $u(x_j)$ is the value of the exact solution of integral equation at the sinc points x_j. There for we need the following lemma.

Lemma 4.1:

Let u(x) be the exact solution of the integral (1.1) and $u(x) \in L_\alpha(D_E)$ let

$$h = (\pi d/\alpha N)^{\frac{1}{2}} and \frac{k_1}{\varphi'}\gamma_j \in L_\alpha(D_E), \frac{k_2}{\varphi'}\gamma_j \in L_\alpha(D_E),$$

for $x \in [a,b]$, then there exists a constant C_5 independent of N, such that

$$\|T\tilde{u} - p\| \le C_5 N^{\frac{1}{2}} \exp(-\pi d\alpha N)^{\frac{1}{2}} \quad (4.1)$$

Proof: See [10].

Theorem 4.1:

Let us consider all assumptions of Lemma 3.1 and let $u_n(x)$ be the approximate solution of Fredholme-Volterra integral equation given by (3.3) then we have

$$\sup_{x\in\Gamma} |u(x) - u_n(x)| \le C_6 \mu N^{\frac{1}{2}} \exp(-\pi d\alpha N)^{\frac{1}{2}} \quad (4.2)$$

where C_6 a constant independent of N, and $\mu = \|T^{-1}\|$

Proof:

Suppose $\beta_n(x)$ defined this following form:

$$\beta_n(x) = \sum_{j=-N}^{N} u(x_j)\gamma_j(x), \quad (4.3)$$

$$\gamma_j(x) = \begin{cases} w_a(x), & j = -N, \\ S(j,h) o\varphi(x), & j = -N+1, \cdots, N-1, \\ w_b(x), & j = N, \end{cases}$$

So we have

$$|u(x) - u_n(x)| \le |u(x) - \beta_n(x)| + |\beta_n(x) - u_n(x)| \quad (4.4)$$

By using Lemma 3.1, we obtain

$$\sup_{x\in\Gamma} |u(x) - \beta_n(x)| \le C_4 \exp(-\pi d\alpha N)^{\frac{1}{2}}$$

Obviously by using Equations (4.3) and (3.3) we have.

$$\begin{aligned} |\beta_n(x) - u_n(x)| &= \left| \sum_{j=-N}^{N} u(x_j)\gamma_j(x) - \sum_{j=-N}^{N} u_j\gamma_j(x) \right| \\ &\le \left(\sum_{j=-N}^{N} |u(x_j) - u_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=-N}^{N} |\gamma_j(x)|^2 \right)^{\frac{1}{2}} = E_1 \end{aligned} \quad (4.5)$$

And we have from definition of the γ_j We obtain

$$\left(\sum_{j=-N}^{N} |\gamma_j(x)|^2 \right)^{\frac{1}{2}} \le C_7 \quad (4.6)$$

That C_7 a constant independent of N.

Now, by using Equations (4.5) and (4.6) we get

$$E_1 \le C_7 \|\tilde{U} - U\| \quad (4.7)$$

In this case by using the system (3.12) and lemma 4.1 we obtain

$$\begin{aligned} \|\tilde{U} - U\| &= \|T^{-1}T(\tilde{U} - U)\| \le \|T^{-1}\| \|T\tilde{U} - P\| \\ &\le \mu \cdot C_5 N^{\frac{1}{2}} \exp(-\pi d\alpha N)^{\frac{1}{2}}, \end{aligned} \quad (4.8)$$

Now by using Equations (4.7) and (4.8) we get

$$|B_n(x) - u_n(x)| \le C_8 \mu \cdot N^{\frac{1}{2}} \cdot \exp(-\pi d\alpha N)^{\frac{1}{2}} \quad (4.9)$$

Obviously by using Equations (4.9) and (4.2) we obtain

$$\begin{aligned} &\sup_{x\in\Gamma} |u(x) - u_n(x)| \le C_4 \exp(-\pi d\alpha N)^{\frac{1}{2}} \\ &+ C_8 \mu N^{\frac{1}{2}} \exp(-\pi d\alpha N)^{\frac{1}{2}} \\ &\le C_6 N^{\frac{1}{2}} \exp(-\pi d\alpha N)^{\frac{1}{2}} \\ &C_6 = \max\{C_4, \mu C_8\}. \end{aligned}$$

5. Numerical Examples

In this section, we apply the sinc collocation method for solving Fredholm-Volterra integral equation example.

