Existence and Uniqueness of the Positive Solution to a Predator-prey Model with Stochastic Perturbation

XU Wen-qin, ZHANG Jian-xun*

(Faculty of Science, Ningbo University, Ningbo 315211, China)

Abstract: In this paper, a predator-prey model is established with the modified Holling-type II schemes with stochastic perturbation. We mainly use Lyapunov function and comparison theorem to show that there is a unique positive solution to the system with positive initial value.

Key words: Itôs formula; Brownian motion; existence and uniqueness

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1 Introduction

The dynamic relationship between the predator and the prey has long been and will continue to be one of the dominant themes in population dynamics due to its universal existence and importance [1]. The functional response, which is given by $ax/(b+x)$, is a familiar nonlinear factor. Recently, the dynamics of a predator-prey system with Holling-type II functional response has been well studied [2-6].

In fact, mutual interference between predator and prey is also a factor affect the population dynamics. Leslie [7] introduced a predator-prey model where the carry capacity of the predator’s environment is proportional to the number of prey and the predator dynamics, which can be described by the equation:

$$\mathrm{d}y/\mathrm{d}t = ry(1-y/\alpha x).$$

He also thinks that the predator y can switch over to other population when the prey population is severely scarce. But its growth will be limited, because there is the fact that its most favorite food, the prey x is not in abundance. Therefore, a positive constant can be added to the denominator and the equation above becomes:

$$\mathrm{d}y/\mathrm{d}t = ry(1-y/(\alpha x+d)).$$

Based on paper [8], has proposed a first study of the following two-dimensional system of autonomous differential equation:

$$\begin{cases}\mathrm{d}x/\mathrm{d}t = x(t)(a-bx(t)-cy(t)/(m_1+x(t)), \\ \mathrm{d}y/\mathrm{d}t = y(t)(r-fy(t)/(m_2+x(t)).\end{cases} \tag{1}$$

However, population dynamics is inevitably affected by environmental white noise which is an important component in an ecosystem. Therefore, lots of authors introduced stochastic perturbation. Recently, studied the predator-prey model (1) with stochastic perturbation and successfully proved the existence and uniqueness of the positive solution [9].

Motivated by these, we can suppose that the carry capacity of the prey’s environment is inversely proportional to the number of predator. So, we can describe this case by the following equation:

$$\mathrm{d}x/\mathrm{d}t = rx(1-xy/\alpha),$$

where $\alpha>0$ is the conversion factor of predator into prey. The term xy/α measures the increment in the

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The first author: XU Wen-qin (1985−), female, Hanchuan Hubei, post of graduate student, research domain: biological mathematics.
E-mail: xuwenqin512@163.com
***Corresponding author:** ZHANG Jian-xun (1959−), male, Qishan Shanxi, professor, research domain: biological mathematics.
E-mail: zhangjianxun@nbu.edu.cn

prey population due to the rarity of the predator. We can predicate that other predators will switch over to the prey when the predator population is severely scare, but the prey's reduction will be limited, because the prey is not the most favorite food for other predators. Similarly in this situation, we can add a positive constant, then the equation becomes $\mathrm{d}x/\mathrm{d}t = rx(1-x/(\alpha/y+1/d))$, and thus, $\mathrm{d}x/\mathrm{d}t = x(r - rd\cdot xy/(y+\alpha d))$.

Then, we can consider a predator-prey model with modified Holling-type II:

$$\begin{cases} \dot{x}(t) = x(t)(r - \frac{fx(t)y(t)}{m+y(t)}), \\ \dot{y}(\mathrm{t}) = y(t)(-a - by(t) + \frac{cx(t)y(t)}{m+y(t)}), \end{cases} \tag{2}$$

with initial value $x(0) = x_0 > 0, y(0) = y_0 > 0$, where $x(t)$ and $y(t)$ represent the population densities at time t, and parameters a, b, c, f, r and m are all positive. These parameters are defined as follows: a is the death rate of predator y, b measures the strength of competition among individuals of species y, m measures the extent to which the environment provides protection to prey x and to the predator y, r describes the growth rate of prey x.

