THE MAXIMUM LIKELIHOOD DRIFT ESTIMATOR FOR MIXED FRACTIONAL BROWNIAN MOTION

C. CAI, P. CHIGANSKY, AND M. KLEPTSYNA

Abstract. The paper is concerned with the maximum likelihood estimator (MLE) of the unknown drift parameter $\theta \in \mathbb{R}$ in the continuous-time regression model

$$X_t = \theta t + B_t + B_t^H, \quad t \in [0, T]$$

where B_t is the Brownian motion and B_t^H is independent fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$. We derive the exact formula for the MLE in terms of the solution of an integral equation and find the asymptotic distribution of the estimation error. In particular, it turns out that the Brownian part does not contribute to the asymptotic variance of the MLE.

1. Introduction and the main result

Consider the continuous-time regression model

$$X_t = \theta t + \sigma B_t + B_t^H, \quad t \in [0, T], \tag{1.1}$$

where B_t is the Brownian motion and B_t^H is independent fractional Brownian motion (fBm) with the Hurst parameter $H \in (\frac{1}{2}, 1)$, i.e. zero mean Gaussian process with the correlation function

$$\mathbb{E}B_t^H B_s^H = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big), \quad s, t \in [0, T].$$

As is well known, for $H \in (\frac{1}{2}, 1)$ the process B_t^H exhibits the long-range dependence property

$$\sum_{j=1}^{\infty} \mathbb{E}B_1^H\big(B_{j+1}^H - B_j^H\big) = \infty,$$

and hence $\xi_t := \sigma B_t + B_t^H$, called in [5] the *mixed* fractional Brownian motion (fBm), can be thought of as observation noise with both "white" and heavily correlated components. The mixed fBm has a number of peculiar probabilistic properties, studied in e.g. [5], [2], [15], which have some relevance to mathematical finance (see e.g. [3]).

The constant $\sigma > 0$, controlling intensity of the Brownian part, and the Hurst parameter H can be reconstructed precisely from the trajectory $X^T := \{X_t, t \in [0, T]\}$ (see e.g. [1]) and hence are assumed to be known.

Date: 20/08/2012.

Key words and phrases. mixed fractional Brownian motion, maximum likelihood estimator, large sample asymptotic.

P.Chigansky is supported by ISF grant 314/09.

Given the sample path X^T, it is required to estimate the unknown drift parameter $\theta \in \mathbb{R}$. The parameter estimation problems in models with mixed fBm have been considered in the recent monographs [8] and [12], where construction of the maximum likelihood estimator (MLE) of θ appears as an open problem (see Remark (iii) page 181 in [12] and the discussion on page 354 in [8]). Our main result aims at filling this gap:

Theorem 1.1. *The MLE of θ is given by*

$$\hat{\theta}_T = \sigma \frac{\int_0^T g(t,T)dX_t}{\int_0^T g(t,T)dt}, \tag{1.2}$$

where[1] *the function $g(t,T)$, $t \in [0,T]$ is the unique $L^2[0,T]$ solution of the integral equation*

$$\sigma^2 g(t,T) + H(2H-1)\int_0^T g(s,T)|s-t|^{2H-2}ds = \sigma, \quad \textit{for a.a. } t \in [0,T]. \tag{1.3}$$

The corresponding estimation error is normal

$$\hat{\theta}_T - \theta \sim N\left(0, \frac{\sigma}{\int_0^T g(t,T)dt}\right) \tag{1.4}$$

with the asymptotic variance

$$\lim_{T\to\infty} T^{2-2H}\mathbb{E}_\theta(\hat{\theta}_T - \theta)^2 = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)}{\Gamma(\frac{3}{2}-H)} \tag{1.5}$$

where $\Gamma(x)$ is the standard Gamma function.

Remark 1.2. The asymptotic variance in (1.5) is independent of σ and coincides with the asymptotic variance of the MLE in the problem with $\sigma = 0$, i.e. estimating the drift of fBm without additional Brownian component (see Section 5.1 in [9]). This means that the Brownian part is asymptotically negligible. The MLE for the model with $\sigma = 0$ is given by (see Remark 2 page 270 in [6]):

$$\hat{\theta}_T = \frac{\ell_H}{k_H T^{2-2H}} \int_0^T t^{\frac{1}{2}-H}(T-t)^{\frac{1}{2}-H}dX_t, \tag{1.6}$$

where ℓ_H and k_H are some constants. It is easy to check that this estimator is applicable to the data X, generated by the model with any $\sigma > 0$ and its asymptotic variance coincides with (1.5). In other words, the estimator (1.6) has the same asymptotic accuracy as the genuine MLE.

The proof of Theorem 1.1 suggests an approximation procedure for the function $g(t,T)$ (see (2.4) and (2.5)). Its typical form, depicted in Figure 1 versus the weight function from the estimator (1.6), indicates significant difference in the non-asymptotic regime.

