Statistics, Causality and Bell's theorem

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Abstract

Bell's (1964) theorem is popularly supposed to establish the non-locality of quantum physics as a mathematical-physical theory. Building from this, observed violation of Bell's inequality in experiments such as that of Aspect and coworkers (1982) is popularly supposed to provide empirical proof of non-locality in the real world. This paper reviews recent work on Bell's theorem, linking it to issues in causality as understood by statisticians. The paper starts with a new proof of a strong (finite sample) version of Bell's theorem which relies only on elementary arithmetic and (counting) probability. This proof underscores the fact that Bell's theorem tells us that quantum theory is incompatible with the conjunction of *three* cherished and formerly uncontroversial physical principles, nicknamed here *locality*, *realism*, and *freedom*. The first, locality, is obviously connected to causality: causal influences need time to propagate spatially. Less obviously, the other two principles, realism and freedom, are also founded on two ideas central to modern statistical thinking on causality: counterfactual reasoning, and the distinction between *do*-ing $X = x$ and *selecting* on $X = x$, respectively. I will argue that (accepting quantum theory) Bell's theorem should lead us to seriously consider relinquishing not locality, but realism, as a fundamental physical principle. The paper goes on to discuss statistical issues, in the interpretation of state-of-the-art Bell type experiments, related to post-selection in observational studies. Finally I state an open problem concerning the design of a quantum Randi challenge: a computer challenge to Bell-deniers.

1 Introduction

In this paper I want to discuss Bell's (1964) theorem from the point of view of causality as understood in statistics and probability. The paper complements and extends the work of Robins, VanderWeele, and Gill (2011), whose general aim is identical.

Bell's theorem states that certain predictions of quantum mechanics are incompatible with the conjunction of *three* fundamental principles of classical physics which are sometimes given the short names "realism", "locality" and "freedom". Corresponding real world experiments, Bell experiments, are supposed to demonstrate that this incompatibility is a property not just of the the theory of quantum mechanics, but also of Nature itself. The consequence is that we are forced to reject one (or more) of these three principles.

Both theorem and experiment hinge around an inequality constraining joint probability distributions (note plural!) of outcomes of measurements on spatially separated physical systems; an inequality which must hold if all three fundamental principles are true. In a nutshell, the inequality is an empirically verifiable consequence of the idea that the outcome of one measurement on one system cannot depend on *which* measurement is performed on the other. This principle, called *locality* or, less succinctly, *relativistic local causality*, is just one of the three principles. Its formulation refers to outcomes of measurements which are not actually performed, so we have to assume their existence, alongside of the outcomes of those actually performed: the principle of *realism*, or more precisely, *counterfactual definiteness*. Finally we need to assume that we have complete *freedom* to choose *which* of several measurements to perform – this is the third principle, also called the *no-conspiracy* principle.

We shall implement the freedom assumption as the assumption of statistical independence between the randomization in a randomized experimental design (i.e., the choice of experiment), and the outcomes of all the possible experiments combined. This includes the "counterfactual" outcomes of those experiments which were not actually performed, as well as the "factual" outcome of the experiment actually chosen. By existence of the outcomes of not actually performed experiments, we mean their mathematical existence within a mathematical physical theory of the phenomenon in question. The concepts of realism and locality together are often considered as one principle called *local realism*. Local realism is implied by the existence of *local hidden variables*, whether deterministic or stochastic. In a precise mathematical sense, the reverse implication is also true: local realism implies that we can construct a local hidden variable model for the phenomenon under study. However one likes to think of this assumption (or pair of assumptions), the important thing to realize is that it is a completely unproblematic notion in classical physics; freedom (no conspiracy) even more so.

To begin with I will establish a new version of the famous Bell inequality (more precisely: Bell-CHSH inequality) using very elementary logic, arithmetic and (discrete) probability. My version will not be an inequality involving (theoretical) expectation values of physical quantities, but it will be a probabilistic inequality involving experimentally observed averages. Moreover, the probabilistic component does not refer to random variation in the outcome of a given measurement on a physical system, but to the experimenter's freedom to choose which measurement to perform: i.e., to the randomness involved in implementing a randomized experimental design. Proving Bell's theorem in this way avoids reliance on abstract concepts (theoretical expectation values) and expresses the result directly in terms of observational data. The proof is new, complete, and completely elementary – it can be explained to a science journalist or to an intelligent teenager. It brings out the importance of the "realism" and "freedom" assumptions alongside of the "locality" assumption, making clear that a violation of Bell's inequality implies that one or more of the three assumptions must fail, without determining which of the three is at fault.

In view of the experimental support for violation of Bell's theorem (despite shortcomings of experiments done to date, to be described later in the paper), the present writer prefers to imagine a world in which "realism" is not a fundamental principle of physics but only an emergent property in the familiar realm of daily life (including the world of applied statisticians). In this way we can have quantum mechanics and locality and freedom. He believes that within this position, the measurement problem (Schrödinger cat problem) has a decent mathematical solution. This position does entail taking quantum randomness very seriously: it becomes an irreducible feature of the physical world, a "primitive notion"; it is not "merely" an emergent feature. It is moreover fundamentally connected with an equally fundamental arrow of time ... and all this is the logical consequence of demanding that quantum physics respect temporal and spatial *causality*.

2 Bell's inequality

We will derive a version of Bell's inequality in three steps involving elementary logic, arithmetic and probability respectively. Throughout this section I use the word *mean* as shorthand for *arithmetic mean*; thus: "mean", as in "$\overline{x}$". It is not meant to imply taking theoretical expectation values, but is simply a synonym for *average*.

2.1 Logic

Lemma 1 *For any four numbers A, A', B, B' each equal to ± 1,*

$$AB + AB' + A'B - A'B' \;=\; \pm\, 2. \tag{1}$$

Proof of Lemma 1 Notice that

$$AB + AB' + A'B - A'B' \;=\; A(B + B') + A'(B - B').$$

B and B' are either equal to one another or unequal. In the former case, $B - B' = 0$ and $B + B' = \pm 2$; in the latter case $B - B' = \pm 2$ and $B + B' = 0$. Thus $AB + AB' + A'B - A'B'$ equals ± 2 times A or times A', which both equal ± 1. Either way, $AB + AB' + A'B - A'B' = \pm 2$. □

2.2 Arithmetic

Consider a spreadsheet containing a $4N \times 4$ table of numbers ± 1. The rows will be labelled by an index $j = 1, \ldots, 4N$. The columns are labelled with names A, A', B and B'. I will denote the four numbers in the jth row of the table by A_j, A'_j, B_j and B'_j. Denote by $\langle AB \rangle = (1/4N) \sum_{j=1}^{4N} A_j B_j$, the mean over the $4N$ rows of the product of the elements in the A and B columns. Define $\langle AB' \rangle$, $\langle A'B \rangle$, $\langle A'B' \rangle$ similarly.

Lemma 2

$$\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \;\leq 2. \tag{2}$$

Proof of Lemma 2 By (1),

$$\begin{aligned}&\langle AB\rangle + \langle AB'\rangle + \langle A'B\rangle - \langle A'B'\rangle \\ &= \ \langle AB + AB' + A'B - A'B'\rangle \ \in \ [-2, 2] \qquad \square\end{aligned}$$

Formula (2) is "just" the CHSH inequality (Clauser, Horne, Shimony and Holt, 1969), the basis of standard proofs of Bell's theorem, espoused by Bell in his later work, and the *raison d'être* of many modern experiments confirming Bell's theorem; see next section.

2.3 Probability

Now suppose that for each row of the spreadsheet, two fair coins are tossed independently of one another, independently over all the rows. Suppose that depending on the outcomes of the two coins, we either observe A or A', and we either observe B or B'. We are therefore able to observe just one of the four products AB, AB', $A'B$, and $A'B'$, each with equal probability $\frac{1}{4}$, for each row of the table. Denote by $\langle AB\rangle_{\mathrm{obs}}$ the mean of the observed products of A and B ("undefined" if the sample size is zero). Define $\langle AB'\rangle_{\mathrm{obs}}$, $\langle A'B\rangle_{\mathrm{obs}}$ and $\langle A'B'\rangle_{\mathrm{obs}}$ similarly.