In this paper, we also consider the effect of randomly fluctuating environment. So, we can incorporate white noise in each equations of the system (2). We assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the growth rate of the prey population and the predator population, specifically,

$$r \to r + \alpha \mathrm{d}B_1(t), \quad a \to a + \beta \mathrm{d}B_2(t),$$

where $B_1(t), B_2(t)$ are mutually independent Brownian motions, α and β represent the intensities of the white noise. Then corresponding to the deterministic model system (2), the stochastic system takes the following form:

$$\begin{cases} \mathrm{d}x(t) = x(t)(r - \frac{fx(t)y(t)}{m+y(t)})\mathrm{d}t + \alpha x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y(t)(-a - by(t) + \frac{cx(t)y(t)}{m+y(t)})\mathrm{d}t - \beta y(t)\mathrm{d}B_2(t). \end{cases} \tag{3}$$

We mainly use Lyapunov function, Itôs formula, the theory of linear stochastic equation to estimate the positive solution of system (3). As we all know, for system (2), there is a trivial equilibrium point $E_0 = (0, 0)$, and a unique interior equilibrium point $E^* = (x^*, y^*)$, if the condition $r/f > a/c$ is satisfied, here $x^* = r(m+y^*)/fy^*, y^* = (rc - af)/bf$. While for stochastic system (3), there is not positive time independent equilibrium point as a solution of the corresponding equation.

Throughout this paper, unless otherwise specified, we let $(\Omega, F, \{F_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and F_0 contains all P-null sets). Let $B_1(t), B_2(t)$ denote the independent standard Brownian motions defined on this probability space.

2 Existence and uniqueness of the positive solution

As the $x(t), y(t)$ in Eq. (3) are population densities of the prey and the predator at time t respectively, we are only interested in the positive solutions. Moreover, for a stochastic differential equation, in order to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy local Lipschitz condition and the linear growth condition (cf. Arnold [1]; Friedman [10]; Mao [11]). However, the coefficients of Eq. (3) neither satisfy local Lipschitz condition, nor the linear growth condition. In this section, we will show existence and uniqueness of the solution by making the change of variables, comparison theorem and Lyapunov functions for stochastic equation [12-13].

Theorem There exists a unique positive global solution $x(t), y(t)$ for $t \in [0, \tau_e)$ to Eq. (3) a.s. with the initial value $x_0 > 0, y_0 > 0$.

Proof Consider the following system:

$$\begin{cases} \mathrm{d}u = (r - \frac{\alpha^2}{2} - \frac{fe^u e^v}{m+e^v})\mathrm{d}t + \alpha \mathrm{d}B_1(t), \\ \mathrm{d}v = (-a - \frac{\beta^2}{2} - be^v + \frac{ce^u e^v}{m+e^v})\mathrm{d}t - \beta \mathrm{d}B_2(t), \end{cases} \tag{4}$$

with initial value $u_0 = \ln x_0, v_0 = \ln y_0$. Obviously, the coefficients of Eq. (4) satisfy local Lipschitz condition, then there is a unique local solution $u(t), v(t)$ on $t \in$

$[0,\tau_e)$, where τ_e is the explosion time (see Arnold [1] and Friedman [10]). Therefore, by Itôs formula, it is easy to check $x(t)=e^{u(t)}, y(t)=e^{v(t)}$ is the unique positive local solution to Eq. (3) with the initial value $x_0>0$, $y_0>0$.

Next, we will use two different methods to prove this solution is global, i.e. $\tau_e=\infty$.

Method one Let $k_0 \geqslant 0$ be sufficiently large in order that both x_0 and y_0 lie within the interval $[1/k_0, k_0]$. For each integer $k \geqslant k_0$, define the stopping time:

$$\tau_k=\inf\{t\in[0,\tau_e):\min\{x(t),y(t)\}\leqslant 1/k \text{ or } \max\{x(t),y(t)\}\geqslant k\},$$

where throughout this paper, we set $\inf\varnothing=\infty$ (as usual $\varnothing$ denotes the empty set). Obviously, τ_k is increasing as $k\to\infty$. Set $\tau_\infty=\lim_{k\to\infty}\tau_k$, whence $\tau_\infty\leqslant\tau_e$ a.s., then $\tau_e=\infty$ and $x(t)>0, y(t)>0$ a.s. for all $t\geqslant 0$. That is to say, to complete the proof, all we need to show is that $\tau_\infty=\infty$ a.s.. If this conclusion is not right, then there are a pair of constants $T>0$ and $\varepsilon\in[0,1)$ such that $P\{\tau_\infty\leqslant T\}>\varepsilon$.

