Causal band-limited approximation and forecasting for discrete time processes

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Abstract

We study causal dynamic approximation of non-bandlimited discrete time processes by band-limited discrete time processes such that a part of the historical path of the underlying process is approximated in Euclidean norm by the trace of a band-limited process. We obtain some conditions of solvability and uniqueness of optimal solution for this problem. An unique extrapolation to future times of the optimal approximating band-limited process can be interpreted as an optimal forecast.

Key words: band-limited processes, discrete time processes, causal filters, sampling, low-pass filters, forecasting.

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1 Introduction

We study causal dynamic approximation of non-bandlimited discrete time processes by band-limited discrete time processes. This task has many practical applications and was studied intensively, mostly in continuous time setting. It is known that it is not possible to find an ideal low-pass causal linear time-invariant filter. In continuous time setting, it is known that the distance of the set of ideal low-pass filters from the set of all causal filters is positive [1] and that the optimal approximation of the ideal low-pass filter is not possible [3]. Our goal is to substitute the solution of these unsolvable problems by solution of an easier problem in discrete time setting where the filter is not necessary time invariant. Our motivation is that, for some problems, time invariancy for a filter is not crucial. For example, a typical approach to forecasting in finance is to approximate the known path of the stock price process by a process that has an unique extrapolation. This extrapolation can be used as a forecast. This procedure has to be done at current time; it is not required that the same forecasting rule will be applied at future times. The present paper suggests to approximate discrete time processes by the discrete time band-limited processes. More precisely, we suggest to approximate the known historical path of the process by the trace of a band-limited process. In this setting, the approximating sequence does not necessary match the underlying process at sampling points. This is different from classical sampling approach (see, e.g., [9],[6],[7]). In [6]-[7], the estimate of the error norm is given. In our setting, it is guaranteed that the approximation generates the error of the minimal norm.

We obtain sufficient conditions of existence and uniqueness of an optimal approximating process. The optimal process is derived in time domain in a form of sinc series. To accommodate the current flow of observations, the coefficients of these series and have to be changed dynamically. The approximating band-limited process can be interpreted as a causal and linear filter that is not time invariant. An unique extrapolation to future times of the optimal approximating band-limited process can be interpreted as an optimal forecast.

This paper develops further the approach suggested in [2], where the continuous time setting was considered. We extend now this approach on discrete time processes.

2 Definitions

For a Hilbert space H, we denote by $(\cdot,\cdot)_H$ the corresponding inner product. We use notation $\mathrm{sinc}\,(x) = \sin(x)/x$.

Let $\mathbb{Z}$ be the set of all integers, and let $\mathbb{Z}^+$ be the set of all positive integers. We denote by ℓ_r the set of all sequences $x = \{x(t)\}_{t\in\mathbb{Z}} \subset \mathbf{R}$, such that $\|x\|_{\ell_r} = \left(\sum_{t=-\infty}^{\infty} |x(t)|^r\right)^{1/r} < +\infty$ for $r \in [1,\infty)$ or $\|x\|_{\ell_\infty} = \sup_t |x(t)| < +\infty$ for $r = +\infty$.

Let ℓ_r^+ be the set of all sequences $x \in \ell_r$ such that $x(t) = 0$ for $t = -1, -2, -3, ...$

For $x \in \ell_1$ or $x \in \ell_2$, we denote by $X = \mathcal{Z}x$ the Z-transform

$$X(z) = \sum_{t=-\infty}^{\infty} x(t) z^{-t}, \quad z \in \mathbf{C}.$$

Respectively, the inverse Z-transform $x = \mathcal{Z}^{-1}X$ is defined as

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{i\omega}\right) e^{i\omega t} d\omega, \quad t = 0, \pm 1, \pm 2, ...$$

If $x \in \ell_2$, then $X|_{\mathbb{T}}$ is defined as an element of $L_2(\mathbb{T})$.