When N is large one would expect $\langle AB\rangle_{\mathrm{obs}}$ to be not much different from $\langle AB\rangle$, and the same for the other three means of observed products. Hence, equation (2) should remain approximately true when we replace the means of the four products over all $4N$ rows with the means of the four products in each of four disjoint sub-samples of expected size N each.

This intuitively obvious fact can be put into numbers through use of the Hoeffding (1963) inequality, a uniform exponential bound to the tails of the binomial distributions and hypergeometic distributions (i.e., sampling with and without replacement). The rest of this subsection will provide the details of a proof of such a probabilistic ("finite statistics") version of the Bell-CHSH inequality:

Theorem 1 *Given a $4N \times 4$ spreadsheet of numbers ± 1 with columns A, A', B and B', suppose that, completely at random, just one of A and A' is observed and just one of B and B' are observed in every row. Then, for any $0 \le \eta \le 2$,*

$$\begin{aligned}\Pr\big(\, \langle AB\rangle_{\mathrm{obs}} + \langle AB'\rangle_{\mathrm{obs}} + \langle A'B\rangle_{\mathrm{obs}} - \langle A'B'\rangle_{\mathrm{obs}} \ \le \ 2 + \eta \,\big) \\ \ge \ 1 - 8e^{-4N(\frac{\eta}{16})^2}. \qquad (3)\end{aligned}$$

Traditional presentations of Bell's theorem focus on what could be called the large N limit of this result. If it is true that as $N \to \infty$, experimental averages converge to some kind of theoretical mean values, then these must satisfy

$$\langle AB\rangle_{\mathrm{lim}} + \langle AB'\rangle_{\mathrm{lim}} + \langle A'B\rangle_{\mathrm{lim}} - \langle A'B'\rangle_{\mathrm{lim}} \ \le \ 2. \qquad (4)$$

Like (2), this inequality is also called the CHSH inequality. There is actually quite a lot in between, as we have seen.

The proof of (3) will use the following two results:

Fact 1 (Hoeffding's theorem, binomial case) *Suppose $X \sim Bin(n,p)$ and $t > 0$. Then*

$$\Pr(X/n \ \geq \ p + t) \ \leq \ \exp(-2nt^2).$$

Fact 2 (Hoeffding's theorem, hypergeometric case) *Suppose X is the number of red balls found in a sample without replacement of size n from a vase containing pM red balls and $(1-p)M$ blue balls and $t > 0$. Then*

$$\Pr(X/n \ \geq \ p + t) \ \leq \ \exp(-2nt^2).$$

Hoeffding's paper gives other, even sharper results – variants of Bennet's inequality, for instance. I have gone for simplicity in the expression of the probability inequality (3), not for the sharpest result possible.

Proof of Theorem 1 In each row of our $4N \times 4$ table of numbers ± 1, the product AB equals ± 1. For each row, with probability 1/4, the product is either observed or not observed. Let N^{obs}_{AB} denote the number of rows in which both A and B are observed. Then $N^{\text{obs}}_{AB} \sim \text{Bin}(4N, 1/4)$, and hence by Fact 1, for any $\delta > 0$,

$$\Pr\Bigl(N^{\text{obs}}_{AB} \leq (1-4\delta)N\Bigr) \ = \ \Pr\Bigl(\frac{N^{\text{obs}}_{AB}}{4N} \ \leq \ \tfrac{1}{4} - \delta\Bigr) \ \leq \ \exp(-8N\delta^2).$$

Let N^{+}_{AB} denote the total number of rows (i.e., out of $4N$) for which $AB = +1$, define N^{-}_{AB} similarly. Let $N^{\text{obs},+}_{AB}$ denote the number of rows such that $AB = +1$ among those selected for observation of A and B. Conditional on $N^{\text{obs}}_{AB} = n$, $N^{\text{obs},+}_{AB}$ is distributed as the number of red balls in a sample without replacement of size n from a vase containing $4N$ balls of which N^{+}_{AB} are red and N^{-}_{AB} are blue. Therefore by Fact 2, conditional on $N^{\text{obs}}_{AB} = n$, for any $\epsilon > 0$,

$$\Pr\Bigl(\frac{N^{\text{obs},+}_{AB}}{N^{\text{obs}}_{AB}} \geq \frac{N^{+}_{AB}}{4N} + \epsilon\Bigr) \leq \exp(-2n\epsilon^2).$$

We introduced above the notation $\langle AB \rangle$ for the mean of the product AB over the whole table; this can be rewritten as

$$\langle AB \rangle \ = \ \frac{N^{+}_{AB} - N^{-}_{AB}}{4N} \ = \ 2\frac{N^{+}_{AB}}{4N} - 1.$$

Similarly, $\langle AB \rangle_{\text{obs}}$ denoted the mean of the product AB just over the rows of the table for which both A and B are observed; this can be rewritten as

$$\langle AB \rangle_{\text{obs}} \ = \ \frac{N^{\text{obs},+}_{AB} - N^{\text{obs},-}_{AB}}{N^{\text{obs}}_{AB}} \ = \ 2\frac{N^{\text{obs},+}_{AB}}{N^{\text{obs}}_{AB}} - 1.$$

Given $\delta > 0$ and $\epsilon > 0$, all of N^{obs}_{AB}, $N^{\text{obs}}_{AB'}$, $N^{\text{obs}}_{A'B}$ and $N^{\text{obs}}_{A'B'}$ are at least $(1-4\delta)N$ with probability at least $1 - 4\exp(-8N\delta^2)$. On the event where this happens, the conditional probability that $\langle AB \rangle_{\text{obs}}$ exceeds $\langle AB \rangle + 2\epsilon$ is bounded by

$$\exp(-2N^{\text{obs}}_{AB}\epsilon^2) \leq \exp(-2N(1-4\delta)\epsilon^2).$$

The same is true for the other three means (for the last one we first exchange the roles of $+$ and $-$ to get a bound on $\langle -A'B'\rangle_{\mathrm{obs}}$). Combining everything, we get that

$$\langle AB\rangle_{\mathrm{obs}} + \langle AB'\rangle_{\mathrm{obs}} + \langle A'B\rangle_{\mathrm{obs}} - \langle A'B'\rangle_{\mathrm{obs}} \ \leq \ 2 + 8\epsilon,$$

except possibly on an event of probability at most

$$p \ = \ 4\exp(-8N\delta^2) + 4\exp(-2N(1-4\delta)\epsilon^2).$$

Choosing $\delta = \epsilon/2\sqrt{2}$ and restricting attention to $2(1-4\delta) \geq 1$, i.e., $\delta \leq 1/8$, $\epsilon \leq 1/2\sqrt{2}$, we can bound p by the simpler expression

$$p \ \leq \ 8\exp(-N\epsilon^2).$$

Replacing 8ϵ by η gives us (3), since $\eta \leq 2$ implies $\epsilon \leq 1/4 < 1/2\sqrt{2}$. $\square$

3 Bell's Theorem

Formulas (2) and (4) (the physics literature often does not notice the difference) are both commonly called Bell's inequality, or the Bell-CHSH inequality, or just the CHSH inequality. The inequality goes back to Clauser, Horne, Shimony and Holt (1969), and is a variant of a similar and earlier inequality of Bell (1964). Both original Bell inequality, and Bell-CHSH inequality, can be used to prove *Bell's theorem*: quantum mechanics is incompatible with the principles of realism, locality and freedom. In other words, if we want to hold on to all three principles, quantum mechanics must be rejected. Alternatively, if we want to hold on to quantum theory, we have to relinquish at least one of those three principles.

The *executive summary* of the proof consists of the following remark: certain models in quantum physics predict

$$\langle AB\rangle_{\mathrm{lim}} + \langle AB'\rangle_{\mathrm{lim}} + \langle A'B\rangle_{\mathrm{lim}} - \langle A'B'\rangle_{\mathrm{lim}} \ = \ 2\sqrt{2} \ \gg \ 2. \tag{5}$$

Moreover, as far as most authorities in physics are concerned, the prediction (5) has been amply confirmed by experiment: therefore, whatever we may think of quantum theory, at least one of locality, realism or freedom has to go.