Remark 1.3. The equation (1.3) is known as Fredholm type two equation with weakly singular kernel (see [11]). Sometimes it is also referred to as the Wiener-Hopf equation on the finite interval. It's solution can be expressed in terms of the solution to a particular instance of the Riemann boundary value problem, which unfortunately doesn't seem to be

[1] the stochastic integral in the numerator is defined through the usual limit procedure, recalled in Subsection 2.2

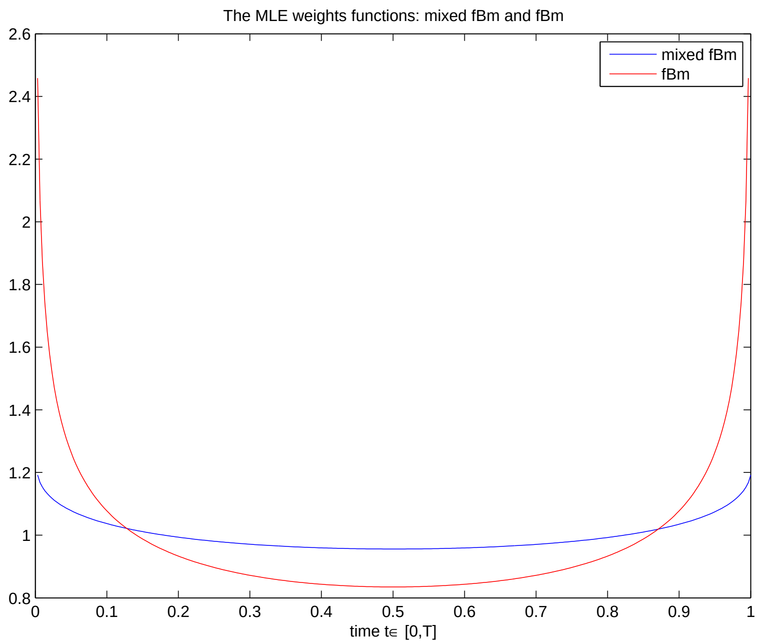

FIGURE 1. The MLE weight function for mixed fBM versus fBm ($\sigma = 1$, $T = 1$, $H = 3/4$)

helpful in our case. It is well known, however, that (1.3) has a unique continuous solution, which even enjoys some regularity properties (see [14]).

Some preliminary calculations show that the asymptotic properties of the MLE in other models with mixed fBm depend on the way the solution of (1.3) approaches the solution of the corresponding first type equation. In particular, the analysis of the MLE for the drift parameter of the stable mixed fractional Ornstein-Uhlenbeck process depends on how fast $g(0,T)$ diverges to $+\infty$ as $T \to \infty$. The latter in turn reduces to the analysis of a singular perturbed second type integral equation with weak singularity such as (2.13) below. This is an interesting problem on its own, which to the best of our knowledge, has never been considered.

2. Proof of Theorem 1.1

2.1. **The likelihood function and the MLE.** Let $\widetilde{B} = (\widetilde{B}_t)_{t\in[0,T]}$ and $B = (B^H_t)_{t\in[0,T]}$ be processes defined on a measurable space $(\Omega, \mathcal{F})$ and $\mathbb{P}_\theta$ be a probability, under which $\widetilde{B}$ and B^H are independent, B^H is the fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\widetilde{B}$ is the Brownian motion with drift $\frac{\theta}{\sigma}$, i.e.

$$\sigma \widetilde{B}_t = \theta t + \sigma B_t, \quad t \in [0,T].$$

Under $\mathbb{P}_\theta$, the process $X = \sigma\widetilde{B} + B^H$ is the mixed fBm with drift θ as defined in (1.1). By Girsanov's theorem and independence of $\widetilde{B}$ and B^H

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} = \exp\left(\frac{\theta}{\sigma}\widetilde{B}_T - \frac{1}{2}\frac{\theta^2}{\sigma^2}T\right).$$

Denote by μ_θ the probability induced by X on the space of continuous functions with the usual supremum topolgy. Then for a measurable set A,

$$\begin{aligned}\mu_\theta(A) =&\mathbb{P}_\theta(X\in A) = \mathbb{E}_0\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}\mathbf{1}_{\{X\in A\}} = \mathbb{E}_0\Big(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}\big|\mathcal{F}^X_T\Big)\mathbf{1}_{\{X\in A\}} = \\ &\int_A \Phi(x)\mu_0(dx),\end{aligned}$$

where $\mathcal{F}^X_T = \sigma\{X_t, t\in[0,T]\}$ and $\Phi(x)$ is a measurable functional, such that

$$\Phi(X) = \Big(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}\big|\mathcal{F}^X_T\Big), \quad \mathbb{P}_0 - a.s.$$

