A spectral construction of the extremal t process

T. Opitz

Université Montpellier 2
Place Eugène Bataillon
34095 Montpellier cedex 5

Abstract

The extremal t process was proposed in the literature for modelling spatial extremes within a copula framework based on the extreme value limit of elliptical t distributions (Davison, Padoan and Ribatet (2012)). A main drawback of this max-stable model was the lack of a spectral construction such that direct simulation was infeasible. In this work, we propose a spectral construction for the extremal t process that renders direct simulation possible. It further allows to identify the extremal Gaussian process introduced by Schlather (2002) as a special case of the extremal t process. All results naturally also hold within the multivariate domain.

Keywords: elliptical distribution, extremal t process, max-stable process, spectral construction

1. Introduction

Davison et al. (2012) survey the statistical modelling of spatial extremes and provide a global view on available models and their interconnections. Among these models, the extremal t process represents a max-stable process that generalizes the t extreme value copula to infinite dimension. It is well defined, yet no direct construction was known back then which lead the authors to class it among copula models characterized by their motivation from multivariate considerations. In that paper, it was further illustrated in what way the extremal t process can be considered as a generalization of the Brown-Resnick process (Kabluchko et al. (2009)), and an application to Swiss rainfall data bears witness of its versatility for extremal dependence modelling. In the following, we will see that the extremal t process provides a natural connection between two prominent max-stable model classes, namely Schlather's extremal Gaussian process (Schlather (2002)) and the Brown-Resnick process. We conceive a spectral construction that generalizes the one of the extremal Gaussian process. It renders direct simulation possible for moderately large general degrees of freedom.

In the remainder of this introduction, we carry together some elementary definitions and results. Section 2 establishes and recalls useful results on convergence and extremal dependence structures, before we present the new spectral construction in Section 3 and conclude with a discussion in Section 4.

1.1. Notational conventions

If not stated otherwise, operations on vectorial arguments like maxima or arithmetic operations must be interpreted componentwise in the following presentation.

Email address: `thomas.opitz@math.univ-montp2.fr` (T. Opitz)
URL: `http://ens.math.univ-montp2.fr/~opitz/` (T. Opitz)

Vectors are typeset in bold face, in particular the vector constants $\mathbf{0} = (0, ..., 0)^T$ and $\mathbf{1} = (1, ..., 1)^T$. Rectangular bounded or unbounded sets are given according to notations like $[\mathbf{u}, \mathbf{v}] = [u_1, v_1] \times \cdots \times [u_d, v_d]$ or $(\mathbf{0}, \boldsymbol{\infty}) = (0, \infty) \times \cdots \times (0, \infty)$. The complementary set of a set B in $\mathbb{R}^d$ is written B^c. The truncation operator $x^+ = \max(x, 0)$ maps negative values to 0. The indicator function of a set B is denoted by $\chi_B(\cdot)$.

1.2. *Some general extreme value theory*

For a more detailled account of max-stability and extreme value theory in general we refer the reader to the textbooks of Beirlant et al. (2004) and de Haan and Ferreira (2006).

1.2.1. *Max-stability*

Let $\mathbf{Z}, \mathbf{Z_1}, \mathbf{Z_2}, ...$ be a sequence of independent and identically distributed (iid) random vectors in $\mathbb{R}^d$ ($d \geq 1$) with nondegenerate univariate marginal distributions. We say that $\mathbf{Z}$ follows a max-stable distribution if sequences of normalizing vectors $\mathbf{a_n} > \mathbf{0}$ and $\mathbf{b_n}$ ($n = 1, 2, ...$) exist such that the equality in distribution

$$\max_{i=1,...,n} \mathbf{a_n}^{-1}(\mathbf{Z_i} - \mathbf{b_n}) \overset{d}{=} \mathbf{Z} \tag{1}$$

holds for the componentwise maximum. A full characterization of multivariate max-stable distributions leads to rather technical expressions. For our purposes, it is convenient to focus on the case where the distribution G of $\mathbf{Z}$ has common α-Fréchet marginal distributions $G_j(z_j) = \Phi_\alpha(z_j) = \exp(-z_j^{-\alpha})\chi_{(0,\infty)}(z_j)$ ($j = 1, ..., d$) for some *tail index* $\alpha > 0$. Monotone and parametric marginal transformations allow to reconstruct all admissible univariate max-stable marginal scales in (1) from this particular marginal scale. More precisely, the class of univariate max-stable distributions is partitioned into the class of α-Fréchet distribution under strictly increasing linear transformations and further the so-called Gumbel and Weibull classes.