Let $\tau \in \mathbb{Z} \cup \{+\infty\}$ and $\theta < \tau$; the case where $\theta = -\infty$ is not excluded. We denote by $\ell_2(\theta, \tau)$ the Hilbert space of complex valued sequences $\{x(t)\}_{t=\theta}^{\tau}$ such that $\|x\|_{\ell_2(\theta,\tau)} = \left(\sum_{t=\theta}^{\tau} |x(t)|^2\right)^{1/2} < +\infty$.

Let $\mathcal{U}_{\Omega,\infty}$ be the set of all mappings $X : \mathbb{T} \to \mathbf{C}$ such that $X\left(e^{i\omega}\right) \in L_2(-\pi, \pi)$ and $X\left(e^{i\omega}\right) = 0$ for $|\omega| > \Omega$. Note that the corresponding processes $x = \mathcal{Z}^{-1}X$ are said to be band-limited.

Let $\mathcal{U}_{\Omega,N}$ be the set of all $X \in U_{\Omega,\infty}$ such that there exists a sequence $\{y_k\}_{k=-N}^{N} \in \mathbf{C}^{2N+1}$ such that $X\left(e^{i\omega}\right) = \sum_{k=-N}^{N} y_k e^{ik\omega/\Omega}\mathbb{I}_{\{|\omega|\le\Omega\}}$, where $\mathbb{I}$ is the indicator function.

We assume that we are given $\Omega \in (\pi/2, \pi)$, $N \in \mathbb{Z}^+ \cup \{+\infty\}$, $s \in \mathbb{Z}$ and $q < s$. The case of $q = -\infty$ is not excluded.

We assume that if $N = +\infty$ then $q = -\infty$.

Let $\mathcal{T} = \{t \in \mathbb{Z} : \ q \le t \le s\}$ if $q > -\infty$ and $\mathcal{T} = \{t \in \mathbb{Z} : \ t \le s\}$ if $q = -\infty$.

Let Z_N be the set of all integers k such that $|k| \le N$ if $N < +\infty$, and let Z_N be the set $\mathbb{Z}$ of all integers if $N = +\infty$.

Let $\mathcal{Y}_N$ be the Hilbert space of sequences $\{y_k\}_{k=-N}^{N} \subset \mathbf{C}$ provided with the Euclidean norm, i.e., such that $\|y\|_{\mathcal{Y}_N} = \left(\sum_{k\in Z_N} |y_k|^2\right)^{1/2} < +\infty$.

Consider the Hilbert spaces of sequences $\mathcal{X} = \ell_2$ and $\mathcal{X}_- = \ell_2(q, s)$.

Let $\mathcal{X}_{\Omega,N}$ be the subset of $\mathcal{X}_-$ consisting of sequences $\{x(t)\}|_{t\in\mathcal{T}}$, where $x \in \mathcal{X}$ are such that $x(t) = (\mathcal{Z}^{-1}X)(t)$ for $t \in \mathcal{T}$ for some $X\left(e^{i\omega}\right) \in \mathcal{U}_{\Omega,N}$.

Up to the end of this paper, we assume that the following condition is satisfied.

Condition 2.1 *Either $N = +\infty$ or $N < +\infty$ and the matrix $\{\mathrm{sinc}\,(k\pi + \Omega m)\}_{k,m=-N}^{N}$ is nondegenerate.*

Proposition 2.1 *Let $N < +\infty$, and let $\Omega_0 \in (\pi/2, \pi)$ be selected such that there exists $p \in (0, 1)$ such that*

$$\min_{k\in Z_N} |\mathrm{sinc}\,(\pi k - \Omega k)| \ge p,$$
$$\max_{k,m\in Z_N,\ t\neq -k} |\mathrm{sinc}\,(\pi k + \Omega m)| < \frac{p}{2N} \quad \textit{for all} \quad \Omega \in [\Omega_0, \pi). \tag{2.1}$$

Then the matrix $\{\mathrm{sinc}\,(k\pi + \Omega m)\}_{k,m=-N}^{N}$ is nondegenerate for all $\Omega \in [\Omega_0, \pi)$.