Example 5.1: Consider the following Fredholm-Volterra integral equation of the second kind with exact solution $u(x) = x$.

$$u(x) = \frac{2}{3}x - \frac{x^4}{3} + \int_0^1 (tx)u(t)\,dt + \int_0^x (tx)u(t)\,dt$$

We solved Example 5.1 for different of

$$N, \alpha = \frac{1}{2} \, and \, h = \pi/\sqrt{N},$$

And we consider the sinc grid points as:

$$s = \{x_{-N}, x_{-N+1}, \cdots, x_{N-1}, x_N\},$$

where

$$x_k = \frac{a + be^{kh}}{1 + e^{kh}}, k = -N, \cdots, N$$

The errors on the given points are denoted by

$$\|E_s(h)\|_\infty = \max |u(x_j) - u_n(x_j)| \quad -N \le j \le N \tag{5.1}$$

Computational results are given in **Tables 1-5**.

Table 1. Results for Example 1 (N = 5).

x i	*Exact solution*	*Approximation solution*
0.0009	0.0009	0.0010
0.0036	0.0036	0.0045
0.0146	0.0146	0.0177
0.0568	0.0568	0.0662
0.1970	0.1970	0.2175
0.5000	0.5000	0.5269
0.8030	0.8030	0.8082
0.9432	0.9432	0.9054
0.9854	0.9854	0.8836
0.9964	0.9964	0.8332
0.9991	0.9991	0.7733

Table 2. Results for Example 1 (N = 10).

x i	*Exact solution*	*Approximation solution*
0.0000	0.0000	0.0001
0.0001	0.0001	0.0002
0.0004	0.0004	0.0004
0.0010	0.0010	0.0011
0.0026	0.0026	0.0029
0.0069	0.0069	0.0080
0.0185	0.0185	0.0223
0.0483	0.0483	0.0585
0.1206	0.1206	0.1419
0.2702	0.2702	0.3044
0.5000	0.5000	0.5381
0.7298	0.7298	0.7520
0.8794	0.8794	0.8634
0.9517	0.9517	0.8791
0.9815	0.9815	0.8465
0.9931	0.9931	0.8009
0.9974	0.9974	0.7850
0.9990	0.9990	0.8269
0.9996	0.9996	0.8753
0.9999	0.9999	0.8777
1.0000	1.0000	0.8443

Table 3. Results for Example 1 (N = 15).

x i	*Exact solution*	*Approximation solution*
0.0000	.0000	.00000
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0001	0.0001	0.0001
0.0001	0.0001	0.0002
0.0003	0.0003	0.0004
0.0007	0.0007	0.0008
0.0015	0.0015	0.0018
0.0034	0.0034	0.0039
0.0076	0.0076	0.0086
0.0170	0.0170	0.0197
0.0375	0.0375	0.0447
0.0807	0.0807	0.0968
0.1649	0.1649	0.1942
0.3076	0.3076	0.3496
0.5000	0.5000	0.5452
0.6924	0.6924	0.7237
0.8351	0.8351	0.8327
0.9193	0.9193	0.8656
0.9625	0.9625	0.8491
0.9830	0.9830	0.8155
0.9924	0.9924	0.8001
0.9966	0.9966	0.8232
0.9985	0.9985	0.8581
0.9993	0.9993	0.8673
0.9997	0.9997	0.8461
0.9999	0.9999	0.8129
0.9999	0.9999	0.7987
1.0000	1.0000	0.8226
1.0000	1.0000	0.8579
1.0000	1.0000	0.8672

Example 5.2: we consider the following Fredholm-Volterra integral equation of the second kind with exact solution $u(x) = \frac{x}{2}$.

$$u(x) = -\frac{x^3}{12} - \frac{x^4}{6} + \frac{x}{4} + \int_0^1 \frac{3}{2}(xt)u(t)\,\mathrm{d}t + \int_0^x \left(\frac{x+t}{2}\right)u(t)\,\mathrm{d}t$$