The latter means that $\mu_\theta \ll \mu_0$ for any $\theta\in\mathbb{R}$ and, since $\widetilde{B} = B$ under $\mathbb{P}_0$, the corresponding likelihood function is given by

$$\begin{aligned}L_T(X;\theta) =&\mathbb{E}_0\Big(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}\big|\mathcal{F}^X\Big) = \mathbb{E}_0\left(\exp\left(\frac{\theta}{\sigma}B_T - \frac{1}{2}\frac{\theta^2}{\sigma^2}T\right)\Big|\mathcal{F}^X_T\right) = \\ &\exp\left(\frac{\theta}{\sigma}M_T + \frac{1}{2}\frac{\theta^2}{\sigma^2}\big(V_T - T\big)\right).\end{aligned}$$

The latter equality holds with $M_T := \mathbb{E}_0(B_T|\mathcal{F}^X_T)$ and $V_T = \mathbb{E}_0\big(B_T - M_T\big)^2$, since the process (B,X) is Gaussian and hence the conditional distribution of B_T given $\mathcal{F}^X_T$ is Gaussian as well.

Let $\mathcal{F}_t$ and $\mathcal{F}^X_t$ be the natural filtrations of (B, B^H) and X respectively and set

$$M_t = \mathbb{E}_0\big(B_t|\mathcal{F}^X_t\big), \quad t\in[0,T].$$

Since B is an $\mathcal{F}$-martingale and $\mathcal{F}^X\subset\mathcal{F}$, the process M_t is an $\mathcal{F}^X$-martingale. Moreover, since $V_t = \mathbb{E}_0(B_t^2|\mathcal{F}^X_t) - M_t^2$ and $B_t^2 - t$ is an $\mathcal{F}$-martingale, for $s\le t$,

$$\begin{aligned}&\mathbb{E}_0\big(M_t^2 - (t - V_t)|\mathcal{F}^X_s\big) = \mathbb{E}_0\big(\mathbb{E}_0(B_t^2|\mathcal{F}^X_t) - t|\mathcal{F}^X_s\big) = \mathbb{E}_0(B_t^2 - t|\mathcal{F}^X_s) = \\ &\mathbb{E}_0(B_s^2|\mathcal{F}^X_s) - s = M_s^2 - (s - V_s),\end{aligned}$$

i.e. the quadratic variation process of the martingale M is $\langle M\rangle_t = t - V_t$ and the likelihood function reads

$$L_T(X;\theta) = \exp\left(\frac{\theta}{\sigma}M_T - \frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right).$$

The MLE of θ, being the maximizer the above expression, is given by

$$\hat\theta_T := \sigma\frac{M_T}{\langle M\rangle_T}.$$

This estimator is unbiased:

$$
\begin{aligned}
&\mathbb{E}_\theta \sigma \frac{M_T}{\langle M\rangle_T} = \sigma \mathbb{E}_0 L_T(X;\theta)\frac{M_T}{\langle M\rangle_T} = \\
&\sigma^2 \frac{1}{\langle M\rangle_T} \exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\frac{d}{d\theta}\mathbb{E}_0 \exp\left(\frac{\theta}{\sigma}M_T\right) = \\
&\sigma^2 \frac{1}{\langle M\rangle_T} \exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\frac{d}{d\theta} \exp\left(\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right) = \theta,
\end{aligned}
$$

with the variance

$$
\begin{aligned}
&\mathbb{E}_\theta\big(\hat{\theta}_T-\theta\big)^2 = \mathbb{E}_\theta \hat{\theta}_T^2 - \theta^2 = \sigma^2 \mathbb{E}_\theta \frac{M_T^2}{\langle M\rangle_T^2} - \theta^2 = \\
&\sigma^2 \exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\mathbb{E}_0 \exp\left(\frac{\theta}{\sigma}M_T\right)\frac{M_T^2}{\langle M\rangle_T^2} - \theta^2 = \\
&\frac{\sigma^4}{\langle M\rangle_T^2} \exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\frac{d^2}{d\theta^2}\mathbb{E}_0 \exp\left(\frac{\theta}{\sigma}M_T\right) - \theta^2 = \\
&\frac{\sigma^4}{\langle M\rangle_T^2}\left(\frac{\theta^2}{\sigma^4}\langle M\rangle_T^2 + \frac{\langle M\rangle_T}{\sigma^2}\right) - \theta^2 = \frac{\sigma^2}{\langle M\rangle_T}
\end{aligned}
\tag{2.1}
$$

To recap, the MLE error is a zero mean Gaussian random variable with variance $\sigma^2/\langle M\rangle_T$. Next we shall derive an explicit characterization of the martingale M in terms of the solution of the integral equation (1.3) and will find the appropriate asymptotic as $T\to\infty$.