Clearly, (2.1) holds for any Ω_0 that is close enough to π, since $\mathrm{sinc}\,(x) \to 1$ as $x \to 0$ and $\mathrm{sinc}\,(x) \to 0$ as $x \to \pi m$, where $m \in \mathbb{Z}$, $m \neq 0$. Therefore, Condition 2.1 can be satisfied with selection of Ω being close enough to π.

Proposition 2.2 *(i) If $N = +\infty$ and $q = -\infty$, then for any $x \in \mathcal{X}_{\Omega,N}$ there exists an unique $X \in \mathcal{U}_{\Omega,N}$ such that $x(t) = (\mathcal{Z}^{-1}X)(t)$ if $t \le s$.*

(ii) If N is finite and $s - q \ge 2N + 1$, then for any any $x \in \mathcal{X}_{\Omega,N}$, there exists an unique $X \in \mathcal{U}_{\Omega,N}$ such that $x(t) = (\mathcal{Z}^{-1}X)(t)$.

(iii) Assume that N is finite and $s - q \le 2N + 1$. In this case, $\{x(t)\}_{t\in\mathcal{T}} \in \mathcal{X}_{\Omega,N}$ for any $x \in \ell_2$ and any $\Omega \in [\Omega_0, \pi)$. If, in addition, $s - q = 2N + 1$, then there is an unique $X \in \mathcal{U}_{\Omega,N}$ such that $x(t) = (\mathcal{Z}^{-1}X)(t)$. If $s - q < 2N + 1$, then there are many $X \in \mathcal{U}_{\Omega,N}$ such that $x(t) = (\mathcal{Z}^{-1}X)(t)$; they form a linear manifold in $\mathcal{X}_{\Omega,N}$.

By Proposition 2.2(i), the future $\{x(t)\}_{t>s}$ of a band-limited process $x \in \mathcal{X}_\Omega$ is uniquely defined by its entire history $\{x(t),\ t \le s\}$ for any $\Omega \in (0, \pi)$. By Proposition 2.2(ii)-(iii), the future of even more "smooth" processes from $\mathcal{X}_{\Omega,N}$ is uniquely defined by a finite set of historical values that has at least $2N + 1$ elements for any $N < +\infty$ and $\Omega \in [\Omega_0, \pi)$.

3 Main results

3.1 Optimal band-limited approximation

Let $x \in \mathcal{X}$ be a process. We assume that the sequence $\{x(t)\}_{t\in\mathcal{T}}$ represents available historical data. Let Hermitian form $F : \mathcal{X}_{\Omega,N} \times \mathcal{X}_- \to \mathbf{R}$ be defined as

$$F(\widehat{x}, x) = \sum_{t=q}^{s} |\widehat{x}(t) - x(t)|^2.$$

Theorem 3.1 *(i) For any $N \le +\infty$, there exists an optimal solution $\widehat{x}$ of the minimization problem*

$$\text{Minimize} \qquad F(\widehat{x}, x) \quad \text{over} \quad \widehat{x} \in \mathcal{X}_{\Omega,N}. \tag{3.1}$$

(ii) If either $N = +\infty$ and $s = -\infty$ or N is finite and $s - q \ge 2N + 1$, then the corresponding optimal process $\widehat{x}$ is uniquely defined.

(iii) If N is finite and $s - q < 2N + 1$ then there are many optimal processes $\widehat{x}$; they form a linear manifold in $\mathcal{X}_{\Omega,N}$.

Remark 3.1 *By Proposition 2.2, there exists an unique extrapolation of the band-limited solution $\widehat{x}$ of problem (3.1) on the future times $t > s$, under the assumptions of Theorem 3.1(ii). It can be interpreted as the optimal forecast (optimal given Ω and N).*

3.2 Optimal sinc coefficients

Up to the end of this section, we assume that either $N = +\infty$ or $N < +\infty$, $s - q \geq 2N + 1$.

To solve problem (3.1) numerically, it is convenient to expand $X\left(e^{i\omega}\right)$ via Fourier series.