Almost no-one is prepared to abandon freedom. The consensus in recent years was that locality was at fault, but it seems to be shifting in recent years towards putting the blame on realism. In essence, this is "back to Bohr" (Copenhagen interpretation): not in the form of a dogma, a prohibition to speak of "what is actually going on behind the scenes", but positively embraced: intrinsic quantum randomness is what happens! There is nothing behind the scenes!

It is important to note that experiments do not exhibit a violation of (2): that would be a logical impossibility. However, an experiment could certainly in principle give strong evidence that (3) or (4) is false. We'll engage in some nit-picking about whether or not such experiments have already been done, later in the paper.

Fortunately, one can understand quite a lot of (5) without any understanding of quantum mechanics: we just need to know certain simple statistical predictions which follow from a particular special model in quantum physics called the EPR-B model. The initials refer here to the celebrated paradox of Einstein, Podolsky and Rosen (1935) in a version introduced by David Bohm (1951).

Recall that quantum physics is a *stochastic* theory (the physists say: a *statistical* theory): it allows us to predict the probabilities of outcomes of measurements on quantum systems, not (in general) the actual outcomes. The EPR paradox and Bell's theorem are two landmarks in the history of the ongoing struggle of many physicists over the last century to find a more classical-like theory *behind* quantum theory: to explain the statistics by explaining what is actually happening behind the scenes, as it were. The purpose of this paper is to review that program from a statistician's point of view.

The EPR-B model is a model which predicts the statistics of the measurement of *spin* on an *entangled* pair of *spin-half* quantum systems or *particles* in the *singlet state*. For our purposes it is not necessary to explain this terminology at all: all we need to know are the (statistical) predictions of the model, in the context of an informal description of the corresponding experiment. This informal description will use the word "particle" many times, but only to help the reader to visualize the experimental set-up. The concept of "particle" might or might not be part of a *theoretical* explanation of the experiment, depending on what kind of physics we believe is going on here; but what kind of physics it might be, is precisely the question we are investigating.

In one *run* of the experiment, two *particles* are generated together at a *source*, and then travel to two distant locations. Here, they are measured by two experimenters Alice and Bob. Alice and Bob are each in possession of a measurement apparatus which can "measure the spin of a particle in any chosen direction". Alice (and similarly, Bob) can freely choose (and set) a *setting* on her measurement apparatus. Alice's setting is an arbitrary direction in real three-dimensional space represented by a unit vector $\vec{a}$, say. Her apparatus will then register an observed outcome ± 1 which is called the observed "spin" of Alice's particle in direction $\vec{a}$. At the same time, far away, Bob chooses a direction $\vec{b}$ and also gets to observe an outcome ± 1, the observed "spin" of Bob's particle in direction $\vec{b}$. This is repeated many times – i.e., the complete experiment will consist of many, say $4N$, runs. We'll imagine Alice and Bob repeatedly choosing new settings for each new run, in the same fashion as in Section 2: each tossing a fair coin to make a binary choice between two possible settings, $\vec{a}$ and $\vec{a}'$ for Alice, $\vec{b}$ and $\vec{b}'$ for Bob. First we will complete our description of the quantum mechanical predictions for each run separately.

For pairs of particles generated in a particular quantum state, and with perfectly implemented measurement apparatus, the prediction of quantum mechanics is that in whatever directions Alice and Bob perform their measurements, their outcomes ± 1 are perfectly random (i.e., equally likely $+1$ as -1). However, there is a correlation between the outcomes of the two measurements at the two locations, depending on the two settings: the expected value of the product of the outcomes is given by the inner-product $-\vec{a}\cdot\vec{b} = -\cos(\theta)$ where θ is the angle between the two directions.

In particular, if the two measurement directions are the *same*, the two outcomes will always be *opposite* to one another. If the measurement directions are *orthogonal*, the two outcomes will be statistically *independent* of one another. If the two directions are *opposite* to one another, the two outcomes will always be *equal*. The just mentioned cosine rule is the smoothest possible way one can imagine that Nature could choose to interpolate between these three special cases, which in themselves do not seem surprising at all: one can easily invent a simple mechanistic theory which conforms to those predictions, in which the two particles "agree with one another" at the source, what their responses will be to any possible setting which they will separately encounter at the two measurement stations.

With this information we can write down the complete 2×2 table for the probabilities of the outcomes

$$\begin{array}{cc} ++ & +- \\ -+ & -- \end{array}$$

at the two locations, given two settings differing in direction by the angle θ: the four probabilities are

$$\begin{array}{cc} \frac{1}{4}(1-\cos(\theta)) & \frac{1}{4}(1+\cos(\theta)) \\ \frac{1}{4}(1+\cos(\theta)) & \frac{1}{4}(1-\cos(\theta)). \end{array}$$

Both marginals of the table are uniform. The "correlation", or expectation of the product, equals the probability the outcomes are equal minus the probability they are different, hence is equal to $\frac{2}{4}(1-\cos(\theta)) - \frac{2}{4}(1+\cos(\theta)) = -\cos(\theta)$.

Consider now the following experimental set-up. Alice is allocated in advance two fixed directions $\vec{a}$ and $\vec{a}'$; Bob is allocated in advance two fixed directions $\vec{b}$ and $\vec{b}'$. The experiment is built up of $4N$ runs. In each run, Alice and Bob are each sent one of a new pair of particles in the singlet state. While their particles are *en route* to them, they each toss a fair coin in order to choose one of their two measurement directions. In total $4N$ times, Alice observes either $A = \pm 1$ or $A' = \pm 1$ say, and Bob observes either $B = \pm 1$ or $B' = \pm 1$. At the end of the experiment, four "correlations" are calculated; these are simply the four sample means of the products AB, AB', $A'B$ and $A'B'$. Each correlation is based on a different subset, of expected size N runs, and determined by the $8N$ fair coin tosses.

Under *realism* we can imagine, for each run, alongside of the outcomes of the actually measured pair of variables, also the outcomes of the not measured pair. Now, the outcomes in Alice's wing of the experiment might in principle depend on the choice of which variable is measured in Bob's wing, but under *locality* this is excluded. Thus, for each run there is a suite of potential outcomes A, A', B and B', but only one of A and A', and only one of B and B' actually gets to be observed. By *freedom* the choices are statistically independent of the actual values of the four.

I'll assume furthermore that the suite of counterfactual outcomes in the jth run does not actually depend on which particular variables were observed in previous runs. This *memoryless* assumption, or *Ansatz* as physicists like to say, can be completely avoided by using the martingale version of Hoeffding's inequality, Gill (2003). But the present analysis is already applicable (i.e.,without adding a fourth assumption) in we imagine $4N$ copies of the experiment each with only a single run, all being done simultaneously in widely separated locations. A *Gedankenexperiment* if ever there was one, but perfectly legitimate for present purposes. (Although: possibly cheaper than finding the Higgs boson, and in my opinion a whole lot more exciting).

The assumptions of realism, locality and freedom have put us full square in the situation of the previous section. Therefore by Theorem 1, the four sample correlations (empirical raw product moments) satisfy (3).