Hence there is an integer $k_1'\geqslant k_0$ such that

$$P\{\tau_\infty\leqslant T\}\geqslant\varepsilon \text{ for all } k\geqslant k_1'. \tag{5}$$

Define a C^2-function $V: R_+^2\to\overline{R}_+$ by

$$V(x,y)=c(x-\ln x)+f(y-\ln y).$$

The non-negativity of this function can be seen from $u-\ln u\geqslant 1, \forall u>0$. Using Itôs formula, we get

$$\begin{aligned}\mathrm{d}V=&c(x-1)[(r-\frac{fxy}{m+y})\mathrm{d}t+\alpha\mathrm{d}B_1(t)]+c\alpha^2/2\mathrm{d}t+\\&f(y-1)[(-a-by+\frac{cxy}{m+y})\mathrm{d}t-\beta\mathrm{d}B_2(t)]+\\&f\beta^2/2\mathrm{d}t:=LV\mathrm{d}t+c(x-1)\alpha\mathrm{d}B_1(t)-\\&f(y-1)\beta\mathrm{d}B_2(t),\end{aligned}$$

where

$$\begin{aligned}LV=&cr(x-1)+c\alpha^2/2+f\beta^2/2+f(y-1)\\&(-a-by)+cfxy(y-x)/(m+y)\leqslant crx+\\&c(-r+\alpha^2/2)+fy(b-a)+f(a+\beta^2/2)-\\&bfy^2+cfxy^2/(m+y)\leqslant c(-r+\alpha^2/2)+f(a+\\&\beta^2/2)+crx+f(b-a)y-bfy^2+cfxy\leqslant M,\end{aligned}$$

and M is a positive constant. Therefore

$$\int_0^{\tau_k\wedge T}\mathrm{d}V(x(t),y(t))\leqslant\int_0^{\tau_k\wedge T}M\mathrm{d}t+\int_0^{\tau_k\wedge T}c\alpha(x-1)\mathrm{d}B_1(t)-\int_0^{\tau_k\wedge T}f\beta(y-1)\mathrm{d}B_2(t),$$

which implies that

$$\begin{aligned}&E\left[V(x(\tau_k\wedge T),y(\tau_k\wedge T))\right]\leqslant V(x(0),y(0))+\\&E\int_0^{\tau_k\wedge T}M\mathrm{d}t\leqslant V(x(0),y(0))+MT.\end{aligned}\tag{6}$$

Set $\Omega_k=\{\tau_k\leqslant T\}$ for $k\geqslant k_1'$, then by Eq. (5), we know $P(\Omega_k)\geqslant\varepsilon$. Note that for every $\omega\in\Omega_k$, there is at least one of $x(\tau_k,\omega), y(\tau_k,\omega)$ equaling either k or $1/k$, then $V(x(\tau_k),y(\tau_k))$ is no less than $k-\ln k$ or $1/k-\ln 1/k=1/k+\ln k$. Consequently,

$$V(x(\tau_k),y(\tau_k))\geqslant(k-\ln k)\wedge(1/k+\ln k).$$

It then follows from Eq. (5) and Eq. (6) that

$$\begin{aligned}&V(x(0),y(0))+MT\geqslant E[1\Omega_k(w)V(x(\tau_k),y(\tau_k))]\geqslant\\&\varepsilon[(k-\ln k)\wedge(1/k+\ln k)],\end{aligned}$$

where $1\Omega_k(w)$ is the indicator function of Ω_k. Letting $k\to\infty$, we get $\infty>V(x(0),y(0))+MT=\infty$, which leads to the contradiction. So we must have $\tau_k=\infty$ a.s..