Consider the mapping $\mathcal{Q} : \mathcal{Y}_N \to \mathcal{X}_{\Omega,N}$ such that $\widehat{x} = \mathcal{Q}y$ is such that $\widehat{x}(t) = (\mathcal{Z}^{-1}\widehat{X})(t)$ for $t \in (q, s]$, where

$$\widehat{X}\left(e^{i\omega}\right) = \sum_{k \in Z_N} y_k e^{ik\omega/\Omega} \mathbb{I}_{\{|\omega| \leq \Omega\}}. \tag{3.2}$$

Clearly, this mapping is linear and continuous.

Let Hermitian form $G : \mathcal{Y}_N \times \mathcal{X}_- \to \mathbf{R}$ be defined as

$$G(y, x) = F(\mathcal{Q}y, x) = \sum_{t=q}^{s} |\widehat{x}(t) - x(t)|^2, \quad \widehat{x} = \mathcal{Q}y. \tag{3.3}$$

Corollary 3.1 *There exists an unique solution y of the minimization problem*

$$\text{Minimize} \qquad G(y, x) \quad \text{over} \quad y \in \mathcal{Y}_N. \tag{3.4}$$

Problem (3.1) can be solved via problem (3.4); its solution can be found numerically if $N < +\infty$.

3.3 Solution of problem (3.4)

Let $\widehat{X}$ be defined by (3.2), where $\{y_k\} \in \mathcal{Y}_N$. Let $\widehat{x} = \mathcal{Z}^{-1}\widehat{X}$. We have that

$$\begin{aligned}\widehat{x}(t) &= \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left(\sum_{k \in Z_N} y_k e^{ik\omega\pi/\Omega}\right) e^{i\omega t} d\omega = \frac{1}{2\pi}\sum_{k \in Z_N} y_k \int_{-\Omega}^{\Omega} e^{ik\omega\pi/\Omega + i\omega t} d\omega \\ &= \frac{1}{2\pi}\sum_{k \in Z_N} y_k \frac{e^{ik\pi + i\Omega t} - e^{-ik\pi - i\Omega t}}{ik\pi/\Omega + it} = \frac{\Omega}{\pi}\sum_{k \in Z_N} y_k \mathrm{sinc}\,(k\pi + \Omega t).\end{aligned} \tag{3.5}$$

We have that

$$G(y,x) = \sum_{t=q}^{s} |\widehat{x}(t) - x(t)|^2 = \sum_{t=q}^{s} \left| \frac{\Omega}{\pi} \sum_{k\in Z_N} y_k \mathrm{sinc}\,(k\pi + \Omega t) - x(t) \right|^2$$
$$= (y, Ry)_{\mathcal{Y}_N} - 2\mathrm{Re}\,(y, rx)_{\mathcal{X}_-} + (\rho x, x)_{\mathcal{X}_-}. \qquad (3.6)$$

Here $R : \mathcal{Y}_N \times \mathcal{Y}_N \to \mathcal{Y}_N$ is a linear bounded Hermitian operator, $r : \mathcal{X}_- \to \mathcal{Y}_N$ is a bounded linear operator, $\rho : \mathcal{X}_- \times \mathcal{X}_- \to \mathcal{X}_-$ is a linear bounded Hermitian operator.

It follows from the definitions that the operator R is non-negatively defined (it suffices to substitute $x(t) \equiv 0$ into the Hermitian form).

3.4 The case when $N < +\infty$

Up to the end of this section, we assume that $N < +\infty$ and $s - q \geq 2N + 1$. In this case, the space $\mathcal{Y}_N$ is finite dimensional. It follows that the operator R can be represented via a matrix $R = \{R_{km}\} \in \mathbf{C}^{2N+1,2N+1}$, where $R_{km} = R_{mk}$. In this setting, $(Ry)_k = \sum_{k=-N}^{N} R_{km} y_m$.

Theorem 3.2 *(i) For any $N < +\infty$, the operator R is positively defined.*

(ii) Problem (3.4) has a unique solution $\widehat{y} = R^{-1} rx$.