2.2. **The martingale representation.** Let us recall briefly some relevant properties of the integrals with respect to fractional Brownian motion. Following the notations of [10], define the spaces

$$
\begin{aligned}
L^2[0,T] &:= \left\{ f:[0,T]\mapsto \mathbb{R} \text{ such that } \int_0^T f^2(u)du<\infty\right\} \\
|\Lambda|_T^{H-\frac{1}{2}} &:= \left\{ f:[0,T]\mapsto \mathbb{R} \text{ such that } \int_0^T\int_0^T |f(u)||f(v)||u-v|^{2H-H}dudv<\infty\right\} \\
\Lambda_T^{H-\frac{1}{2}} &:= \left\{ f:[0,T]\mapsto \mathbb{R} \text{ such that } \int_0^T \Big(s^{\frac{1}{2}-H}\big(\mathbf{I}_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}}f(u)\big)(s)\Big)^2 ds<\infty\right\},
\end{aligned}
$$

where $\mathbf{I}_{T-}^{H-\frac{1}{2}}$ is the fractional integral operator, whose definition is recalled in Subsection 2.3 below. For $H\in(\frac{1}{2},1)$ the inclusions $L^2[0,T]\subset|\Lambda|_T^{H-\frac{1}{2}}\subset\Lambda_T^{H-\frac{1}{2}}$ hold (see Remark 4.2 in [10]).

For the simple function of the form,

$$
f(u)=\sum_{k=1}^n f_k \mathbf{1}_{\{[u_k,u_{k+1})\}}(u),\quad f_k\in\mathbb{R},\quad 0=u_1<u_2<...<u_k=T
$$

the stochastic integral with respect to B^H is defined by

$$\int_0^T f(t)dB_t^H := \sum_{k=1}^n f_k\big(B^H_{u_{k+1}} - B^H_{u_k}\big).$$

Since the simple functions are dense in $\Lambda_T^{H-\frac{1}{2}}$ (see Theorem 4.1 in [10]) , the definition of $\int_0^T f(t)dB_t^H$ is extended to $f\in\Lambda_T^{H-\frac{1}{2}}$ through the limit

$$\int_0^T f(t)dB_t^H := \lim_n \int_0^T f_n dB^H,$$

where f_n is any sequence of the simple functions, such that $\lim_n \|f-f_n\|_{\Lambda_T^{H-\frac{1}{2}}} = 0$.

It turns out however (see Section 5 of [10]), that the image of $\Lambda_T^{H-\frac{1}{2}}$ under the map $f\mapsto \int_0^T f(t)dB_t^H$ is a strict subset of $\overline{\mathrm{sp}}_{[0,T]}(B^H)$, the closure in $L^2(\Omega,\mathcal{F},\mathbb{P}_0)$ of all possible linear combinations of the increments of B^H. In other words, some linear functionals of B^H cannot be realized as stochastic integrals of the above type. Nevertheless we have the following:

Lemma 2.1. *Assume $H\in(\frac{1}{2},1)$ and let η be a Gaussian random variable, such that (η, X_t), $t\in[0,T]$ is a Gaussian random process. Then there exists a function $g(\cdot,T)\in L^2[0,T]$, such that*

$$\mathbb{E}_0(\eta|\mathcal{F}_T^X) = \mathbb{E}_0\eta + \int_0^T g(s,T)dX_s, \quad \mathbb{P}_0 - a.s \tag{2.2}$$

Proof. Following the arguments of the proof of Lemma 10.1 in [7], let (t_i), $i=0,...,2^n$ be the dyadic partition of $[0,T]$, i.e. $t_i = i2^{-n}$, $i=0,...,2^n$ and $\mathcal{F}^X_{T,n} = \sigma\{X_{t_i}-X_{t_{i-1}}, i=1,...,2^n\}$. Then $\mathcal{F}^X_{T,n}\nearrow \mathcal{F}^X_T$ and by the martingale convergence

$$\lim_n \mathbb{E}_0(\eta|\mathcal{F}^X_{T,n}) = \mathbb{E}_0(\eta|\mathcal{F}^X_T), \quad \mathbb{P}_0 - a.s. \tag{2.3}$$

as well as in $L^2(\Omega,\mathcal{F},\mathbb{P}_0)$, since $\mathbb{E}_0(\eta|\mathcal{F}^X_{T,n})$ are uniformly integrable. By the Normal Correlation theorem,

$$\mathbb{E}_0(\eta|\mathcal{F}^X_{T,n}) = \mathbb{E}_0\eta + \sum_{i=1}^{2^n} g^n_{i-1}\big(X_{t_i}-X_{t_{i-1}}\big), \tag{2.4}$$

with constants g^n_{i-1}, $i=1,...,2^n$. Define

$$g_n(t,T) := \sum_{i=1}^{2^n} g^n_{i-1}\mathbf{1}_{\{[t_{i-1},t_i)\}}(t), \tag{2.5}$$

then

$$\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,n}\big) = \mathbb{E}_0\eta + \sigma\int_0^T g_n(t,T)dB_t + \int_0^T g_n(t,T)dB_t^H,$$

