Projected Flat *m*-th Root Finsler Metrics

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Abstract: Projected flat Finsler metrics on an open subset in R^n are the regular solution to Hilbert's Fourth Problem. We study locally projected flat *m*-th root Finsler metrics and its generalized metrics in this paper. We prove that they must be locally Minkowskian if they are irreducible.

Key words: Finsler metric; projected flat; *m*-th root metric; locally Minkowskian

CLC number: O186 **Document code:** A **Article ID:** 1001-5132（2011）03-0048-05

The famous Hilbert's Fourth Problem in the regular case is to study and characterize locally projectively flat Finsler metrics. On an open domain in R^n, a Finsler metric is said to be projectively flat if its geodesics are straight lines. In the Riemannian case, by Beltrami's theorem, we know that a Riemannian metric with constant sectional curvature if and only if it is locally projectively flat. There also exist many non-Riemannian projectively flat Finsler metrics such as the famous Hilbert metric and Funk metric on a strongly convex domain. In [1], the author characterized projectively flat (α,β)-metrics, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$. There are two special examples as following,

$$F=\frac{\sqrt{|y|^2-(|x|^2|y|^2-<x,y>^2)}}{1-|x|^2}+\frac{<x,y>}{1-|y|^2}, \quad (1)$$

$$F=\frac{(\sqrt{|y|^2-(|x|^2|y|^2-<x,y>^2)}+<x,y>)^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-<x,y>^2)}}. \quad (2)$$

The metric in (1) is the well-known Funk metric and the metric in (2) is constructed by Berwald L[2].

In this paper, we will discuss the following important two classes of Finsler metrics,

$$F=A^{1/m}, \quad (3)$$

$$F=\sqrt{A^{2/m}+B}, \quad (4)$$

where $A=a_{i_1i_2\ldots i_m}(x)y^iy^{i_2}\ldots y^{i_m}$ and $B=b_{ij}(x)y^iy^j$. The forms in (3) and (4) are called an m-th root metric and a generalized m-th root metric respectively. Obviously, they both are reversible Finsler metrics. The m-th root metrics in the form $F=\sqrt[m]{a_{i_1i_2\ldots i_m}(x)y^iy^{i_2}\ldots y^{i_m}}$ was studied by Matsumoto M[3], Okubo K[4] and Shimada H[5], etc[6-8]. Shen Z and Cheng X studied projectively flat with some special curvature[9-10]. In recent years, physicists are interested in fourth-root of metrics. Some geometers have obtained some results about projectively flat fourth-root Finsler metric Brinzei N has derived some equations to characterize projectively flat m-th root metrics. In [6], Kim B and Park H studied the m-th root Finsler metrics admitting (α,β)-types. Thus these are very important to the properties of m-th root metrics for further research.

If at every point, there is a local coordinate domain and the metric $F=F(y)$ is independent of its position

Received date: 2010–09–24. JOURNAL OF NINGBO UNIVERSITY (NSEE): http://3xb.nbu.edu.cn
Foundation item: Supported by the NNSFC (10801080); Natural Science Foundation of Ningbo (2008A610014); K. C. Wang Magna Fund in Ningbo University.
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x, then a Finsler metric $F=F(x,y)$ is said to be locally Minkowskian. In this case, all geodesics are linear lines $x^i(t)=ta^i+b^i$. But it is called locally projectively flat, if $x^i(t)=f(t)a^i+b^i$, at every point, there is a local coordinate domain in which the geodesics are straight lines. The main purpose of this paper is to study locally projectively flat m-th root metrics and its generalized metrics.

In this paper, we consider the condition $m>4$. Obviously, by the definition of Finsler metric m must be even. The case when $m=4$ has been studied by Li B L and Shen Z M. In this paper we obtain the result in general case.

We prove the following.

Theorem 1 Let $F=A^{1/m}$ $(m>4)$ be an m-th root metric on a manifold of dimension $n\geqslant 3$. Assume that A is irreducible. Then F is locally projectively flat if and only if it is locally Minkowskian.