(iii) The components of the matrix R can be found from the equality

$$R_{km} = \frac{\Omega^2}{\pi^2} \sum_{t=q}^{s} \mathrm{sinc}\,(m\pi + \Omega t) \mathrm{sinc}\,(k\pi + \Omega t). \qquad (3.7)$$

(iv) The components of the vector $rx = \{(rx)_k\}_{k=-N}^{N}$ can be found from the equality

$$(rx)_k = \frac{\Omega}{\pi} \sum_{t=q}^{s} \mathrm{sinc}\,(k\pi + \Omega t) x(t). \qquad (3.8)$$

Corollary 3.2 *Let $\widehat{y}$ be the vector calculated as in Theorem 3.2, $\widehat{y} = \{\widehat{y}_k\}_{k=-N}^{N}$. The process*

$$\widehat{x}(t) = \widehat{x}(t,q,s) = \frac{\Omega}{\pi} \sum_{k\in Z_N} y_k \mathrm{sinc}\,(k\pi + \Omega t)$$

represents the output of a causal filter that is linear but not time invariant.

4 Numerical experiments

In the numerical experiments described below, we have used MATLAB.

The experiments show that some eigenvalues of R are quite close to zero despite the fact that, by Theorem 3.2, $R > 0$. Respectively, the error $E = \|R\widehat{y} - rx\|_{\ell_2(q,s)}$ for the MATLAB solution of the equation $R\widehat{y} = rx$ does not vanish. Further, in our experiments, we found that the error E can be decreased by the replacing R in the equation $\widehat{x} = R^{-1}rx$ by $R_\varepsilon = R + \varepsilon I$, where I is the unit matrix and where $\varepsilon > 0$ is small. In particular, for $\varepsilon = 0.001$, the corresponding error $E(\varepsilon) = \|R_\varepsilon^{-1}rx - \widehat{y}\|_{\ell_2(q,s)} < \|R^{-1}rx - \widehat{y}\|_{\ell_2(q,s)}$, i.e., the approximation for $q \le t \le s$ is better for $\widehat{y} = R_\varepsilon^{-1}rx$ calculated for $\varepsilon = 0.001$ than for $\widehat{y} = R^{-1}rx$ calculated for $\varepsilon = 0$. We have used $\varepsilon = 0.001$ and $N = 15$.

Figures 5.1-5.2 show a example of a process $x(t)$ and the corresponding band-limited process $\widehat{x}(t)$ approximating $x(t)$ at times $t \in \{-25, ..., 15\}$ (i.e, with $q = -25$, $s = 15$).

Figure 5.1 shows the result for $\Omega = 0.4$; Figure 5.2 shows the result for $\Omega = 1$. The values of $\widehat{x}(t)$ for $t > 15$ were calculated using the history $\{x(s)\}_{-25 \le s \le 15}$ and can be considered as an optimal forecast of $x(t)$.

We have verified numerically that the matrix $\{\mathrm{sinc}\,(k\pi + \Omega m)\}_{k,m=-N}^{N}$ is nondegenerate in both cases. Therefore, Condition 2.1 is satisfied. In fact, we found that this matrix was nondegenerate in all experiments for all kinds of Ω and N.

By Remark 3.1, the extrapolation of the process $\widehat{x} \in \mathcal{X}_{\Omega,N}$ to the future times $t > s$ can be interpreted as the optimal forecast (optimal given Ω and N).

Remark 4.1 We have used the procedure of replacement R by $R_\varepsilon = R + \varepsilon I$ with small $\varepsilon > 0$ to reduce the error of calculation of the inverse matrix for the matrix R that is positively defined but is close to a degenerate matrix. It can be noted that the same replacement could lead to a meaningful setting for the case when $\varepsilon > 0$ is not small. More precisely, it leads to optimization problem

$$\text{Minimize} \qquad G(y, x) + \varepsilon^2 \sum_{k=-N}^{N} |y_k|^2 \quad \text{over} \quad y \in \mathcal{Y}_N. \tag{4.1}$$

The solution restrains the norm of y, and, respectively, the norm of $\widehat{x}$.