and

$$\mathbb{E}_0\Big(\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,n}\big)-\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,m}\big)\Big)^2=\sigma^2\int_0^T\big(g_n(t,T)-g_m(t,T)\big)^2dt+$$
$$c_H\int_0^T\int_0^T\big(g_n(t,T)-g_m(t,T)\big)\big(g_n(s,T)-g_m(s,T)\big)|s-t|^{2H-2}dsdt,$$

where $c_H:=H(2H-1)$. Since the kernel in the last integral is positive definite

$$\lim_n\sup_{m\ge n}\sigma^2\int_0^T\big(g_n(t,T)-g_m(t,T)\big)^2dt\le\lim_n\sup_{m\ge n}\mathbb{E}_0\Big(\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,n}\big)-\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,m}\big)\Big)^2=0,$$

where the latter equality holds by (2.3), since $L^2(\Omega,\mathcal{F},\mathbb{P}_0)$ is complete. Since $L^2[0,T]$ is a complete space, there exists a function $g(t,T)\in L^2[0,T]$, such that $\lim_n\|g-g_n\|_2=0$. Then

$$\mathbb{E}_0\bigg(\mathbb{E}_0(\eta|\mathcal{F}^X_T)-\mathbb{E}_0\eta-\sigma\int_0^T g(t,T)dB_t-\int_0^T g(t,T)dB^H_t\bigg)^2\le$$
$$3\mathbb{E}_0\Big(\mathbb{E}_0(\eta|\mathcal{F}^X_T)-\mathbb{E}_0(\eta|\mathcal{F}^X_{T,n})\Big)^2+3\sigma^2\int_0^T\big(g_n(t,T)-g(t,T)\big)^2dt+$$
$$3c_H\int_0^T\int_0^T\big(g_n(t,T)-g(t,T)\big)\big(g_n(s,T)-g(s,T)\big)|s-t|^{2H-2}dsdt\xrightarrow{n\to\infty}0,$$

where the latter convergence holds, since $L^2[0,T]\subset|\Lambda|_T^{H-\frac12}$. $\square$

Applying the above lemma, we obtain the claimed formulas (1.2) and (1.4):

Lemma 2.2. *For $H\in(\frac12,1)$ the equation (1.3) has a unique solution $g(\cdot,T)\in L_2([0,T])$. Moreover,*

$$M_T=\int_0^T g(t,T)dX_t. \tag{2.6}$$

and

$$\langle M\rangle_T=\sigma\int_0^T g(t,T)dt. \tag{2.7}$$

Proof. By Lemma 2.1, there exists $g(\cdot,T)\in L^2[0,T]$, such that

$$M_T=\mathbb{E}(B_T|\mathcal{F}^X_T)=\int_0^T g(t,T)dX_t,\quad \mathbb{P}_0-a.s.$$

holds. For an arbitrary function $h\in L^2[0,T]$,

$$\mathbb{E}_0\left(B_T-\int_0^T g(r,T)dX_r\right)\int_0^T h(s)dX_s=$$
$$\mathbb{E}_0\left(\int_0^T dB_t-\sigma\int_0^T g(t,T)dB_t-\int_0^T g(t,T)dB^H_t\right)\left(\sigma\int_0^T h(t)dB_t+\int_0^T h(t)dB^H_t\right)=$$
$$\int_0^T h(s)\bigg(\sigma-\sigma^2 g(s,T)-c_H\int_0^T g(r,T)|s-r|^{2H-2}dr\bigg)ds.$$

By the orthogonality property of the conditional expectation and by arbitrariness of h, it follows that $g(t,T)$ satisfies (1.3) for almost all $t \in [0,T]$.

To argue the uniqueness, suppose that there is a function $\tilde{g} \in L^2[0,T]$, which also solves (1.3). Then $\Delta(t) := g(t,T) - \tilde{g}(t,T)$ solves

$$\Delta(t) + c_H \int_0^T \Delta(s)|s-t|^{2H-2} ds = 0.$$

Multiplying by $\Delta(t)$ and integrating we get

$$\int_0^T \Delta^2(t)dt + c_H \int_0^T \int_0^T \Delta(s)\Delta(t)|s-t|^{2H-2} ds dt = 0.$$

Since the kernel in the second integral is positive definite, it follows that $\Delta(t) = 0$ a.e., i.e. $\tilde{g}$ coincides with g a.e.

Finally, since M is a Gaussian martingale,

$$\langle M\rangle_T = \mathbb{E}_0 M_T^2 = \mathbb{E}_0\bigg(\int_0^T g(s,T)dX_s\bigg)^2 =$$
$$\int_0^T g(t,T)\bigg(\sigma^2 g(t,T) + c_H\int_0^T g(s,T)|s-t|^{2H-2}ds\bigg)dt = \sigma\int_0^T g(t,T)dt$$

□

2.3. **The explicit solution to an auxiliary equation.** Consider the integral equation

$$\int_0^1 c_H|s-r|^{2H-2}\phi(r)dr = \psi(s), \quad 0 \le s \le 1 \tag{2.8}$$

where ψ is a given function. This equation is known to admit explicit solution (see e.g. Lemma 3 in [6]), which is the key to the closed form asymptotic formula (1.5). For reader's convenience we give a short self-contained derivation.