With α-Fréchet marginal distributions, the *standard exponent measure* $\mathbb{M}$ is defined on $[\mathbf{0}, \boldsymbol{\infty}) \setminus \{\mathbf{0}\}$ by $\mathbb{M}((\mathbf{0}, \mathbf{z}]^c) = -\log \mathbb{P}(\mathbf{Z}^\alpha \leq \mathbf{z})$ with the convention $-\log 0 = \infty$ and characterizes the dependence structure in G on a standardized scale; it is uniquely defined by the dependence function $M(\mathbf{z}) = \mathbb{M}((\mathbf{0}, \mathbf{z}]^c)$ which takes the value ∞ whenever $\min_j z_j = 0$ such that $\mathbf{z} \notin (\mathbf{0}, \boldsymbol{\infty})$. Valid dependence functions are characterized by $M(\mathbf{e_j}) = 1$ for $j = 1, ..., d$, $\mathbf{e_j} = (\infty, ..., \infty, 1, \infty, ..., \infty)^T$ a vector with j-th component 1, and by the homogeneity property $M(r\mathbf{z}) = r^{-1}M(\mathbf{z})$ for $r > 0$. In the independence case when G is the product of its univariate marginal distributions, we observe $M(\mathbf{z}) = \sum_{j=1}^d z_j^{-1}$. The extremal coefficient $M(\mathbf{1}) \in [1, d]$ can serve as an indicator of the strength of extremal dependence, ranging from full dependence associated with the value 1 to independence associated with the value d.

In the infinite-dimensional domain, we call a stochastic process $\mathbf{Z} = \{Z(s), s \in S \subset \mathbb{R}^p\}$ ($p \geq 1$) with a non-empty Borel set S max-stable if its finite-dimensional distributions are max-stable. If $\mathbf{Z_1}, \mathbf{Z_2}, ...$ are iid copies of $\mathbf{Z}$, then sequences of functions $a_n(s) > 0$ and $b_n(s)$ ($n \geq 1$) exist such that $\{\max_{i=1,...,n} a_n(s)^{-1}(Z_i(s) - b_n(s))\} \overset{d}{=} \{Z(s)\}$.

1.2.2. *Domain of attraction*

Let $\mathbf{X}, \mathbf{X_1}, \mathbf{X_2}, ...$ be a sequence of iid random vectors in $\mathbb{R}^d$ with distribution function F. For suitably chosen normalizing sequences, relation (1) can hold asymptotically in the sense of distributional convergence with nondegenerate marginal distributions in the limit $\mathbf{Z}$:

$$\max_{i=1,\ldots,n} \mathbf{a_n}^{-1}(\mathbf{X_i} - \mathbf{b_n}) \xrightarrow{d} \mathbf{Z} \quad (n \to \infty) \ . \tag{2}$$

Then we say that the distribution F of $\mathbf{X}$ is in the max-domain of attraction (MDA) of the max-stable distribution G of $\mathbf{Z}$; or we simply say that $\mathbf{X}$ is in the MDA of $\mathbf{Z}$. Normalizing sequences are not unique and the limit distribution G is unique up to a linear transformation. If normalizing constants can be chosen such that all the univariate marginal distributions G_j are of the same α-Fréchet type, then the particular choice of $\mathbf{b_n} = \mathbf{0}$ is admissible. In this case, the convergence in distribution (2) is equivalent to

$$n\,\mathbb{P}(\mathbf{a_n}^{-1}\mathbf{X} \not\leq \mathbf{z}) \to M(\mathbf{z}^{\alpha}) \quad \text{ for all } \mathbf{z} \in (\mathbf{0}, \boldsymbol{\infty}) \ . \tag{3}$$

For $d = 1$, we have $n\,\mathbb{P}(a_n^{-1}X \geq z) \to z^{-\alpha}$ $(z > 0)$, and then X is said to be *regularly varying at* ∞ *with index* $\alpha > 0$ or just *regularly varying* in the remainder of this paper, denoted as $X \in \mathrm{RV}_{\alpha}$. The normalizing sequence can be chosen as $a_n = \inf\{x : P(X \geq x) \leq n^{-1}\}$.