Let's contrast this prediction with the quantum mechanical predictions obtained with a certain clever selection of directions. We'll take the four vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ to lie in the same plane. It's then enough to specify the angles $\alpha, \alpha', \beta, \beta' \in [0, 2\pi]$ which they make with respect to some fixed vector in this plane. Consider the choice $\alpha = 0$, $\alpha' = \pi/2$, $\beta = 5\pi/4$, $\beta' = 3\pi/4$. The differences $|\alpha - \beta|$, $|\alpha - \beta'|$, $|\alpha' - \beta|$ are all equal to $\pi \pm \pi/4$: these pairs of angles are all "close to opposite" to one another; the corresponding measurements are strongly positively *correlated*. On the other hand, $|\alpha' - \beta'| = \pi/4$: the two angles are "close to equal" and the corresponding measurements are as strongly *anti-correlated*, as the other pairs were strongly correlated. Three of the correlations are equal to $-\cos(3\pi/4) = -(-1/\sqrt{2}) = 1/\sqrt{2}$ and the fourth is equal to $-\cos(\pi/4) = -1/\sqrt{2}$. Thus we would expect to see, up to statistical variation (statistical variation in the coin toss outcomes!),

$$\langle AB \rangle_{\text{obs}} + \langle AB' \rangle_{\text{obs}} + \langle A'B \rangle_{\text{obs}} - \langle A'B' \rangle_{\text{obs}} \approx 4/\sqrt{2} \ = \ 2\sqrt{2} \ \approx \ 2.828 \ \gg \ 2,$$

cf. (5). By Tsirelson's inequality (Csirel'son, 1980), this is actually the largest absolute deviation from the CHSH inequality which is allowed by quantum mechanics.

Now many experiments have been performed confirming the predictions of quantum mechanics, beautifully. The most famous experiments to date are those performed by Aspect et al. (1982) in Orsay, Paris, and by Weihs et al. (1998) in Innsbruck. In these experiments, the choices of which direction to measure are not literally made with coin tosses performed by human beings, but by physical systems which attempted to imitate such processes as closely as possible. In both cases the separation between the locations of Alice and Bob is large; while the time it takes from initiating the choice of direction to measure to completion of the measurement (the time when an outcome ± 1 is irrevocably committed to a computer data base) is small: so small, that Alice's measurement is complete before a signal traveling at the speed of light could possibly transmit Bob's choice to Alice's location. (This depends on what one considers to be the "time of initiation". Aspect's experiment can be thought to be less rigorous than Weihs' in this respect. More details later).

The data gathered from the Innsbruck experiment is available online. It had $4N \approx 15\,000$; and found $\langle AB \rangle_{\text{obs}} + \langle AB' \rangle_{\text{obs}} + \langle A'B \rangle_{\text{obs}} - \langle A'B' \rangle_{\text{obs}} \ = \ 2.73 \pm 0.022$, the statistical accuracy (standard deviation) following from a standard delta-method calculation assuming i.i.d. observations per setting pair. The reader can check that this indeed corresponds to accuracy obtained by a standard computation using the binomial variances of the estimated probabilities of equal outcomes for each of the four subsamples.

By (3), under realism, locality and freedom, the chance that $\langle AB\rangle_{\text{obs}} + \langle AB'\rangle_{\text{obs}} + \langle A'B\rangle_{\text{obs}} - \langle A'B'\rangle_{\text{obs}}$ would exceed 2.73 is less than 10^{-12}.

I will return to some important features of this data in a later section. The experiment deviates in several ways from what has been described so far, and I will just summarize them now.

An *unimportant* detail is the physical system used: polarization of entangled photons rather than spin of entangled spin-half particles (e.g., electrons). The polarization of entangled photons, in the appropriate state, measured in the same direction, is equal; not opposite; and it is after a rotation of $\pi/2$, not π, that the polarization of such photons becomes "opposite". A polarization filter – e.g., the glass in a pair of polaroid sunglasses – distinguishes between "horizontally" and "vertically" polarized light (difference: 90 degrees); a Stern-Gerlach device distinguishes between "spin up" and "spin down" (difference: 180 degrees). Doubling or halving the angle, and reversal of one of the outcomes, takes us from one model to the other.

An *important* and unjustly neglected difference between the idealization and the truth concerns the idea that in advance we decide to create $4N$ individual pairs of entangled particles. In the real world experiment with photons, there is no way to control when a pair of photons will leave the source. Even talking about "pairs of photons" is using classical physical language which can be acutely misleading. In actual fact, all we observe are individual detection events (time, current setting, outcome) at each of the two detectors, i.e., at each measurement apparatus. We do not observe, let alone control, times of emissions from the source!

Also extremely important (but less neglected) is the fact that (in our naive particle picture) many particles fail to be detected at all. One could say that the outcome of measuring one particle is not binary but ternary: $+$, $-$, or *no detection*.

In combination with the previous difficulty, if particles are not being transmitted at fixed times, then, if neither particle is detected we do not even know if there was a corresponding emission of a pair of particles. The data cannot be summarized in a list of pairs of settings and pairs of outcomes (whether binary of ternary), but consists of two lists of the random times of definite measurement outcomes in each wing of the experiment together with the settings in force at the time of the measurements (which are being rapidly switched, at random times).

When detection events occur close together in time they are treated as belonging to a pair of photons, i.e., as belonging to the same "run" of the experiment. Using the language of "runs" and "photons" for convenience (i.e., without wishing to imply that "runs" and "photons" are objective concepts): not every photon which arrives at Alice's or Bob's detector actually gets measured at all. Only about one in twenty times that there is a measurement event in one wing of the experiment, is there also an event in the other wing, within such a short time interval that the pair can be considered as belonging together. Relative to a naive picture of pairs of particles leaving the source, individually some getting measured, some not, the observed statistics suggest that only one in twenty photons gets detected (and hence measured), only one in four hundred pairs of photons get measured together.

As just mentioned, the choice of measurement direction was not made "anew" for each new pair of particles. It was simply being constantly "chosen anew". Many times after a switch of measurement setting, no particle (apparently) arrives at all; there is no measurement outcome in that cycle. In the Orsay experiment measurement directions in the two wings of the experiment varied rapidly and cyclically, with periods whose ratio was not close to a ratio of small integers. Thus in fact, the switches between measurement settings were fixed in advance. Measurement times are effectively random, and widely spread in time relative to the period of the switches between measurement directions. In the Innsbruck experiment, the switches between measurement direction were made locally, at each measurement station, by a very rapid (quantum!) random number generator.

We will return to the issue of whether the idealized picture $4N$ pairs of particles, each separately being measured, each particle in just one of two ways, is really appropriate, in a later section. However, the point is that quantum mechanics does seem to promise that experiments of this nature could in principle be done, and if so, there seems no reason to doubt they could violate the CHSH inequality. Three correlations more or less equal to $1/\sqrt{2}$ and one equal to $-1/\sqrt{2}$ have been measured in the lab. Not to mention that the whole curve $\cos(\theta)$ has been experimentally recovered. What would this mean if the experiments had been perfect? What is the chance they'll ever be perfected?

4 Realism, locality, freedom

The EPR-B correlations have a second message, beyond the fact that they violate the CHSH inequality. They also exhibit perfect anti-correlation in the case that the two directions of measurement are exactly equal – and perfect correlation in the case that they are exactly opposite. This brings us straight to the EPR argument not for the non-locality of quantum mechanics, but for the incompleteness of quantum mechanics.

Einstein, Podolsky and Rosen (1935) were revolted by the idea that the "last word" in physics would be a "merely" statistical theory. Physics should explain why, in each individual instance, what actually happens does happen. The belief that every "effect" *must* have a "cause" has driven Western science since Aristotle. Now according to the singlet correlations, if Alice were to measure the spin of her particle in direction $\vec{a}$, it's certain that if Bob were to do the same, he would find exactly the opposite outcome. Since it is inconceivable that Alice's choice has any immediate influence on the particle over at Bob's place, it must be that the outcome of measuring Bob's particle in the direction $\vec{a}$ is predetermined "in the particle" as it were. The measurement outcomes from measuring spin in all conceivable directions on both particles must be predetermined properties of those particles. The observed correlation is merely caused by their origin at a common source.

Thus Einstein uses *locality* and the predictions of quantum mechanics itself to infer *realism*, more properly called *counterfactual definiteness*, the notion that the outcomes of measurements on physical systems are predefined properties of those systems, merely revealed by the act of measurement, to argue for the *incompleteness* of quantum mechanics

– it describes some aggregate properties of collectives of physical systems, but does not even deign to talk about physically definitely existing properties of individual systems.