5 Proofs

Proof of Proposition 2.1. Let $\alpha_{m,k} = \mathrm{sinc}\,(\pi k + \Omega m)$. By (2.1), there exists $p \in (0, 1)$ and

$k \in Z_N$ such that

$$|a_{k,-k}| = |\mathrm{sinc}\,(-\pi k + \Omega k)| \geq p,$$
$$\sum_{k \in Z_N,\ k \neq -m} |a_{m,k}| = \sum_{k \in Z_N,\ k \neq -m} |\mathrm{sinc}\,(\pi k + \Omega m)| < 2N \frac{p}{2N} = p.$$

It follows that the matrix $\{\alpha_{m,k}\}_{m,k=-N}^{N}$ can be transformed into a strictly diagonally dominant matrix, i.e., a non-degenerate matrix. $\square$

Proof of Proposition 2.2. The statement of this proposition for $N = +\infty$ and $q = -\infty$ is known in principle. It suffices to consider $s = 0$ only. Without a loss of generality, we assume that $s = 0$. Further, we assume that $\mathcal{T} = \{t : \ t \leq 0\}$, i.e., it is defined for $q = -\infty$. It suffices to prove that if $x(\cdot) \in \mathcal{X}_{\Omega,N}$ is such that $x(t) = 0$ for $t \in \mathcal{T}$, then $x(t) = 0$ for $t > 0$. For the sake of completeness, we give below a proof based on Theorem 1 [4]. By this theorem, processes $x(\cdot) \in \mathcal{X}_{\Omega,N}$ are weakly predictable in the following sense: for any $T > 0$, $\varepsilon > 0$, and $\kappa \in \ell_\infty(0,T)$, there exists $\widehat{\kappa}(\cdot) \in \ell_2(0,+\infty) \cap \ell_\infty(0,+\infty)$ such that

$$\|y - \widehat{y}\|_{\ell_2} \leq \varepsilon,$$

where

$$y(t) \triangleq \sum_{m=t}^{t+T} \kappa(t-m)x(m), \qquad \widehat{y}(t) \triangleq \sum_{m=-\infty}^{t} \widehat{\kappa}(t-m)x(m).$$

Let us apply this to a process $x(\cdot) \in \mathcal{X}_{\Omega,N}$ such that $x(t) = 0$ for $t \in \mathcal{T}$. Let us observe first that

$$\widehat{y}(t) = 0 \quad \forall t < 0. \tag{5.1}$$

Let $T > 0$ be given. Let us show that $x(t) = 0$ if $0 \leq t \leq T$. Let $\{\kappa_i(\cdot)\}_{i=1}^{+\infty}$ be a basis in $\ell_2(-T,0)$. Let $y_i(t) \triangleq \sum_{m=t}^{t+T} \kappa_i(t-m)x(m)$. It follows from (5.1) that $y_i(t) = 0$ if $t \leq 0$. Since $y_i(t)$ is a continuous function, it follows that $y_i(t) = 0$ for $t \leq 0$. It follows that $x(t) = 0$ if $t \leq T$.

Further, let us apply the proof given above to the function $x_1(t) = x(t+T)$. Clearly, $x_1(\cdot) \in \mathcal{X}_{\Omega,N}$ and $x_1(t) = 0$ for $t < 0$. Similarly, we obtain that $x_1(t) = 0$ for all $t \leq T$, i.e., $x(t) = 0$ for all $t < 2T$. Repeating this procedure n times, we obtain that $x(t) = 0$ for all $t < nT$ for all $n \geq 1$. This completes the proof of Proposition 2.2 for $N = +\infty$ and $q = -\infty$.

It can be noted that, instead of [4], we could use predictability of band-limited processes established in [5].

Let us prove the statements (ii) for a finite N. Let us consider first the case when $s - q = 2N + 1$. Without a loss of generality, we assume that $s = N$. It suffices to consider $q = -N$

only; in this case, the set $\mathcal{T} = \{t : \ q \le t \le s\} = \{t : \ -N \le t \le N\}$, i..e, $\mathcal{T} = Z_N$ and it has $2N-1$ elements. It suffices to prove that if $x(\cdot) \in \mathcal{X}_{\Omega,N}$ is such that $x(t) = 0$ for $t \in \mathcal{T}$, then $x(t) = 0$ for $t > 0$. By (3.5), we have that

$$\sum_{k \in Z_N} a_{t,k} y_k = 0, \quad -N \le t \le N, \tag{5.2}$$

for some set $\{y_k\}$. By Proposition 2.1, linear system (5.2) is a system with a non-degenerate matrix. Hence $y_k = 0$ for all k. This completes the proof of Proposition 2.2 (ii) for the case when $s - q = 2N + 1$.