For stochastic processes, the notion of MDA is defined in the sense of the convergence of all finite-dimensional distributions according to (2).

1.2.3. A spectral construction for max-stable processes

The commonly used models for max-stable processes are generated with the help of so-called spectral constructions. Schlather (2002) proposes to use a Poisson process $\{V_i\} \sim \mathrm{PRM}(v^{-2}dv)$ on $(0, \infty)$ and iid replicates $\mathbf{Q_i}$ of an integrable random process $\mathbf{Q}$, independent of $\{V_i\}$ and with $\mathbb{E}Q^{+}(s) = 1$ $(s \in S)$, in order to construct the max-stable process

$$\mathbf{Z} = \{Z(s)\} = \left\{ \max_{i=1,2,\ldots} V_i Q_i(s) \right\} \quad (s \in S) \tag{4}$$

with univariate marginal distributions of type Φ_1. In particular, as a consequence of the infinite number of points in the Poisson process $\{V_i\}$, it is possible to replace $Q_i(s)$ by the zero-truncated value $Q_i^{+}(s)$ in this construction. Subsequently, without loss of generality we assume that the points V_i are in descending order such that $V_1 \geq V_2 \geq \ldots$ and $V_1 \sim \Phi_1$. We obtain extremal Gaussian processes by choosing a centered and appropriately scaled Gaussian process $\mathbf{W}$ for $\mathbf{Q}$. Other choices of $\mathbf{Q}$ were considered (cf. Davison et al. (2012)), leading for instance to so-called Brown-Resnick processes (Brown and Resnick (1977); Kabluchko et al. (2009)). The dependence function of $\mathbf{Z}$ for a finite number of points $s_1, \ldots, s_d \in S$ is

$$M_{s_1,\ldots,s_d}(\mathbf{z}) = \mathbb{E} \max_{j=1,\ldots,d} \left(z_j^{-1} Q^{+}(s_j) \right) \ . \tag{5}$$

1.3. Elliptical distributions

Definition 1.1 (Elliptically distributed random vectors). *A random vector* $\mathbf{X}$ *in* $\mathbb{R}^d$ *is said to follow a* (non-singular) elliptical distribution *if it allows for a stochastic representation* $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + R_d A \mathbf{U}$ *with a deterministic location vector* $\boldsymbol{\mu}$*, an invertible* $d \times d$ *matrix* A *that defines the dispersion matrix* $\Sigma = AA^T = (\sigma_{j_1 j_2})_{1 \leq j_1, j_2 \leq d}$ *and a nondegenerate random variable* $R_d \geq 0$ *independent from a random vector* $\mathbf{U}$ *uniformly distributed on the Euclidean unit sphere* $\mathcal{S}_{d-1} = \{\mathbf{z} \in \mathbb{R}^d \mid \mathbf{z}^T\mathbf{z} = 1\}$*. We call* R_d *the* radial variable.

Contrary to more general definitions (cf. Anderson and Fang (1990)), we prefer to remain within the framework of quadratic and non-singular A to avoid an overly technical presentation for the more general cases. A prominent example are elliptical multivariate t distributions.

Definition 1.2 (Multivariate t distribution). *We say that an elliptically distributed random vector* $\mathbf{X}$ *in* $\mathbb{R}^d$ *follows the* multivariate t distribution with $\nu > 0$ (general) degrees of freedom *if* $d^{-1}R_d^2 \sim F_{d,\nu}$*, where* $F_{d,\nu}$ *is the* F*-distribution with density function*

$$x \mapsto [x\,\mathrm{Beta}(0.5d, 0.5\nu)]^{-1}\sqrt{(dx+\nu)^{-(d+\nu)}(dx)^d\nu^\nu}\cdot\chi_{[0,\infty)}(x)$$

and Beta *refers to the Beta-function. We write* $\mathbf{X} \sim t_\nu(\boldsymbol{\mu}, \Sigma)$ *and* $\mathbb{P}(\mathbf{X} \le \mathbf{x}) = t_\nu(\mathbf{x} \mid \boldsymbol{\mu}, \Sigma)$.