Whether it needed external support or not, the notion of counterfactual definiteness is nothing strange in all of physics (at least, prior to the invention of quantum mechanics). It belongs with a deterministic view of the world as a collection of objects blindly obeying definite rules. Note however that the CHSH proof of Bell's theorem does not start by inferring counterfactual definiteness from other properties. A wise move, since in actual experiments, we would never observe exactly perfect correlation (or anti-correlation). And even if we have observed it one thousand times, this does not prove that the "true correlation" is $+1$; it only proves, statistically, that it is very close to $+1$.

Be that as it may, Bell's theorem uses three assumptions to derive the CHSH inequality, and the first is counterfactual definiteness. Only after we agree that, even if only, say, A' and B were actually measured in one particular run, that A and B' also exist at least in some mathematical sense alongside of the two other, does it make sense to discuss *locality*: the assumption that *which* variable is being observed at Alice's location does not influence the values taken by the other two at Bob's location. To go further still, only after we have assumed both counterfactual definiteness and locality, does it make sense to assume *freedom*: the assumption that we can freely choose to observe either A or A', and either B or B'.

Some writers here like to associate the freedom assumption with the free will of the experimenter, others with the existence of "true" randomness in other physical processes. Thus one metaphysical assumption is justified by another. I would rather focus on practical experience and on Occam's razor. We understand fairly well, mathematically and physically, how small uncontrollable variations in the initial conditions of the toss of a coin lead, for all practical purposes, to completely random binary outcomes "head" or "tail". In a quite similar way, we know how the arbitrary choice of seed of a pseudo random number generator can lead to a sequence of binary digits which for all practical purposes behaves like outcomes of a sequence of coin tosses. Now imagine that both Alice and Bob choose the detector settings according either to a physical random number generator such as a human being tossing a coin, or a pseudo random number generator. Let's accept counterfactual definiteness and locality. Do we really believe that the observed correlations $1/\sqrt{2}$, $1/\sqrt{2}$, $1/\sqrt{2}$, $-1/\sqrt{2}$ occur through some as yet unknown physical mechanism by which the outcomes of Alice's random generators were exquisitely tuned to the measurement outcomes of Bob's photons? Sure, if the universe is completely deterministic, then everything which goes on today was already settled at the time of the big bang, including which measurement settings were going to be fed into which photo-detectors. But if you really want to believe this, how come we never see a bigger violation of CHSH than the $2\sqrt{2}$ which we observe in the Aspect experiment, and how come we never see any evidence for dependence between coin tosses and outcomes of distant photo-detectors except in the specific scenario of a Bell-CHSH type experiment?

The idea that we can save local realism by adopting "super-determinism" has not been taken seriously by many physicists, except perhaps Gerhard 't Hooft, who argues that at the Planck scale we do not have freedom to choose measurement settings. Indeed, so far, we do no experiments at all at that scale. This is taken by 't Hooft as legitimization for trying to build a local realistic theory of the physical world at an "underlying scale" with the idea that the statistical model of quantum mechanics would "emerge" at a higher level of description. But it's ludicrous to me that super-determinism at the Planck scale could delicately constrain the statistical dependence between coin tosses and distant photo-detector clicks.

But this means we have to make a choice between two other inconceivable possibilities: do we reject locality, or do we reject realism?

Here I would like to call on Occam's principle again. Suppose realism is true. Despite the fact that the two particles, if measured in the same way, would exhibit equal and opposite spins, it can't be the case that those spins were somehow embedded in the particles at the source. If we believe the predictions of quantum mechanics for the EPR-B experiment, we have to imagine that the act of measuring one of the particles in a particular way had an instant effect far away. We don't have any theory for *how* that effect happens. Well – there is a theory called Bohmian mechanics which does create a mathematical framework in which this is exactly what does happen. It is a mechanistic *description* of what goes on "under the surface" which exactly reproduces the statistical predictions of quantum theory, but is it an *explanation*? It has further defects: it is not relativistically invariant, and can't be; even though what happens "on the surface" does have this property. It requires an preferred space-time reference frame. Since it "merely" reproduces the predictions of quantum mechanics, which we have anyway, we don't actually need it, though as a mathematical device it can provide clever tricks for afficionados for solving some problems. So far, it has not caught on.

It seems to me that we are pretty much forced into rejecting *realism*, which, please remember, is actually a highly idealistic concept. I hasten to add that I am not alone in this, and could easily cite a number of very authoritative voices in modern quantum physics (I will mention just one name now: Nicolas Gisin; another name later). However, I admit it somehow goes against all instinct. In the case of equal settings, how can it be that the outcomes are equal and opposite, if they were not predetermined at the source? I freely admit, *there is simply no way to imagine otherwise.*

Possibly lamely I would like here to appeal to the limitations of our own brains, the limitations we experience in our "understanding" of physics due to our own rather special position in the universe. According to recent research in neuroscience our brains are already at birth hardwired with various basic conceptions about the world. These "modules" are nowadays called *systems of core knowledge*. The idea is that we cannot acquire new knowledge from our sensory experiences (including learning from experiments: we cry, and food and/or comfort is provided!) without having a prior framework in which to interpret the data of experience and experiment. It seems that we have modules for elementary algebra and for analysis: basic notions of number and of space. But we also have modules for causality. We distinguish between *objects* and *agents* (we learn that we ourselves are agents). Objects are acted on by agents. Objects have continuous existence in space time, they are local. Agents can act on objects, also at a distance. Together this seems to me to be a built-in assumption of determinism; we have been created (by evolution) to operate in an Aristotelian world, a world in which every effect has a cause.

The argument (from physics and from Occam's razor, not from neuroscience!) for abanding realism is made eloquently by Boris Tsirelson in an internet encyclopedia article on entanglement[1]. It was Tsirelson from whom I borrowed the terms counterfactual definiteness, relativistic local causality, and no-conspiracy. He points out that it's a mathematical fact that quantum physics is consistent with relativistic local causality and with no-conspiracy. In all of physics, there is no evidence *against* either of these two principles. This is for him a good argument to reject counterfactual definiteness.

I would like to close this section by just referring to a beautiful paper by Masanes, Acin and Gisin (2006) who argue in a very general setting (i.e., not assuming quantum theory, or local realism, or anything) that *quantum non-locality*, by which they mean the violation of Bell inequalities, together with *non-signalling*, which is the property that the marginal probability distribution seen by Alice of A does not depend on whether Bob measures B of B', together implies indeterminism: that is to say: that the world is stochastic, not deterministic.

5 Resolution of the Measurement Problem

The measurement problem, also known as Schrödinger's cat problem) is the problem to reconcile two apparently mutually contradictory parts of quantum mechanics. When a quantum system is isolated from the rest of the world, its quantum state (a vector, normalized to have unit length, in Hilbert space) evolves unitarily, deterministically. When we look at a quantum system from outside, by making a measurement on it in a laboratory, the state collapses to one of the eigenvectors of an operator corresponding to the particular measurement, and it does so with probabilities equal to the squared lengths of the projections of the original state vector into the eigenspaces. Yet the system being measured together with the measurement apparatus used to probe it form together a much larger quantum system, supposedly evolving unitarily and deterministically in time.

For practical purposes, physicists know how to model which parts of their experiments in which way, so as to get results which are confirmed by experiment, and many are not concerned with the measurement problem. However, cosmologists wanting to build a physical model for the evolution of the whole universe based on quantum physics, have a problem. The universe is not just a wave function in an enormous Hilbert space. As we experience it, it consists of more or less definite objects following more or less definite trajectories in real space-time.