Recall the definition of the Riemann-Liouville fractional integrals (see [13]): for a function $\varphi \in L^1(a,b)$ and $\mu > 0$, the left-sided and right-sided integrals are defined as

$$\mathbf{I}^\mu_{a+}\varphi(x) := \frac{1}{\Gamma(\mu)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\mu}}dt, \quad x > a$$

and

$$\mathbf{I}^\mu_{b-}\varphi(x) := \frac{1}{\Gamma(\mu)}\int_x^b \frac{\varphi(t)}{(t-x)^{1-\mu}}dt, \quad x < b,$$

respectively. The corresponding left and right fractional derivatives of a function f on $[a,b]$ are defined for $\mu \in (0,1)$:

$$\mathbf{D}^\mu_{a+}f(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^\mu}dt$$

and

$$\mathbf{D}^\mu_{b-}f(x) = -\frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^\mu}dt.$$

The following composition rules hold: for any $\mu > 0$ and any integrable φ

$$\mathbf{D}^\mu_{a\pm}\mathbf{I}^\mu_{a\pm}\varphi(x) = \varphi(x). \tag{2.9}$$

We shall also need the representation formula (eq. (2.1) page 24 in [4])

$$|r-u|^{2H-2}=\frac{(ru)^{H-\frac{1}{2}}}{\beta(2-2H,H-\frac{1}{2})}\int_0^{r\wedge u} v^{1-2H}(r-v)^{H-3/2}(u-v)^{H-3/2}dv,$$

where $\beta(\alpha,\gamma)=\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}$.

Plugging this expression into the left hand side of (2.8) and setting $C_1(H):=c_H/\beta(2-2H,H-\frac{1}{2})$, we obtain

$$\begin{aligned}
&\int_0^1 c_H|s-r|^{2H-2}\phi(r)dr=\\
&C_1(H)\int_0^1(rs)^{H-\frac{1}{2}}\int_0^{s\wedge r}(s-\tau)^{H-3/2}(r-\tau)^{H-3/2}\tau^{1-2H}d\tau\phi(r)dr=\\
&C_1(H)\int_0^s(rs)^{H-\frac{1}{2}}\int_0^{r}(s-\tau)^{H-3/2}(r-\tau)^{H-3/2}\tau^{1-2H}d\tau\phi(r)dr+\\
&C_1(H)\int_s^1(rs)^{H-\frac{1}{2}}\int_0^{s}(s-\tau)^{H-3/2}(r-\tau)^{H-3/2}\tau^{1-2H}d\tau\phi(r)dr=\\
&C_1(H)\int_0^s\int_\tau^s(rs)^{H-\frac{1}{2}}(s-\tau)^{H-3/2}(r-\tau)^{H-3/2}\tau^{1-2H}\phi(r)drd\tau+\\
&C_1(H)\int_0^s\int_s^1(rs)^{H-\frac{1}{2}}(s-\tau)^{H-3/2}(r-\tau)^{H-3/2}\tau^{1-2H}\phi(r)drd\tau=\\
&C_1(H)s^{H-\frac{1}{2}}\int_0^s\tau^{1-2H}(s-\tau)^{H-3/2}\int_\tau^1 r^{H-\frac{1}{2}}(r-\tau)^{H-3/2}\phi(r)drd\tau=\\
&C_1(H)\Gamma^2(H-\frac{1}{2})s^{H-\frac{1}{2}}\mathbf{I}_{0+}^{H-\frac{1}{2}}g(s),
\end{aligned}$$

where

$$g(\tau)=\tau^{1-2H}\mathbf{I}_{1-}^{H-\frac{1}{2}}f(\tau)\quad\text{and}\quad f(r)=r^{H-\frac{1}{2}}\phi(r).$$