If $A=\sqrt[m]{a_{ij}(x)y^iy^j}$ is the square of a Riemannian metric of constant sectional curvature $K=\mu$, then $F=A^{1/m}=\sqrt{a_{ij}(x)y^iy^j}$ is locally projectively flat. But it is not locally Minkowskian when $\mu\neq 0$. Thus the assumption in Theorem 1 on A being irreducible can not be removed.

As a general case of Theorem 1, we obtain the following theorem.

Theorem 2 Let $F=\sqrt{A^{2/m}+B}$ $(m>4)$ be a generalized m-th root metric on a manifold of dimension $n\geqslant 3$. Assume that A is irreducible and $B\neq 0$. Then F is locally projectively flat if and only if it is locally Minkowskian.

If $A=(a_{ij}(x)y^iy^j)^{m/2}$ is reducible such that $A^{2/m}+B=a_{ij}(x)y^iy^j+b_{ij}(x)y^iy^j$ is the square of a Riemannian metric of constant sectional curvature $K=\mu$, then $F=\sqrt{A^{2/m}+B}$ is locally projectively flat. But if $\mu\neq 0$, F is not locally Minkowskian. Therefore, the condition that A is irreducible can not be dropped.

Li B L and Shen Z M have studied the case when $m=4$, they give an special example. Their example is projectively flat but not Minkowskian. In the example the irreducible condition on $A-B^2$ can not be dropped. In this paper, we consider the case when $m>4$. We find that when $m>4$, A should be irreducible. However, there is no special condition on $A-B^2$.

1 Preliminaries

In this section, we will introduce some basic knowledge about projective flat Finsler metric. Let $F=F(x,y)$ be a Finsler metric on an open subset $U\subset R^n$ and F is positive define, the matrix $g_{ij}=g_{ij}(x,y)$ is positive define, where

$$g_{ij}(x,y):=1/2[F^2]_{y^iy^j}(x,y)\ (y\neq 0).$$

The following equation

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+2G^i(x,\frac{\mathrm{d}x}{\mathrm{d}t})=0,$$

is the geodesics of F, where

$$G^i=(1/4)g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\}.$$

F is called a Berwald metric if $G^i=G^i(x,y)$ are quadratic in y. It is called a Landsberg metric if $F_{y^i}[G^i]_{y^jy^ky^l}=0$. Thus every Riemannian metric must be a Berwald metric and every Berwald metric must be a Landsberg metric.

The Riemann curvature $R_y=R^i_k\frac{\partial}{\partial x^i}\otimes \mathrm{d}x^k:T_xM\to T_xM$ is defined by $R_y(v)=R^i_k(x,y)v^k\frac{\partial}{\partial x^i}|_x$, $v=v^k\frac{\partial}{\partial x^k}|_x$, where

$$R^i_k=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For each tangent plane $\Pi\subset T_xM$ and $y\in P$, the flag curvature of (Π,y) is defined by

$$K(\Pi,y):=\frac{g_{il}(x,y)R^i_k(x,y)u^ku^l}{F(x,y)^2g_{ij}(x,y)u^iu^j-[g_{ij}(x,y)y^iu^j]^2},$$

where $u\in\Pi$ such that $\Pi=span\{y,u\}$, A Finsler metric whose flag curvature $K(\Pi,y)=K(x,y)$ is independent of tangent planes Π containing $y\in T_xM$ is said to be of scalar flag curvature. If it is a Riemannian metric, the flag curvature $K(\Pi,y)=K(\Pi)$ is independent of $y\in T_xM$. Therefore it is of scalar flag curvature $K=K(x,y)$ if and only if is of isotropic sectional curvature $K=$

$K(x)$. There are two famous theorems we will be used in this paper.

Theorem 3[11] Let F be a Landsberg metric of scalar flag curvature on a manifold of dimension $n \geqslant 3$. If the flag curvature $K \neq 0$, then it is Riemannian.