Let us consider the case when $s - q > 2N + 1$. We assume again that $s = N$ In this case, the linear system (5.2) has considered jointly with the system

$$\sum_{k \in Z_N} a_{t,k} y_k = 0, \quad -q \le t < -N. \tag{5.3}$$

Clearly, system (5.2)-(5.3) admits only zero solution again. This completes the proof of Proposition 2.2 (ii).

Let us prove statement (iii). Let us consider first the case when $s - q = 2N + 1$. Since homogeneous linear system (5.2) allows only zero solution for $\Omega \in [\Omega_0, \pi)$ for some Ω_0, it follows that the non-homogeneous system

$$\sum_{k \in Z_N} a_{t,k} y_k = x(t_k), \quad -N \le t \le N \tag{5.4}$$

admits a unique solution $\{y_k\}$ for any set $\{x(t_k)\}$, Therefore, we proved that $\{x(t)\}_{t \in \mathcal{T}} \in \mathcal{X}_{\Omega,N}$ for any $x \in \ell_2$. If $s - q < 2N + 1$, then there are many solutions of (5.4), and these solutions form a linear manifold. This completes the proof of Proposition 2.2. □

Proof of Theorem 3.1. Let us prove statement (i). It suffices to prove that $\mathcal{X}_{\Omega,N}$ is a closed linear subspace of $\ell_2(q, s)$. In this case, there exists a unique projection $\widehat{x}$ of $\{x(t)\}_{t \in \mathcal{T}}$ on $\mathcal{X}_{\Omega,N}$, and the theorem is proven.

Clearly, for any $N \le +\infty$, the set $U_{\Omega,N}$ is a closed linear subspace of $L_2(-\pi, \pi)$. Consider a mapping $Q : \mathcal{U}_{\Omega,N} \to \mathcal{X}_{\Omega,N}$ such that $x(t) = (QX)(t) = (\mathcal{Z}^{-1}X)(t)$ for $t \in \mathcal{T}$. It is a linear continuous operator. By Proposition 2.2, it is a bijection. Since this mapping is continuous, it follows that the inverse mapping $Q^{-1} : \mathcal{X}_{\Omega,N} \to U_{\Omega,N}$ is also continuous (see Corollary in Ch.II.5 [10], p.77). Since the set $U_{\Omega,N}$ is a closed linear subspace of $L_2(-\pi, \pi)$, it follows that $\mathcal{X}_{\Omega,N}$ is a closed linear subspace of $\mathcal{X}_-$. This completes the proof of Theorem 3.1(i).

Statements (ii)-(iii) follows immediately from Proposition 2.2. This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Let us prove statement (i). We know that $R \geq 0$. Suppose that there exists $\bar{y} \in \mathbf{C}^{2N+1}$ such that $\bar{y} \neq 0$ and $R\bar{y} = 0$. Let $r^* : \mathcal{Y}_N \to \mathcal{X}_-$ be the adjoint operator to the operator $r^* : \mathcal{X}_- \to \mathcal{Y}_N$. If $r^*\bar{y} \neq 0$ then there exists $x \in \mathcal{X}_-$ such that $G(\bar{y}, x) < 0$, which is not possible since $G(y, x) \geq 0$ for all y, x. Therefore, $r^*\bar{y} = 0$, i.e., $G(\bar{y}, x) = (\rho x, x)_{\mathcal{X}_-}$. Further, let $\widehat{y}$ be a solution of problem (3.4). We have that $G(\widehat{y}, x) = G(\widehat{y} + \bar{y}, x)$. Hence $\widehat{y} + \bar{y} \neq \widehat{y}$ is another solution of problem (3.4). This contradicts to Corollary 3.1 that states that this problem has an unique solution. Statement (ii) follows from (i) and from classical theory of quadratic forms. Statements (iii)-(iv) follow immediately from representation (3.6). This completes the proof of Theorem 3.2. □