Method two Since the solution is positive, we have $\mathrm{d}x(t)\leqslant rx(t)\mathrm{d}t+\alpha\mathrm{d}B_1(t)$.

Let

$$\Phi_2(t)=\frac{e^{(r-\frac{\alpha^2}{2})t+\alpha B_1(t)}}{\frac{1}{x_0}},\tag{7}$$

then $\Phi_2(t)$ is the unique solution of the equation:

$$\begin{cases}\mathrm{d}\Phi_2(t)=r\Phi_2(t)\mathrm{d}t+\alpha\Phi_2(t)\mathrm{d}B_1(t),\\\Phi_2(0)=x_0.\end{cases}\tag{8}$$

The comparison theorem for stochastic equations yields $x(t)\leqslant\Phi_2(t), t\in[0,\tau_e)$, a.s..

Besides,

$$\mathrm{d}x(t)\geqslant x(t)(r-fx(t))\mathrm{d}t+\alpha x(t)\mathrm{d}B_1(t).$$

Let

$$\Phi_1(t)=\frac{e^{(r-\frac{\alpha^2}{2})t+\alpha B_1(t)}}{\frac{1}{x_0}+f\int_0^t e^{(r-\frac{\alpha^2}{2})s+\alpha B_1(s)}\mathrm{d}s}.\tag{9}$$

Similarly, we can get $x(t)\geqslant\Phi_1(t), t\in[0,\tau_e)$, a.s, on the other hand,

$$\mathrm{d}y(t)\geqslant y(t)(-a-by(t))\mathrm{d}t-\beta y(t)\mathrm{d}B_2(t).$$

Obviously,

$$\Psi_1(t)=\frac{e^{(-a-\frac{\beta^2}{2})t-\beta B_2(t)}}{\frac{1}{y_0}+b\int_0^t e^{(-a-\frac{\beta^2}{2})s-\beta B_2(s)}\mathrm{d}s},\tag{10}$$

is the solution to the equation:

$$\begin{cases} d\Psi_1(t) = \Psi_1(t)(-a - by(t))\mathrm{d}t - \beta y(t)\mathrm{d}B_2(t), \\ \Psi_1(0) = y_0, \end{cases} \quad (11)$$

and $y(t) \geqslant \Psi_1(t), t \in [0, \tau_e)$, a.s..

Now, we have

$$\mathrm{d}y(t) \leqslant y(t)(-a - by(t) + c\Phi_2(t))\mathrm{d}t - \beta y(t)\mathrm{d}B_2(t).$$

By the arguments as above, we can get

$$y(t) \leqslant \frac{e^{(-a-\frac{\beta^2}{2})t - \beta B_2(t) - c\int_0^t \Phi_2(s)\mathrm{d}s}}{\frac{1}{y_0} + b\int_0^t e^{(-a-\frac{\beta^2}{2})s - c\int_0^s \Phi_2(u)\mathrm{d}u - \beta B_2(s)}\mathrm{d}s} := \Psi_2(t),$$

$$t \in [0, \tau_e), \text{ a.s.}. \quad (12)$$

To sum up, there are $\Phi_1(t) \leqslant x(t) \leqslant \Phi_2(t)$ and $\Psi_1(t) \leqslant y(t) \leqslant \Psi_2(t), t \in [0, \tau_e)$, a.s..

Noting that $\Phi_1(t)$, $\Phi_2(t)$, $\Psi_1(t)$ and $\Psi_2(t)$ are all exist on $t \geqslant 0$, hence we have there is a unique positive solution $x(t), y(t), t \geqslant 0$ to Eq. (3) a.s. for any initial value $x_0 > 0, y_0 > 0$.

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一类带有随机扰动的捕食-食饵模型正解的存在唯一性

许温琴，张建勋*

（宁波大学 理学院，浙江 宁波 315211）

摘要:主要讨论了一类带有随机扰动和改进了的功能性反应函数 II 的捕食-食饵模型，通过构造 Lyapunov 函数和使用比较定理两种不同的方法证明了其正解的存在唯一性.

关键词：伊藤公式；布朗运动；存在唯一性

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