Theorem 4[12] Every Berwald metric with K=0 is locally Minkowskian. As we know, a Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is locally projectively flat if and only if

$$F_{x^l y^k} y^l = F_{x^k}, \tag{5}$$

which is found by Hamel G in [13]. In this case, $P = F_{x^l} y^l / (2F)$ and the metric is of scalar flag curvature shown by

$$K = \frac{P^2 - P_{x^l} y^l}{F^2}. \tag{6}$$

Then, locally projectively flat Finsler metrics are of scalar flag curvature.

For using Theorem 3 and Theorem 4 in the following proof, let's consider the special case that $G^i = Py^i$, where $P = P_i(x) y^i$ is a local 1-form and the dimension $n \geqslant 3$. Let

$$U := \{ x \in M \mid K(x, y) \neq 0 \quad \text{for some} \quad y \in T_x M \}.$$

Assume that $U \neq \varnothing$. If $K \neq 0$, by Theorem 3, we get F is Riemannian on U. By continuity, we conclude that $U = M$, that is to say, F is Riemannian on the whole manifold. Since F is a Berwald metric, by Theorem 4, we easily get it must be locally Minkowskian if $K = 0$ on M. The case of $n = 2$ was solved by Berwald L.

2 Projectively flat *m*-th root metrics

In this section, we will discuss projectively flat *m*-th root metrics $F = A^{1/m}$ on an open subset $U \subset R^n$. For simplicity, we let

$$A_0 := A_{x^l} y^l, \ A_{x^k} := \frac{\partial A}{\partial x^k},$$

$$A_{x^l y^k} := \frac{\partial^2 A}{\partial x^l \partial y^k}, \quad A_{x^l y^k} y^l := A_{0k}.$$

Lemma 1 Let $F = A^{1/m}$ be an *m*-th root metric on an open subset $U \subset R^n$. It is projectively flat if and only if

$$mA(A_{0k} - A_{x^k}) = (m-1) A_0 A_{y^k}. \tag{7}$$

Proof By a direct computation, we have

$$F_{x^k} = \frac{A^{1/m} A_{x^k}}{mA},$$

$$F_{x^l y^k} = \frac{A^{1/m} A_{y^k} A_{x^l}}{m^2 A^2} + \frac{A^{1/m} A_{x^l y^k}}{mA} - \frac{A^{1/m} A_{x^l} A_{y^k}}{mA^2}.$$

Then by the equation $F_{x^l y^k} y^l - F_{x^k} = 0$, we have

$$\left(\frac{A^{1/m} A_{y^k} A_{x^l}}{m^2 A^2} + \frac{A^{1/m} A_{x^l y^k}}{mA} - \frac{A^{1/m} A_{x^l} A_{y^k}}{mA^2}\right) y^l - \frac{A^{1/m} A_{x^k}}{mA} = 0.$$

Simplify the above equation, yields

$$\frac{(A_{x^k} - A_{x^l y^k} y^l) A}{m} + \frac{(m-1) A_0 A_{y^k}}{m^2} = 0.$$

Then we obtain (7).

Proof of Theorem 1 Assume that F is projectively flat. Because of A is irreducible and $\deg(A_{y^k}) = m-1$ is less than $\deg(A)$, by (7), we have A_0 is divisible by A, that is, there is a 1-form η such that

$$A_0 = 2m\eta A. \tag{8}$$

Thus the spray coefficients $G^i = Py^i$ are given by

$$P = \frac{A_0}{2mA} = \eta.$$

We can obtain that $G^i = \eta y^i$ are quadratic in y. Then F is a Berwald metric.

Assume that $n \geqslant 3$. By Theorem 3, if the scalar flag curvature $K \neq 0$, then F is Riemannian. Thus A is a perfect square of a Riemannian metric. This contradicts our assumption, so $K = 0$. That is, F is a Berwald metric with $K = 0$. It just satisfies the condition of Theorem 4, F is locally Minkowskian. The converse is obvious. If F is locally Minkowskian, by the definition of Minkowskian, F is independent of its position x, then the equation (5) is satisfied. Thus F is projectively flat.