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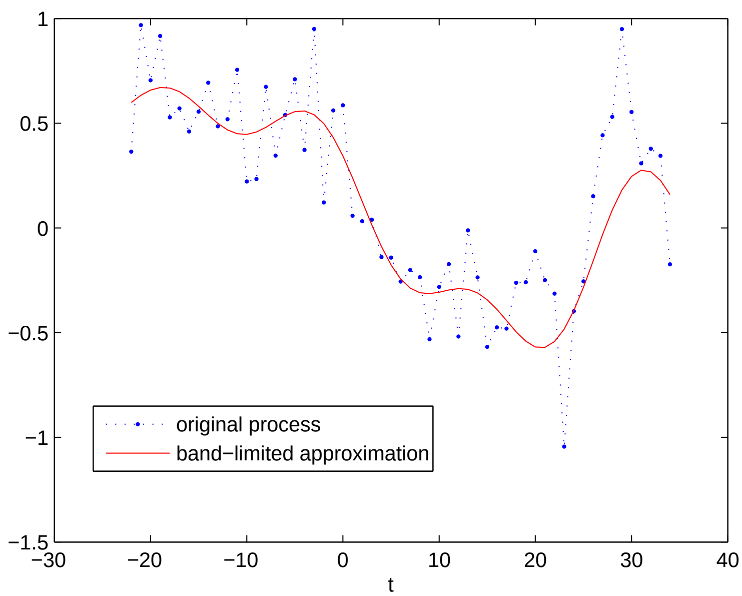

Figure 5.1: Example of $x(t)$ and band-limited process $\widehat{x}(t)$ approximating $x(t)$ for $t \in \{-25, .., 15\}$, with $\Omega = 0.4$, and $N = 15$. The values of $\widehat{x}(t)$ for $t > 15$ were calculated using $\{x(s)\}_{s \le 15}$ and can be considered as an optimal forecast of $x(t)$.

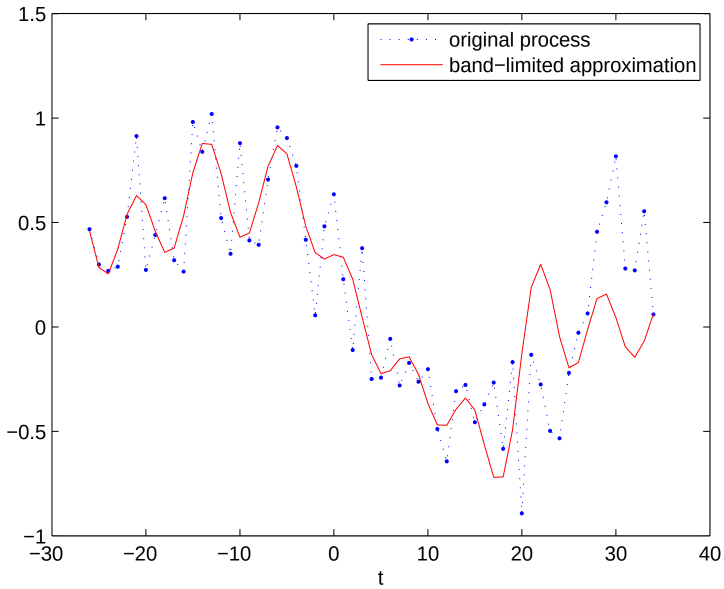

Figure 5.2: Example of $x(t)$ and band-limited process $\widehat{x}(t)$ approximating $x(t)$ for $t \in \{-25, .., 15\}$, with $\Omega = 1$, and $N = 15$. The values of $\widehat{x}(t)$ for $t > 15$ were calculated using $\{x(s)\}_{s \leq 15}$ and can be considered as an optimal forecast of $x(t)$.