Computational results are given in **Tables 6-9.**

Table 4. Results for Example 1 (N = 20).

x i	*Exact solution*	*Approximation solution*
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0000	0.0000	0.0000
0.0001	0.0001	0.0001
0.0001	0.0001	0.0001
0.0002	0.0002	0.0002
0.0004	0.0004	0.0005
0.0009	0.0009	0.0011
0.0018	0.0018	0.0021
0.0036	0.0036	0.0043
0.0073	0.0073	0.0084
0.0146	0.0146	0.0166
0.0290	0.0290	0.0335
0.0568	0.0568	0.0671
0.1084	0.1084	0.1291
0.1970	0.1970	0.2317
0.3313	0.3313	0.3782
0.5000	0.5000	0.5499
0.6687	0.6687	0.7064
0.8030	0.8030	0.8103
0.8916	0.8916	0.8520
0.9432	0.9432	0.8477
0.9710	0.9710	0.8240
0.9854	0.9854	0.8106
0.9927	0.9927	0.8238
0.9964	0.9964	0.8488
0.9982	0.9982	0.8588
0.9991	0.9991	0.8452
0.9996	0.9996	0.8205
0.9998	0.9998	0.8083
0.9999	0.9999	0.8227
0.9999	0.9999	0.8483
1.0000	1.0000	0.8585
1.0000	1.0000	0.8451
1.0000	1.0000	0.8205
1.0000	1.0000	0.8082
1.0000	1.0000	0.8226
1.0000	1.0000	0.8483

Table 5. Results for Example 1.

N	h	$\|E_s(h)\|_\infty$
5	1.405	0.2258
10	0.99346	0.2124
15	0.81116	0.2012
20	0.70248	0.1918

Table 6. Results for Example 2 (N = 5).

x_i	*Exact solution*	*Approximation solution*
0.00090	0.00040	0.04600
0.00360	0.00180	0.06170
0.01460	0.00730	0.06090
0.05680	0.02840	0.07140
0.19700	0.09850	0.13000
0.50000	0.25000	0.27650
0.80300	0.40150	0.39830
0.94320	0.47160	0.41230
0.98540	0.49270	0.38000
0.99640	0.49820	0.34390
0.99910	0.49960	0.31110

Table 7. Results for Example 2 (N = 7).

x_i	*Exact solution*	*Approximation solution*
0.0002	0.0001	0.0286
0.0008	0.0004	0.0266
0.0026	0.0013	0.0442
0.0086	0.0043	0.0612
0.0276	0.0138	0.0684
0.0851	0.0426	0.0887
0.2337	0.1169	0.1535
0.5000	0.2500	0.2804
0.7663	0.3831	0.3888
0.9149	0.4574	0.4110
0.9724	0.4862	0.3849
0.9914	0.4957	0.3502
0.9974	0.4987	0.3212
0.9992	0.4996	0.3266
0.9998	0.4999	0.3684

Table 8. Results for Example 2 (N = 9).

x_i	*Exact solution*	*Approximation solution*
0.0001	0.0000	0.0520
0.0002	0.0001	0.0424
0.0007	0.0003	0.0311
0.0019	0.0009	0.0291
0.0053	0.0026	0.0440
0.0149	0.0075	0.0620
0.0414	0.0207	0.0757
0.1096	0.0548	0.1033
0.2598	0.1299	0.1703
0.5000	0.2500	0.2833
0.7402	0.3701	0.3810
0.8904	0.4452	0.4078
0.9586	0.4793	0.3873
0.9851	0.4925	0.3548
0.9947	0.4974	0.3283
0.9981	0.4991	0.3309
0.9993	0.4997	0.3638
0.9998	0.4999	0.3852
0.9999	0.5000	0.3752

Table 9. Results for Example 2.

N	h	$\|E_s(h)\|_\infty$
5	1.4050	0.1885
7	1.1874	0.1775
9	1.0472	0.1690
17	0.7619	0.1608
20	0.7025	0.1586
30	0.5736	0.1540

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