As ν tends to infinity, the limit of the multivariate t distribution is the multivariate normal distribution with $R_d^2 \sim \chi_d^2$. The multivariate t distribution can be constructed as a variance mixture of the multivariate normal distribution: With $\nu Y^{-1} \sim \mathrm{Gamma}(0.5\nu, 2)$ $(\nu > 0)$ and a multivariate normal random vector $\mathbf{W} \sim N(\mathbf{0}, \Sigma)$ that is independent of Y, we obtain $\boldsymbol{\mu} + \sqrt{Y}\mathbf{W} \sim t_\nu(\boldsymbol{\mu}, \Sigma)$, cf. for instance Demarta and McNeil (2005). This construction is easily generalized to the infinite-dimensional setting on a domain S: If $\mathbf{W}$ is a centered Gaussian process with domain S and covariance function Cov and $\nu Y^{-1} \sim \mathrm{Gamma}(0.5\nu, 2)$ independent of $\mathbf{W}$, then we call the random process $\sqrt{Y}\mathbf{W}$ a *(centered) t random process* on S which is characterized by the degree of freedom ν and the dispersion function Cov (cf. Røislien and Omre (2006)).

1.4. The t max-domain of attraction

The multivariate t distribution fulfills the MDA condition (2). For normalizing constants $\mathbf{b_n} = \mathbf{0}$ and $\mathbf{a_n} = n^{1/\nu}(\sigma_{jj}^{1/2}c_\nu^{1/\nu})_{j=1,\dots,d}$ with $c_\nu = \Gamma(0.5(\nu+1))^{-1}\nu^{-0.5\nu+1}\sqrt{\pi}\,\Gamma(0.5\nu)$ (cf. Table 2.1 on page 59 in Beirlant et al. (2004)), we obtain ν-Fréchet marginal distributions in the max-stable limit distribution G of the multivariate t distribution $t_\nu(\boldsymbol{\mu}, \Sigma)$. The dependence function of G was derived by Nikoloulopoulos et al. (2009): Denote by $\Sigma^* = (\sigma^*_{j_1,j_2})_{j_1,j_2}$ the correlation matrix that corresponds to the dispersion matrix Σ, and by $\Sigma^*_{-j,-j} = (\sigma^*_{j_1,j_2})_{j_1\neq j, j_2\neq j}$, $\Sigma_{j,-j} = \Sigma^T_{-j,j} = (\sigma^*_{j_1,j_2})_{j_1=j, j_2\neq j}$ submatrices obtained by removing some of the rows or columns. Similarly for vectors, we write $\mathbf{z}_{-j} = (z_1, ..., z_{j-1}, z_{j+1}, ..., z_d)^T$. Then

$$M_{\nu,\Sigma^*}(\mathbf{z}) = \sum_{j=1}^{d} z_j^{-1}\, t_{\nu+1}\left((\mathbf{z}_{-j}/z_j)^{\nu^{-1}} \mid \Sigma^*_{-j,j}, (\nu+1)^{-1}\left(\Sigma^*_{-j,-j} - \Sigma^*_{-j,j}\Sigma^{*T}_{-j,j}\right)\right) . \tag{6}$$

We refer to the max-stable limit as *extremal t distribution*, and we call *extremal t process* the max-stable limit of a t random process which is a generalization of the extremal t distribution. Its dependence structure is characterized by the set of dependence functions for all finite-dimensional distributions, hence by a general degree of freedom $\nu > 0$ and the correlation function

$$\mathrm{Cov}^*(s_{j_1}, s_{j_2}) = \sqrt{\mathrm{Cov}(s_{j_1}, s_{j_1})\mathrm{Cov}(s_{j_2}, s_{j_2})}^{\,-1}\mathrm{Cov}(s_{j_1}, s_{j_2}) \quad (s_{j_1}, s_{j_2} \in S)$$

which corresponds to the dispersion function Cov and determines the matrices Σ^* for all finite-dimensional distributions. Similar to the copula approach in the multivariate domain, it is convenient to call extremal t process any max-stable process whose set of dependence functions is the same. Put differently, we allow for univariate max-stable marginal scales beyond the ν-Fréchet scale.