Accepting that quantum theory is intrinsically stochastic, and accepting the reality of measurement outcomes, has led Slava Belavkin (2007) to a mathematical framework which he calls eventum mechanics which (in my opinion) indeed reconciles the two faces of quantum physics (Schrödinger evolution, von Neumann collapse) by a most simple device. Moreover, it is based on ideas of causality with respect to time. I have attempted to explain this model in as simple terms as possible in Gill (2009). The following words will

[1] http://en.citizendium.org/wiki/Entanglement_%28physics%29

only make sense to those with some familiarity with quantum mechanics though at very elementary level only. The idea is to model the world, as is conventional from the quantum theory point of view, with a Hilbert space, a state on that space, and a unitary evolution. Inside this framework we look for a collection of bounded operators on the Hilbert space which all commute with one another, and which are *causally* compatible with the unitary evolution of the space, in the sense that they all commute with past copies of themselves (in the Heisenberg picture, one thinks of the quantum observables as changing, the state as fixed; each observable corresponds to a time indexed family of bounded operators). We call this special family of operators the *beables*: they correspond to physical properties in a classical-like world which can coexist, all having definite values at the same time, and definite values in the past too. The state and the unitary evolution together determine a joint probability distribution of these time-indexed variables, i.e., a stochastic process. At any fixed time we can condition the state of the system on the past trajectories of the beables. This leads to a quantum state over all bounded operators which commute with all the beables.

The result is a theory in which the deterministic and stochastic parts of traditional quantum theory are combined into one completely harmonious whole. In fact, the notion of restricting attention to a subclass of all observables goes back a long way in quantum theory under the name *superselection rule*; and abstract quantum theory (and quantum field theory) has long worked with arbitary algebras of observables, not necessarily the full algebra of a specific Hilbert space. With respect to those traditional approaches the only novelty is to suppose that the unitary evolution when restricted to the sub-algebra is not invertible. It is an endomorphism, not an isomorphism. There is an arrow of time.

Quantum randomness is just time, the quantum future meeting the classical past in the present.

6 Loopholes

In real world experiments, the ideal experimental protocol of particles leaving a source at definite times, and being measured at distant locations according to locally randomly chosen settings cannot be implemented.

Experiments have been done with pairs of entangled ions, separated only by a short distance. The measurement of each ion takes a relatively long time, but at least it is almost always successful. Such experiments are obviously blemished by the so-called *communication* or *locality* loophole. Each particle can know very well how the other one is being measured.

Many very impressive experiments have been performed with pairs of entangled photons. Here, the measurement of each photon can be performed very rapidly and at huge distance from one another. However, many photons fail to be detected at all. For many events in one wing of the experiment, there is often no event at all in the other wing, even though the physicists are pretty sure that almost all detection events do correspond to (members of) entangled pairs of photons. This is called the *detection* loophole. Popularly it is thought to be merely connected to the efficiency of photo-detectors and that it will be easily overcome by the development of better and better photodetectors. Certainly that is necessary, but not sufficient, as I'll explain.

In Weihs' experiment mentioned earlier, only 1 in 20 of the events in each wing of the experiment is paired with an event in the other wing. Thus of every 400 pairs of photons – if we may use such terminology, and if we assume that detection and non-detection occur independently of one another in the two wings of the experiment – only 1 pair results in a successful measurement of both the photons; there are 19 further unpaired events in each wing of the experiment; and there were 361 pairs of photons not observed at all.

Imagine (anthropocentrically) classical particles about to leave the source and aiming to fake the singlet correlations. If they are allowed to go undetected often enough, they can engineer any correlations they like, as follows. Consider two new photons about to leave the source. They agree between one another with what pair of settings they would like to be measured. Having decided on the desired setting pair, they next generate outcomes ± 1 by drawing them from the joint probability distribution of outcomes given settings, which they want the experimenter to see. Only then do they each travel to their corresponding detector. There, each particle compares the setting it had chosen in advance with the setting chosen by Alice or Bob. If they are not the same, it decides to go undetected.

With probability $1/4$ we will have successful detections in both wings of the experiment. For those detections, the pair of settings according to which the particles are being measured is identical to the pair of settings they had aimed at in advance.

This may seem silly, but it does illustrate that if one wants to experimentally *prove* a violation of local realism without making an untestable assumption of *missing at random*, one has to put limits on the amount of "non-detections".

Jan-Åke Larsson (1998, 1999) has proved variants of the CHSH inequality which take account of the possibility of non-detections. The idea is that under local realism, as the proportion of "missing" measurements increases from zero, the upper bound "2" in the CHSH inequality (4) increases too. We introduce a quantity γ called the *efficiency of the experiment*: this it the minimum over all setting pairs of the probability that Alice sees an outcome given Bob sees an outcome (and vice versa). It is not to be confused with "detector efficiency". The (sharp) bound on $\langle AB\rangle_{\lim} + \langle AB'\rangle_{\lim} + \langle A'B\rangle_{\lim} - \langle A'B'\rangle_{\lim}$ set by local realism is no longer 2 as in (4), but $2 + \delta$, where $\delta = \delta(\gamma) = 4(\gamma^{-1} - 1)$.

In particular, as long as $\gamma \geq 1/\sqrt{2} \approx 0.7071$, the bound $2 + \delta$ is smaller than $2\sqrt{2}$. Weihs' experiment has an efficiency of 5%. If only we could increase it to above 71% *and* simultaneously get the state and measurements even closer to perfection, we could have definitive experimental proof of Bell's theorem.

This would be correct for a "clocked" experiment. Suppose now particles determine themselves the times that they are measured. Thus a local realist pair of particles trying to fake the singlet correlations could arrange between themselves that their measurement times are delayed by smaller or greater amounts depending on whether the setting they see at the detector is the setting they want to see, or not. It turns out that this gives our devious particles even more scope for faking correlations. Gill and Larsson (2004) showed the sharp bound on $\langle AB\rangle_{\lim} + \langle AB'\rangle_{\lim} + \langle A'B\rangle_{\lim} - \langle A'B'\rangle_{\lim}$ set by local realism is $2 + \delta$, where now $\delta = \delta(\gamma) = 6(\gamma^{-1} - 1)$. As long as $\gamma \geq 3(1 - 1/\sqrt{2}) \approx 0.8787$, the bound $2 + \delta$ is smaller than $2\sqrt{2}$. We need to get experimental efficiency above 88%, and have everything else perfect, at the very limits allowed by quantum physics. We have a long way to go.

7 Bell's theorem without inequalities

In recent years new proofs of Bell's theorem have been invented which appear to avoid probability or statistics altogether, such as the famous GHZ (Greenberger, Horne, Zeilinger) proof. Experiments have already been done implementing the set-up of these proofs, and physicists have claimed that these experiments prove quantum-nonlocality by the outcomes of a finite number of runs: no statistics, no inequalities (yet their papers do exhibit error bars!).

Such a proof runs along the following lines. Suppose local realism is true. Suppose also that some event $\mathcal{A}$ is certain. Suppose that it then follows from local realism that another event $\mathcal{B}$ has probability zero, while under quantum mechanics it can be arranged that the same event $\mathcal{B}$ has probability one. Paradoxical, but not a contradiction in terms: the catch is that events $\mathcal{A}$ and $\mathcal{B}$ are events under different experimental conditions: it is only under local realism and freedom that the events $\mathcal{A}$ and $\mathcal{B}$ can be situated in the same sample space. Moreover, freedom is needed to equate the probabilities of observable events with those of unobservable events.

As an example, consider the following scenario, generalizing the Bell-CHSH scenario to the situation where the outcome of the measurements on the two particles is not binary, but an arbitrary real number. This situation has been studied by Zohren and Gill (2006), Zohren, Reska, Gill and Westra (2010).

Just as before, settings are chosen at random in the two wings of the experiment. Under local realism we can introduce variables A, A', B and B' representing the outcomes (real numbers) in one run of the experiment, both of the actually observed variables, and of those not observed.

It turns out that it is possible under quantum mechanics to arrange that $\Pr\{B' \leq A\} = \Pr\{A \leq B\} = \Pr\{B \leq A'\} = 1$ while $\Pr\{B' \leq A'\} = 0$. On the other hand, under local realism, $\Pr\{B' \leq A\} = \Pr\{A \leq B\} = \Pr\{B \leq A'\} = 1$ implies $\Pr\{B' \leq A'\} = 1$.

Note that the four probability measures under which, under quantum mechanics, $\Pr\{A \leq B\}$, $\Pr\{A \geq B'\}$, $\Pr\{A' \geq B\}$, $\Pr\{A' \geq B'\}$ are defined, refer to four different experimental set-ups, according to which of the four pairs (A, B) etc. we are measuring.