3 Generalized *m*-th root metrics

In this section, we consider the generalized *m*-th root metric

$$F = \sqrt{A^{2/m} + B}, \tag{9}$$

where $A=a_{i_1i_2\dots i_m}(x)y^iy^{i_2}\dots y^{i_m}$ and $B=b_{ij}(x)y^iy^j$. We denote A_0 as above and let

$$B_0:=B_{x^l}y^l,\ B_{x^k}:=\frac{\partial B}{\partial x^k},$$

$$B_{x^ly^k}:=\frac{\partial^2 B}{\partial x^l y^k},\ B_{x^ly^k}y^l=B_{0k}.$$

The following lemma is obvious.

Lemma 2 Let $m>4$. Assume the following equation holds.

$$\Theta\sqrt[m]{A^2}+\Xi\sqrt[m]{A^4}+P(A)=0,$$

where Θ and Ξ are polynomials in y. $P(A)$ is a polynomial in A, where $A=a_{i_1i_2\dots i_m}y^iy^{i_2}\dots y^{i_m}$. Then $\Theta=\Xi=P(A)=0$.

Lemma 3 Let $F=\sqrt{A^{2/m}+B}$ $(m>4)$ be a generalized *m*-th root metric on an open subset $U\subset R^n$. Assume that $A=a_{i_1i_2\dots i_m}(x)y^iy^{i_2}\dots y^{i_m}$ is irreducible and $B\neq 0$. A generalized *m*-th root metric $F=\sqrt{A^{2/m}+B}$ is projectively flat on an open subset in R^n then F is a Berwald metric.

Proof By a direct computation, we get

$$F_{x^k}=\frac{2A^{2/m}A_{x^k}+mAB_{x^k}}{2mA(A^{2/m}+B)^{1/2}},$$

$$F_{x^ly^k}=-\frac{1}{4}\left(\frac{2A^{2/m}A_{x^l}}{mA}+B_{x^l}\right)\left(\frac{2A^{2/m}A_{y^k}}{mA}+B_{y^k}\right)\cdot (A^{2/m}+B)^{-3/2}+\frac{1}{2}\left(\frac{4A^{2/m}A_{y^k}A_{x^l}}{m^2A^2}+\frac{2A^{2/m}A_{x^ly^k}}{mA}-\frac{2A^{2/m}A_{x^l}A_{y^k}}{mA^2}+B_{x^ly^k}\right)(A^{2/m}+B)^{-1/2}.$$

Then by the equation (5), we obtain

$$\Theta\sqrt[m]{A^2}+\Xi\sqrt[m]{A^4}+P(A)=0, \tag{10}$$

where

$$\Theta=-(1/2)m^2(B_{x^k}-B_{0k})A^2-(m-2)A_0A_{y^k}B-(1/2)m(A_0B_{y^k}-2A_{0k}B+B_0A_{y^k}+2A_{x^k}B)A,$$

$$\Xi=-m(A_{x^k}-A_{0k})A-(m-1)A_0A_{y^k}, \tag{11}$$

$$P(A)=(1/4)m^2A^2(-2B_{0k}B+B_0B_{y^k}+2B_{x^k}B).$$

Because $m>4$, then $\sqrt[m]{A^4}$ is a irrational expression. By Lemma 2, we have

$$\Theta=0, \tag{12}$$

$$\Xi=0, \tag{13}$$

$$P(A)=0.$$

By (13), we have $mA(A_{0k}-A_{x^k})=(m-1)A_0A_{y^k}$, since A is irreducible and $\deg(A_{y^k})=m-1$, then A_0 is divisible by A, that is, there is a 1-form η such that