2. A Breiman-type result and its ramifications

Breiman's theorem (Breiman (1965); Cline and Samorodnitsky (1994)) characterizes the tail behavior of the product of a non-negative regularly varying random variable with a "lighter-tailed" non-negative random variable. We state a generalization adapted to our purposes.

Proposition 2.1. *Let $R \in \mathrm{RV}_\alpha$ $(\alpha > 0)$ be a non-negative random variable such that $n\,\mathbb{P}(a_n^{-1}R \geq r) \sim r^{-\alpha}$ for $r > 0$ as $n \to \infty$. Suppose $\mathbf{X} \geq \mathbf{0}$ is a random vector in $\mathbb{R}^d$, independent of R, such that $\mathbb{E}(\mathbf{X}^\alpha) = \mathbf{1}$. Assume further that $\mathbb{E}(\mathbf{X}^{\alpha+\delta}) < \boldsymbol{\infty}$ for some $\delta > 0$ or that $\mathbb{P}(R \geq r) \sim cr^{-\alpha}$ for some $c > 0$. Then the distribution of the vector $\mathbf{Y} = R\mathbf{X}$ fulfills the MDA condition (2) with a max-stable limit distribution G. The normalizing vector sequences $\mathbf{a_n} = (a_n, ..., a_n)^T$ and $\mathbf{b_n} = \mathbf{0}$ in (2) lead to marginal distributions Φ_α in G. The dependence function of G is*

$$M_{\mathbf{Y}}(\mathbf{z}) = \mathbb{E}\left[\max_{j=1,\ldots,d}(z_j^{-1}X_j^\alpha)\right] \quad (\,\mathbf{z} \in (\mathbf{0}, \boldsymbol{\infty})\,)\;.$$

Proof. We observe $n\,\mathbb{P}(\mathbf{a_n}^{-1}R\mathbf{X} \not\leq \mathbf{z}) = n\,\mathbb{P}(R\max_{j=1,\ldots,d}(z_j^{-1}X_j) > a_n)$ for $\mathbf{z} \in (\mathbf{0}, \boldsymbol{\infty})$. We can apply the classical Breiman theorem if $\mathbb{E}[\mathbf{X}^{\alpha+\delta}] < \boldsymbol{\infty}$ and otherwise a variant from Lemma 2.3 in Davis and Mikosch (2008) if $\mathbb{P}(R \geq r) \sim cr^{-\alpha}$, and we obtain

$$n\,\mathbb{P}(R\max_{j=1,\ldots,d}(z_j^{-1}X_j) > a_n) \sim n\,\mathbb{P}(R > a_n)\,\mathbb{E}\left[\max_{j=1,\ldots,d}(z_j^{-1}X_j)^\alpha\right] \quad (\mathbf{z} \in (\mathbf{0}, \boldsymbol{\infty}))$$

as $n \to \infty$. Since $n\,\mathbb{P}(R > a_n) \to 1$ as $n \to \infty$, we can deduce the α-Fréchet marginal distributions and $M_{\mathbf{Y}}(\mathbf{z}) = \mathbb{E}[\max_{j=1,\ldots,d}(z_j^{-1}X_j^\alpha)]$.

We notice that the dependence function depends on R only through the index of regular variation α. The following corollary can also be deduced from Lemma 3.1 in Segers (2012) or from some related derivations in Remark 3.6 in Davis and Mikosch (2008); for completeness of the presentation, here we provide a proof based on Proposition 2.1.

Corollary 2.1. *The process $\{V_1Q_1(s)\}$ $(s \in S)$ in the spectral construction (4) is in the MDA of the process $\mathbf{Z}$ in (4) with normalizing functions $a_n(s) = n$ and $b_n(s) = 0$ $(n \geq 1)$.*

Proof. It is easily seen that the random vector $V_1(Q_1(s_1), ..., Q_1(s_d))^T$ and its zero-truncated version $\mathbf{Y} = V_1(Q_1^+(s_1), ..., Q_1^+(s_d))^T$ belong to the same MDA (if any) for any finite collection of points s_j $(j = 1, ..., d)$. Hence the statement follows from a straightforward application of Proposition 2.1 with $\alpha = 1$, $R = V_1 \sim \Phi_1$ and $\mathbf{X} = (Q_1^+(s_1), ..., Q_1^+(s_d))^T$ which yields the same dependence function $M_{\mathbf{Y}}$ as the one in (5).