The experiment to verify these quantum mechanical predictions has not yet been performed though some colleagues are interested. Interestingly, though it requires a quantum entangled state, that state should *not* be the maximally entangled state. Maximal "quantum non-locality" is quite different from maximal entanglement. And this is not an isolated example of the phenomenon.

Note that even if the experiment is repeated a large number of times, it can never prove that probabilities like $\Pr\{A \leq B\}$ are exactly equal to 1. It can only give strong statistical evidence, at best, that the probability in question is very close to 1 indeed.

But actually experiments are never perfect and more likely is that after a number of repetitions, one discovers that $\{A > B\}$ actually has positive probability – that event will happen a few times. Thus though the proof of Bell's theorem that quantum mechanics is in conflict with local realism appears to have nothing to do with probability, and only to do with logic, in fact, as soon as we try to convert this into experimental proof that *Nature* is incompatible with local realism, we will be in the business of proving (statistically) violation of inequalities, again (as the next section will make clear).

Finally in this section, I have to admit to having lied in my statement "it turns out to be possible". Actually, Zohren et al. (2010) only showed that one can arrange that those probabilities can be arbitrarily close to 1 and 0: it is not clear that the limiting situation still corresponds to states and measurements belonging to the usual framework of quantum mechanics. However I believe that this "lie for children" does not affect the moral of the story.

8 Better Bell inequalities

Why all the attention to the CHSH inequality? There are others around, aren't there? And are there alternatives to "inequalities" altogether? Well, in a sense the CHSH inequality is the only Bell inequality worth mentioning in the scenario of two parties, two measurements per party, two outcomes per measurement. Let's generalize this scenario and consider p parties, each choosing between one of q measurements, where each measurement has r possible outcomes (further generalizations are possible to unbalanced experiments, multi-stage experiments, and so on). I want to explain why CHSH plays a very central role in the $2\times2\times2$ case, and why in general, *generalized Bell inequalities* are all there is when studying the $p \times q \times r$ case. The short answer is: these inequalities are the bounding hyperplanes of a convex polytope of "everything allowed by local realism". The vertices of the polytope are deterministic local realistic models. An arbitrary local realist model is a mixture of the models corresponding to the vertices. Such a mixture is a *hidden variables model*, the hidden variable being the particular random vertex chosen by the mixing distribution in a specific instance.

From quantum mechanics, after we have fixed a joint p-partite quantum state, and sets of q r-valued measurements per party, we will be able to write down probability tables $p(a, b, ...|x, y, ...)$ where the variables x, y, etc. take values in $1, \ldots, q$, and label the measurement used by the first, second, ... party. The variables a, b, etc., take values in $1, \ldots, r$ and label the possible outcomes of the measurements. Altogether, there are $q^p r^p$ "elementary probabilities" in this list (indexed set) of tables. More generally, any specific instance of a theory, whether local-realist, quantum mechanical, or beyond, generates such a list of probability tables, and defines thereby a point in $q^p r^p$-dimensional Euclidean space.

We can therefore envisage the sets of all local-realist models, all quantum models, and so on, as subsets of $q^p r^p$-dimensional Euclidean space. Now, whatever the theory, for any values of x, y, etc., the sum of the probabilities $p(a, b, \ldots |x, y, \ldots)$ must equal 1. These are called *normalization constraints*. Moreover, whatever the theory, all probabilities must be nonnegative: *positivity constraints*. Quantum mechanics satisfies *locality* (with respect to what it talks about!), which means that the marginal distribution of the outcome of any one of the measurements of any one of the parties does not depend on which measurements are performed by any of the other parties. Since marginalization corresponds again to summation of probabilities, these so-called *no-signaling constraints* are expressed by linear equalities in the elements in the probability tables corresponding to a specific model. Not surprisingly, local-realist models also satisfy the no-signaling constraints.

We will call a list of probability tables restricted only by positivity, normalization and no-signalling, but otherwise completely arbitrary, a *local* model. The positivity constraints are linear inequalities which place us in the positive orthant of Euclidean space. Normalization and no-signalling are linear equalities which place us in a certain affine subspace of Euclidean space. Intersection of orthant and affine subspace creates a convex polytope: the set of all local models. We want to study the sets of local-realist models, of quantum models, and of local models. We already know that local-realist and quantum are contained in local. It turns out that these sets are successively larger, and strictly so: *quantum* includes all *local-realist* and more (that's Bell's theorem); *local* includes all *quantum* and more (that's Tsirelson's inequality combined with an example of a local, i.e., no-signalling model which violates Tsirelson's inequality).

Let's investigate the local-realist models in more detail. A special class of local-realist models are the *local-deterministic* models. A local-deterministic model is a model in which all of the probabilities $p(a, b, \dots | x, y, \dots)$ equal 0 or 1 *and* the no-signalling constraints are all satisfied. In words, such a theory means the following: for each possible measurement by each party, the outcome is prescribed, independently of what measurements are made by the other parties. Now, it is easy to see that any local-realist model corresponds to a probability mixture of local-deterministic models. After all, it "is" a joint probability distribution of simultaneous outcomes of each possible measurement on each system, and thus it "is" a probability mixture of degenerate distributions: fix the random element ω, and each outcome of each possible measurement of each party is fixed; we recover their joint distribution by picking ω at random.

This makes the set of local-realist models a convex polytope: all mixtures of a finite set of extreme points. Therefore it can also be described as the intersection of a finite collection of half-spaces, each half-space corresponding to a boundary hyperplane.

It can also be shown that the set of quantum models is closed and convex, but its boundary is very difficult to describe.

Let's think of these three models from "within" the affine subspace of no-signalling and normalization. Relative to this subspace, the local models form a full (non-empty interior) closed convex polytope. The quantum models form a strictly smaller closed, convex, full set. The local-realist models form a strictly smaller still, closed, convex, full polytope.

Slowly we have arrived at a rather simple picture. Imagine a square, with a circle inscribed in it, and with another smaller square inscribed within the circle. The outer square represents the boundary of the set of all local models. The circle is the boundary of the convex set of all quantum models. The square inscribed within the circle is the boundary of the set of all local-realist models. The picture is oversimplified. For instance, the vertices of the local-realist polytope are also extreme points of the quantum body and vertices of the local polytope.

A generalized Bell inequality is simply a boundary hyperplane, or face, of the local-realist polytope, relative to the normalization and no-signalling affine subspace, but excluding boundaries corresponding to the positivity constraints. I will call these interesting boundary hyperplanes "non-trivial". In the $2 \times 2 \times 2$ case, for which the affine subspace where all the action lies is 8 dimensional, the local-realist polytope has exactly 8 non-trivial boundary hyperplanes. They correspond exactly to all possible CHSH inequalities (obtained by permuting outcomes, measurements and parties). Thus in the $2 \times 2 \times 2$ case, the Bell-CHSH inequality is indeed "all there is".

When we increase p, q or r, new Bell inequalities turn up, and moreover keep turning up ("new" means not obtainable from "old" by omitting parties or measurements or grouping outcomes). It seems a hopeless (and probably pointless) exercise to try to classify them. Incidentally, it's an open question as to whether *every* generalized Bell inequality, as defined here, is violated by quantum mechanics.

Quite a few generalized Bell inequalities have turned out to be of particular interest, for instance, the work of Zohren and Gill concerned the $2 \times 2 \times r$ case and discussed a class of inequalities, one for each r, whose asymptotic properties could be studied as r increased to infinity.

9 Bell's fifth position

There is an interesting and neglected gap in the proof of Bell's theorem, which has been pointed out only by a few authors, among them the present writer Gill (2003). Quantum theory allows for the existence of entangled states, but does this imply their existence in nature? In particular, is it possible to create entangled states "to order" of several spatially distant and individually spatio-temporarily localized, particles? This is what a succesfull loophole-free Bell experiment requires. The experiments on entangled photons are bedevilled by the fact that as the distance between the measurement stations increases, the efficiency of the set-up decreases. Recall: in a Bell-type experiment, "detection efficiency" should be defined as the minimum over all setting combinations, of the probability of having a detection in one wing of the experiment, given a detection in the other (both for Alice given Bob, and for Bob given Alice). It is not just a property of photodectors but of the combined paths from source to detectors). Some of the loss of individual photons takes place already in the emission phase (typically inside a crystal excited by a laser) during which photons are propagating in three-dimensional space, not linearly.