$$A_0=2m\eta A. \tag{14}$$

Substituting (14) into (11) and (13), yields

$$A_{0k}=A_{x^k}+2m\eta A_{y^k}-2\eta A_{y^k}. \tag{15}$$

Plugging (14) and (15) into (12), we get

$$(1/2)m^2A(2\eta B_{y^k}-B_{0k}B_{x^k})=mA_{y^k}(2\eta B-(1/2)B_0). \tag{16}$$

Obviously, the right side of (16) is divisible by A. By the assumption A is irreducible, $\deg(A_{y^k})$ and $\deg(2\eta B-(1/2)B_0)$ are both less than $\deg(A)$, so we have

$$B_0=4\eta B. \tag{17}$$

By (14) and (17), we get the spray coefficients $G^i=Py^i$ with

$$P=\frac{F_{x^l}y^l}{2F}=\frac{2A^{2/m}A_0+mAB_0}{4mA(A^{2/m}+B)}=\frac{4m\eta AA^{2/m}+4m\eta AB}{4mA(A^{2/m}+B)}=\eta.$$

Then F is a Berwald metric.

Proof of Theorem 2 By Lemma 3, we get if F is projectively flat then F is a Berwald metric. When $m>4$, by Theorem 3, if the scalar flag curvature $K\neq 0$, every Berwald metric must be Riemaniann. But the case of $B=0$ and A is a perfect square of a Riemaniann contradicts our assumption. Therefore $K=0$, that is to say F is a Berwald metric with the scalar flag curvature zero, then it is locally Minkowskian by Theorem 4. The sufficiency is obvious.

References:

[1] Shen Z M. On projectively flat (α,β) -metrics[J]. Canadian Mathematical Bulletin, 2009, 52:132-144.

[2] Berwald L. Uber die *n*-dimensionalen geometrien konstanter Krummung, in denen die Geraden die durzesten sind[J]. Math Z, 1929, 30:449-469.

[3] Matsumoto M. Theory of Finsler spaces with *m*-th root

metric[J]. Publl Math Debrecen, 1996, 49:135-155.
[4] Matsumoto M, Okubo K. Theory of Finsler spaces with *m*-th root metric: Connections and main scalars[J]. Tensor (N.S),1995, 56:93-104.
[5] Shimada H. On Finsler space with the metric $L=(a_{i_1i_2\ldots i_m}y^iy^{i_2}\ldots y^{i_m})^{1/m}$ [J]. Tensor (N.S), 1979, 33:365-372.
[6] Kim B, Park H. The m-th root Finsler metrics admitting (α,β)-types[J]. Bull Korean Math Soc, 2004, 41:45-52.
[7] Li B L, Shen Z M. On a class of projectively flat Finsler metrics with constant flag curvature[J]. Int J Math, 2007, 18(7):749-760.
[8] Yu Y Y, You Y. On Einstein *m*-th root metrics[J]. Differential Geometry and its Applications, 2010, 28:290-294.
[9] Shen Z M. Projectively flat Finsler metrics of constant flag curvature[J]. Trans Amer Math Soc, 2003, 355(4): 1713-1728.
[10] Cheng X Y, Shen Z M. Projectively flat Finsler metrics with almost isotrpic S-curvature[J]. Acta Mathematica Scientia, 2006, 26:307-313.
[11] Numata S. On Landsberg spaces of scalar curvature[J]. J Korea Math Soc, 1975, 12:97-100.
[12] Berwald L. Parallelubertragung in allgemeinen Raumen [J]. Atti Congr Intern Mat Bologna, 1928, 4:263-270.
[13] Hamel G. Uber die Geometrieen in denen die Geraden die KUrzesten sind[J]. Math Ann, 1903, 57:231-264.

具有 m 次根的射影平坦芬斯勒度量

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摘要：在 R^n 上的开子集射影平坦芬斯勒度量是希尔伯特第四问题的正则情况．作者研究了 m 次根的芬斯勒度量以及广义的 m 次根的芬斯勒度量，证明了在不可约的条件下这种度量是局部闵科夫斯基的．

关键词：芬斯勒度量；射影平坦；m 次根度量；局部闵科夫斯基

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