Hence the equation (2.8) reads

$$C_2(H)s^{H-\frac{1}{2}}\mathbf{I}_{0+}^{H-\frac{1}{2}}g(s)=1,\quad 0\le s\le t.$$

where $C_2(H):=C_1(H)\Gamma^2(H-\frac{1}{2})$ is set for brevity. Assuming integrability required by the rule (2.9) and applying it to both sides of this equation, we obtain

$$g(\tau)=\frac{1}{C_2(H)}\mathbf{D}_{0+}^{H-\frac{1}{2}}G(\tau),$$

where $G(s)=s^{\frac{1}{2}-H}\psi(s)$. Differentiating once again gives

$$f(r)=\frac{1}{C_2(H)}\mathbf{D}_{1-}^{H-\frac{1}{2}}F(r),$$

with $F(\tau)=\tau^{2H-1}\mathbf{D}_{0+}^{H-\frac{1}{2}}G(\tau)$ and the solution of (2.8) is

$$\phi(r)=r^{\frac{1}{2}-H}f(r).$$

In what follows we will need the solution of (2.8) for $\psi(s)\equiv 1$. In this case, $G(s)=s^{\frac12-H}$ and

$$g(\tau)=\frac{1}{C_2(H)}\mathbf{D}^{H-\frac12}_{0+}G(\tau)=\frac{1}{C_2(H)}\frac{\Gamma(\frac32-H)}{\Gamma(2-2H)}\tau^{1-2H}$$

where we used the formulas from Section 2.5 in [13]. Hence

$$F(\tau)=\tau^{2H-1}\mathbf{D}^{H-\frac12}_{0+}G(\tau)=\frac{\Gamma(\frac32-H)}{\Gamma(2-2H)}$$

and the corresponding solution is

$$\begin{aligned}\phi(r)=&r^{\frac12-H}f(r)=r^{\frac12-H}\frac{1}{C_2(H)}\mathbf{D}^{H-\frac12}_{t-}F(r)=\\ &\frac{1}{C_2(H)}\frac{\Gamma(\frac32-H)}{\Gamma(2-2H)}r^{\frac12-H}\mathbf{D}^{H-\frac12}_{t-}1(r)=\frac{1}{C_2(H)}\frac{1}{\Gamma(2-2H)}r^{\frac12-H}(1-r)^{\frac12-H}.\end{aligned}\tag{2.10}$$

Now with this solution at hand, it is easy to trace back all the integrability assumptions need for (2.9) to apply. Plugging in all the constants, we also obtain

$$\begin{aligned}&\int_0^1\phi(r)dr=\frac{1}{C_2(H)}\frac{1}{\Gamma(2-2H)}\int_0^1 r^{\frac12-H}(1-r)^{\frac12-H}dr=\\ &\frac{1}{H(2H-1)}\frac{\beta(2-2H,H-\frac12)}{\Gamma^2(H-\frac12)}\frac{1}{\Gamma(2-2H)}\beta\Big(\frac32-H,\frac32-H\Big)=\\ &\frac{1}{H(2H-1)}\frac{\Gamma(2-2H)\Gamma(H-\frac12)}{\Gamma(\frac32-H)\Gamma^2(H-\frac12)}\frac{1}{\Gamma(2-2H)}\frac{\Gamma^2(\frac32-H)}{\Gamma(3-2H)}=\\ &\frac{\Gamma(\frac32-H)}{H(2H-1)\Gamma(H-\frac12)\Gamma(3-2H)}=\frac{\Gamma(\frac32-H)}{2H\Gamma(H+\frac12)\Gamma(3-2H)},\end{aligned}\tag{2.11}$$

where we used the property $\Gamma(x+1)=x\Gamma(x)$, $x>0$.

2.4. **The large sample asymptotic.** Finally we are ready to derive the asymptotic announced in (1.5). Let $\mu:=T^{2H-1}$ and define $g_\mu(u):=T^{2H-1}g(uT,T)$, $u\in[0,1]$. Then (1.3) reads

$$\frac{1}{\mu}\sigma^2 g_\mu(u)+c_H\int_0^1 g_\mu(v)|u-v|^{2H-2}dv=\sigma,\quad u\in[0,1],\tag{2.12}$$

and, moreover,

$$\langle M\rangle_T=\sigma\int_0^T g(s,T)ds=\sigma T^{2-2H}\int_0^1 g_\mu(u)du.\tag{2.13}$$

Define the operator

$$Kf(u)=c_H\int_0^1 f(v)|u-v|^{2H-2}dv,\quad f\in|\Lambda|_T^{H-\frac12},$$

and the scalar products

$$\langle f,h\rangle:=\int_0^1 f(s)h(s)ds,\quad f,h\in L^2[0,1]$$

and

$$\langle f,h\rangle_K := c_H\int_0^1\int_0^1 f(v)h(u)|u-v|^{2H-2}dvdu, \quad h,f\in|\Lambda|_T^{H-\frac12}.$$