Succesful experiments have been done of ions (the binary property being excited/not excited) separated at tiny distance from one another in an ion trap. These experiments have close to 100% efficiency but the measurement of the energy level of the ions takes an enormous length of time (relatively speaking) to complete. So far it has proved very difficult to create entanglement between massive objects at macroscopic distance from one another. In view of the duration of their measurement, much larger distances are required than for experiments on photons.

The detection and measurement of a photon is extremely rapid. The quantum physics of the propagation of entangled photons tells us that the two photons will be in the required entangled state at their respective detectors, *given* that they both arrive there, within the right time window. Increasing the separation between the detectors increases *quantum uncertainty* in the time they will arrive at their detectors, as well as increasing their chance (partly quantum mechanical in origin) of not being detected at all. Could it be that quantum physics itself could prevent a loophole-free and succesful Bell-type experiment from ever be being performed?

10 Quantum Randi challenges

A second reason for the specific form of the proof of Bell's theorem which started this paper, is that it lends itself well to design of computer challenges. Every year, new researchers publish, or try to publish, papers in which they claim that Bell made some fundamental errors, and in which they put forward a specific local realist model which allegedly reproduces the quantum correlations. The papers are long and complicated; the author finds it hard to get the work published, and suspects a conspiracy by The Establishment.

Now it is unlikely that someone will ever come up with a disproof of a famous and now 50 years old theorem, especially, as I hope I have made clear, the mathematical essence of that theorem is pretty trivial. A disproof of Bell's theorem is as likely as a proof that the square root of 2 is not an irrational number.

I have found it useful in debates with "Bell-deniers" to challenge them to implement their local realist model as computer programs for a network of classical computers, connected so as to mimic the time and space separations of the Bell-CHSH experiments. A somewhat similar challenge has been independently proposed by Sascha Vongehr[2]), who gave his challenge the name "quantum Randi challenge", inspired by the well known challenge of James Randi (scientific sceptic and fighter against pseudo-science[3]) concerning paranormal claims. Vongehr's challenge differs in a number of significant respects from mine, for various good reasons. My challenge is not a quantum Randi challenge in Vongehr's sense (and he coined the phrase). Some differences will be mentioned in a moment.

The protocol of the challenge I have issued in the past is the following. Bell-denier is to write computer programs for three of his own personal computers, which are to play the roles of source $\mathcal{S}$, measurement station $\mathcal{A}$, and measurement station $\mathcal{B}$. The following is to be repeated say 15 000 times. First, $\mathcal{S}$ sends messages to $\mathcal{A}$ and $\mathcal{B}$. Next, connections between $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ are severed. Next, from the outside world so to speak, I deliver the results of two coin tosses (performed by myself), separately of course, as input setting to $\mathcal{A}$ and to $\mathcal{B}$. Heads or tails correspond to a request for A or A' at $\mathcal{A}$, and for B or B' at $\mathcal{B}$. The two measurement stations $\mathcal{A}$ and $\mathcal{B}$ now each output an outcome ± 1. Settings and

[2]http://www.science20.com/alpha_meme/official_quantum_randi_challenge-80168
[3]http://en.wikipedia.org/wiki/James_Randi

outcomes are collected for later data analysis, Bell-denier's computers are re-connected; next run.

Bell-denier's computers can contain huge tables of random numbers, shared between the three, and of course they can use pseudo-random number number generators of any kind. By sharing the pseudo-random keys in advance, they have resources to any amount of shared randomness they like.

In Gill (2003) I showed how a martingale Hoeffding inequality could be used to generalize the exponential bound (3) to the situation just described. This enabled me to choose $4N$, and a criterion for win/lose (say, halfway between 2 and $2\sqrt{2}$), and a guarantee to Bell-denier (at least so many runs with each combination of settings), such that I would happily bet 3000 Euros any day that the Bell-denier's computers will fail the challenge.

The point (for me) is not to win money for myself, but to enable the Bell-denier who considers accepting the challenge (a personal challenge between the two of us, with adjudicators to enforce the protocol) to discover for him or herself that "it cannot be done". It's important that the adjudicators do not need to look inside the programs written by the Bell-denier, and preferably don't even need to look inside his computers. They are black boxes. The only thing that has to be enforced are the communication rules. However, there are difficulties here. What if Bell-denier's computers are using a wireless network which the adjudicators can't detect?

Sascha Vongehr has proposed a somewhat simpler protocol, for which he believes that a total of 800 runs are enough. In his challenge, *the* quantum Randi challenge so to speak, Bell-denier has to write programs which will run on any decent computers. The computers will be communicating by internet and the Bell-denier's *programs*, not his *computers*, have to beat Bell repeatedly when other persons run Bell-denier's programs, again and again. The point is that if the Bell-denier posts his programs on internet and people test them and they work (and if they work, people will test them!), *The Internet* will spread the news far and wide – there is no way *The Establishment* can prevent the news coming out. Vongehr requires that the Bell-denier's computer programs will succeed "most times" at beating the bound 2; while I required a single success at exceeding a larger bound (though smaller than $2\sqrt{2}$).

Let me return to my "one-on-one" challenge. We'll have to trust one another enough that Bell-denier does only use, say, a specific wireless network, which the adjudicators can switch on and off, for communication between his three computers. Still, the necessity to many times connect and disconnect the three computers in synchrony with delivery of settings and broadcasting of outcomes causes logistic nightmares, and also raises timing problems. The length of time of the computation on $\mathcal{A}$ could be used to signal to $\mathcal{B}$ if we wait for *both* computers to be ready before they are connected to the outside world to output their results. So the network has to be switched on and off at regular time intervals synchronized with delivery of settings and broadcasting of outcomes.

For this and for many other reasons, it would be extremely convenient to allow bulk processing. My bulk processing protocol runs as follows: $\mathcal{S}$ sends messages to both $\mathcal{A}$ and $\mathcal{B}$ corresponding to say $4N$ entangled pairs of particles, all in one go. (This actually makes the $\mathcal{S}$ redundant: Bell-denier just needs to clone two copies of it as a virtual computer inside $\mathcal{A}$ and $\mathcal{B}$). Next, I deliver $4N$ coin tosses each to $\mathcal{A}$ and $\mathcal{B}$ as settings. Finally, after a pre-agreed length of time, $\mathcal{A}$ and $\mathcal{B}$ each deliver $4N$ outcomes ± 1.

Challenge to the reader: *prove a useful inequality like (3) for this situation, or prove that it cannot be done.*

What is a useful inequality; what is the problem here, anyway? A *useful* inequality should have an error probability which, as that in (3), becomes arbitrarily small as N increases. The *problem* is that through having access to all settings simultaneously, the Bell-denier is able to create dependence between outcomes of different runs. It's clear that this can be used to increase the dispersion of the outcomes, though the mean values are not affected. How far can the dispersion be increased?

So far I only succeeded in obtaining results using the boundedness of the summands, a Bell-inequality bound on their mean, and Markov's inequality. The resulting error probability does not depend on N at all, so it's useless for a one-on-one, one shot challenge. It can be used for a mixed bulk/sequential challenge, for instance, 100 batches of size 800 (or as much as Bell-denier feels comfortable with) each, in which the Bell-denier should achieve a high score in a large proportion of the batches. That's something; but can we do better?

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At the time no answer was forthcoming. I now consider that the answer is yes, if you accept counterfactual definiteness, no if not. I consider counterfactual definiteness not an "optional" philosophical or metaphysical position, but rather a physical property, which might hold true in some realms but not others. In the quantum realm I believe there are very good physical grounds to reject counterfactual definiteness. Moreover, doing so leads to resolution of many otherwise problematic aspects of quantum theory.

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