Exhaustiveness of the extremal t dependence structure

In general, regular variation of the radial variable R_d ensures the MDA condition for an elliptical distribution and is a necessary and sufficient condition for the presence of asymptotic dependence (cf. Theorem 4.3 in Hult and Lindskog (2002)). Since we can write the (zero-truncated) components of an elliptical random vector as $R_d(AU)_j^+$ $(j = 1, ..., d)$, these are also regularly varying in the case of $R_d \in \mathrm{RV}_\alpha$ due to Breiman's theorem. The value of the location vector $\boldsymbol{\mu}$ is irrelevant in this context. For the other elliptical distributions in the Gumbel or Weibull MDA, as for instance the multivariate normal distribution (Gumbel), one always obtains independence in the max-stable limit G. If $R_d \in \mathrm{RV}_\alpha$, we can use Proposition 2.1 and write $M(\mathbf{z})$ as

$$M(\mathbf{z}) = M_{\alpha,\Sigma^*} = \left\{\mathbb{E}[(U_1^+)^\alpha]\right\}^{-1}\mathbb{E}\max_{j=1,\ldots,d}\left\{z_j^{-1}[(A^*\mathbf{U})_j^+]^\alpha\right\}$$

with $A^*(A^*)^T = \Sigma^*$, and we see that the radial variable enters the dependence structure only through its index of regular variation. Since the multivariate t distribution covers the full range of tail indices (equal to the general degree of freedom) and correlation matrices Σ^*, the extremal t dependence structure is exhaustive within the class of asymptotically dependent elliptical distributions.

3. A spectral construction of the extremal-*t* process

We first provide a multivariate spectral construction for the extremal t distribution based on elliptical distributions.

Theorem 3.1 (Multivariate spectral construction). *Suppose the following items are given:*

- *a tail index $\alpha > 0$,*
- *iid replications $\mathbf{X_i}$ of an elliptically distributed random vector $\mathbf{X} = (X_1, ..., X_d)$ with dispersion matrix Σ^* ($\sigma^*_{jj} = 1$ for $j = 1, ..., d$) and location vector $\boldsymbol{\mu} = \mathbf{0}$ such that the expectation $m_\alpha = \mathbb{E}[(X_1^+)^\alpha]$ is non-null and finite and*
- *a Poisson process $\{V_i\} \sim \mathrm{PRM}(\alpha v^{-(\alpha+1)} dv)$ on $(0, \infty)$.*

Define the componentwise maximum

$$\mathbf{Z} = m_\alpha^{-\alpha^{-1}} \max_{i=1,2,...} V_i \mathbf{X_i} \ . \tag{7}$$

Then $\mathbf{Z}$ follows the extremal t distribution with α-Fréchet marginal distributions and dependence function M_{α,Σ^*}.

Proof. Due to infinite number of points in the Poisson process $\{V_i\}$, we can replace $\mathbf{X}$ by $\mathbf{X}^+$ in the construction (7). By taking $\mathbf{Z}$ to the power of α, i.e. $m_\alpha^{-1} \max_{i=1,2,...} V_i^\alpha (\mathbf{X_i^+})^\alpha$, we obtain a special case of the construction (4) which proves the max-stability and the α-Fréchet marginal distributions of $\mathbf{Z}$. Since $V_1 \sim \Phi_\alpha$, the radial variable in the elliptical random vector $V_1\mathbf{X_1}$ is regularly varying with index α due to the variant of Breiman's theorem from Lemma 2.3 in Davis and Mikosch (2008). Thus $V_1\mathbf{X_1}$ is in the MDA of the extremal t distribution and at the same time in the MDA of $\mathbf{Z}$ according to Corollary 2.1. We conclude that $\mathbf{Z}$ follows the extremal t distribution with dependence function (6).

As a direct application of Theorem 3.1, we are now able to present one possible spectral representation of extremal t processes via the corresponding Gaussian process.