In terms of these notations, the equation (2.12) becomes

$$\frac{\sigma^2}{\mu}g_\mu + Kg_\mu = \sigma.$$

By linearity, the solution of the equation $Kg=\sigma$ is $g(u)=\sigma\phi(u)$, where ϕ is given by the formula (2.10), and $\delta_\mu := g_\mu - g$ satisfies

$$\frac{\sigma^2}{\mu}\delta_\mu + K\delta_\mu = -\frac{\sigma^2}{\mu}g.$$

Since $g\in L^2[0,1]\subset|\Lambda|_1^{H-\frac12}$, multiplying by δ_μ and integrating we obtain

$$\frac{\sigma^2}{\mu}\|\delta_\mu\|_2^2 + \|\delta_\mu\|_K^2 = \frac{\sigma^2}{\mu}\big|\langle g,\delta_\mu\rangle\big|.$$

and, in particular, $\|\delta_\mu\|_2^2\le\big|\langle g,\delta_\mu\rangle\big|$. On the other hand, by the Cauchy-Schwarz inequality $\big|\langle g,\delta_\mu\rangle\big|\le\|g\|_2\|\delta_\mu\|_2$ and hence $\|\delta_\mu\|_2\le\|g\|_2$. Note that δ_μ also satisfies

$$\frac{\sigma^2}{\mu}g_\mu + K\delta_\mu = 0.$$

Multiplying both sides of this equation by g and integrating, we get

$$\frac{\sigma^2}{\mu}\langle g_\mu,g\rangle + \langle K\delta_\mu,g\rangle = 0.$$

But

$$\big|\langle g_\mu,g\rangle\big|\le\big|\langle\delta_\mu,g\rangle\big| + \|g\|_2^2 \le \|\delta_\mu\|_2\|g\|_2 + \|g\|_2^2 \le 2\|g\|_2^2<\infty$$

and hence

$$\sigma|\langle\delta_\mu,1\rangle| = |\langle\delta_\mu,Kg\rangle| = |\langle K\delta_\mu,g\rangle| = \frac{\sigma^2}{\mu}\big|\langle g_\mu,g\rangle\big| \le \frac{\sigma^2}{\mu}2\|g\|_2^2 \xrightarrow{\mu\to\infty} 0.$$

In other words,

$$\lim_{\mu\to\infty}\int_0^1 g_\mu(u)du = \int_0^1 g(u)du = \sigma\int_0^1\phi(u)du.$$

Finally, by the formulas (2.1) and (2.13)

$$T^{2-2H}\mathbb{E}_\theta\big(\hat\theta_T-\theta\big)^2 = T^{2-2H}\frac{\sigma^2}{\langle M\rangle_T} = \frac{\sigma^2}{\sigma\int_0^1 g_\mu(u)du} \xrightarrow{T\to\infty} \frac{\sigma^2}{\sigma^2\int_0^1 g(u)du},$$

and, in view of (2.11), the asymptotic (1.5) follows.

References

[1] Sophie Achard and Jean-François Coeurjolly. Discrete variations of the fractional Brownian motion in the presence of outliers and an additive noise. *Stat. Surv.*, 4:117–147, 2010.

[2] Fabrice Baudoin and David Nualart. Equivalence of Volterra processes. *Stochastic Process. Appl.*, 107(2):327–350, 2003.

[3] Christian Bender, Tommi Sottinen, and Esko Valkeila. Fractional processes as models in stochastic finance. In *Advanced mathematical methods for finance*, pages 75–103. Springer, Heidelberg, 2011.

[4] Francesca Biagini, Yaozhong Hu, Bernt Øksendal, and Tusheng Zhang. *Stochastic calculus for fractional Brownian motion and applications*. Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2008.

[5] Patrick Cheridito. Mixed fractional Brownian motion. *Bernoulli*, 7(6):913–934, 2001.

[6] Alain Le Breton. Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. *Statist. Probab. Lett.*, 38(3):263–274, 1998.

[7] Robert S. Liptser and Albert N. Shiryaev. *Statistics of random processes. I*, volume 5 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, expanded edition, 2001. General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.

[8] Yuliya S. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*, volume 1929 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.

[9] Ilkka Norros, Esko Valkeila, and Jorma Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 5(4):571–587, 1999.

[10] Vladas Pipiras and Murad S. Taqqu. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli*, 7(6):873–897, 2001.

[11] Andrei D. Polyanin and Alexander V. Manzhirov. *Handbook of integral equations*. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2008.

[12] B. L. S. Prakasa Rao. *Statistical inference for fractional diffusion processes*. Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester, 2010.

[13] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikolʹskiĭ, Translated from the 1987 Russian original, Revised by the authors.

[14] G. Vainikko and A. Pedas. The properties of solutions of weakly singular integral equations. *J. Austral. Math. Soc. Ser. B*, 22(4):419–430, 1980/81.

[15] Harry van Zanten. When is a linear combination of independent fBm's equivalent to a single fBm? *Stochastic Process. Appl.*, 117(1):57–70, 2007.

Laboratoire de Statistique et Processus, Universite du Maine, France
E-mail address: `Chunhao.Cai.Etu@univ-lemans.fr`

Department of Statistics, The Hebrew University, Mount Scopus, Jerusalem 91905, Israel
E-mail address: `pchiga@mscc.huji.ac.il`

Laboratoire de Statistique et Processus, Universite du Maine, France
E-mail address: `marina.kleptsyna@univ-lemans.fr`