Theorem 3.2 (Spectral representation of extremal t processes). *Suppose the following items are given:*

- *a tail index $\alpha > 0$,*
- *iid replications $\mathbf{W_i}$ of a standard Gaussian random field $\mathbf{W}$ on $S \subset \mathbb{R}^p$ with correlation function Cov^* and*
- *a Poisson process $\{V_i\} \sim \mathrm{PRM}(\alpha v^{-(\alpha+1)} dv)$ on $(0, \infty)$.*

Then the process defined by

$$\mathbf{Z} = \{Z(s)\} = \left\{ {m_\alpha}^{-\alpha^{-1}} \max_{i=1,2,...} V_i W_i(s) \right\} \quad (s \in S) \ , \tag{8}$$

with $m_\alpha = \sqrt{\pi}^{-1} 2^{0.5(\alpha-2)} \Gamma(0.5(\alpha+1))$ and $\Gamma(\cdot)$ the Gamma function, is an extremal t process with α-Fréchet marginal distributions. Its dependence structure is characterized by α general degrees of freedom and the dispersion function Cov^*.

Proof. It remains to verify the value of m_α. Using the variable transformation $y = 0.5x^2$ yields

$$m_\alpha = \int_0^\infty x^\alpha (2\pi)^{-0.5} \exp(-0.5x^2) dx = \int_0^\infty (2y)^{0.5\alpha} (2\pi)^{-0.5} \exp(-y)(2y)^{-0.5} dy \ ,$$

and gathering the involved constants leads to the desired representation of m_α.

4. Discussion

Taking $\{Z(s)\}$ in (8) to the power $Z^\alpha(s)$ establishes unit Fréchet marginal distributions as in the construction (4) with $\mathbf{Q} = \mathbf{W}^\alpha$. Clearly, we identify the extremal Gaussian process for $\alpha = 1$ (cf. Schlather (2002)).

Remark 4.1 (Extremal coefficient). *When* $d = 2$ *and* $\sigma_{12}^* = 0$ *in Theorem 3.1, the range of the extremal coefficient covers the open interval* $(1.5, 2)$*: As the degree of freedom* $\nu = \alpha$ *tends to infinity in* (6)*, the univariate t-distribution converges towards the normal distribution and its variance tends to* 0 *such that* $M(\mathbf{1}) = 2\lim_{\nu\to\infty} t_{\nu+1}(1 \mid 0, (1+\nu)^{-1}) = 2$ *in* (6)*. As* ν *tends to* 0*, we observe* $M(\mathbf{1}) = 2\lim_{\nu\to\infty} t_{\nu+1}(1 \mid 0, (1+\nu)^{-1}) = 2t_1(1 \mid 0, 1) = 2(\pi^{-1}\arctan(1) + 0.5) = 1.5$*. This helps understand the long-range dependence structure in models for extremal t processes since the applied correlation functions are usually positive and approach* 0 *as the distance between two points increases to infinity.*

In particular, extremal t processes can be considered more flexible than extremal Gaussian processes or Brown-Resnick processes which are both special cases; see Davison et al. (2012) for the case of the Brown-Resnick process which arises for some correlation structures as α tends to infinity. Moreover, the formulation of the dependence function (6) for the extremal Gaussian process ($\nu = 1$) is more general than the bivariate expressions obtained by Schlather (2002) and lends itself more easily to interpretation. Combining extremal t models with a random set approach as described in Davison and Gholamrezaee (2012) further enlargens the range of available models and dependence structures.

Remark 4.2 (Simulation). *Theorems 3.1 and 3.2 allow to simulate extremal t distributions and processes with the method devised in Theorem 4 of Schlather (2002). When the degree of freedom is large, the computational complexity can become very restrictive, making it difficult to assure a good quality of simulation. However, the Hüsler-Reiss distribution (Hüsler and Reiss (1989); Falk et al. (2011)) in the multivariate case and the Brown-Resnick process in the infinite-dimensional case are adequate proxies for some dispersion structures, as explained in Davison et al. (2012).*

Future research should explore in more detail up to which general degree of freedom α the simulation procedure is numerically feasible, and if the Brown-Resnick process provides an adequate substitute around and beyond the "critical" value of α. The spectral construction (8) further opens the way for tackling conditional simulation in the theoretical framework developed by Dombry and Éyi-Minko (2011) and applied in Dombry et al. (2011) and Dombry and Ribatet (2012).

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