General lower bounds on maximal determinants of binary matrices

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In memory of Warwick Richard de Launey 1958–2010

Abstract

We give general lower bounds on the maximal determinant of $n \times n$ $\{+1, -1\}$ matrices, both with and without the assumption of the Hadamard conjecture. Our bounds improve on earlier results of de Launey and Levin and of Koukouvinos, Mitrouli and Seberry.

1 Introduction

For $n \geq 1$, let $D(n)$ denote the maximum determinant attainable by an $n \times n$ $\{+1, -1\}$ matrix. There are several well-known upper bounds on $D(n)$, such as Hadamard's original bound [15] $D(n) \leq n^{n/2}$, which applies for all positive integers n, and bounds due to Ehlich [10, 11], Barba [3], Wojtas [33] which are stronger but apply only to certain congruence classes of n mod 4.

In this paper we give new lower bounds on $D(n)$, improving on earlier results of Cohn [7], Clements and Lindström [6], Koukouvinos, Mitrouli and Seberry [20, Theorem 2], and de Launey and Levin [23].

We consider only square $\{+1, -1\}$ matrices. The *order* is the number of rows (or columns) of such a matrix. A $\{+1, -1\}$ matrix H with $|\det H| = n^{n/2}$ is called a *Hadamard matrix*. A Hadamard matrix has order 1, 2, or a multiple of 4; the *Hadamard conjecture* is that every positive multiple of 4 is the order of a Hadamard matrix. It is known [19] that every positive multiple of 4 up to and including 664 is the order of a Hadamard matrix.

Our technique for obtaining lower bounds on $D(n)$ is to consider a Hadamard matrix H of order say h as close as possible to n. If $h > n$ we consider minors of order n in H, much as was done by de Launey and Levin [23], although the details differ as we use a theorem of Szöllősi [32] instead of the probabilistic approach of [23]. If $h < n$ we construct a matrix of order n with large determinant having H as a submatrix. By combining both ideas, we improve on the bounds that are attainable using either idea separately.

The distance $\delta(n) = |h - n|$ of n from the (closest) order h of a Hadamard matrix can be bounded by the *prime gap* function $\lambda(x)$ which bounds the maximum distance between successive primes p_i, p_{i+1} with $p_i \le x$. Thus, we can use known results on $\lambda(x)$, such as the theorem of Baker, Harman and Pintz [2], to obtain unconditional lower bounds on $D(n)$. Unfortunately, such results, even on the assumption of the Riemann hypothesis, are much weaker than what is conjectured to be true.

If we are willing to assume the Hadamard conjecture, then $\delta(n) \le 2$, and we can give much sharper lower bounds. In this case we show that the relative gap between the (Hadamard) upper bound and the lower bound is of order $n^{1/2}$. More precisely, our Corollary 4 gives $D(n)/n^{n/2} \ge (3n)^{-1/2}$. This improves on earlier results by de Launey and Levin [23], following Koukouvinos, Mitrouli and Seberry [20, Theorem 2], who obtained $D(n)/n^{n/2} \ge cn^{-3/2}$.

After defining our notation in §2, we give unconditional lower bounds on $D(n)$ in §3. The main result is Theorem 1, which implies that $D(n)/n^{n/2} \ge n^{-\delta(n)/2}$. A consequence (Corollary 3) which improves on a result of Clements and Lindström [6] is that $n \log n - 2 \log D(n) = O(n^{21/40} \log n)$.

In §4 we give stronger lower bounds on the assumption of the Hadamard conjecture.

The lower bound results are weaker than what is conjectured to be true. Numerical evidence for $n \le 120$ supports a conjecture of Rokicki *et al* [27] that $D(n)/n^{n/2} \ge 1/2$. In §4 we come close to this conjecture (on the assumption of the Hadamard conjecture) for five of the eight congruence classes of n mod 8.

2 Notation

The positive integers are denoted by $\mathbb{N}$, and the reals by $\mathbb{R}$.

For $n \in \mathbb{N}$, $\mathcal{H}_n$ denotes the set of Hadamard matrices of order n, and $\mathcal{H} := \{n \in \mathbb{N} \mid \mathcal{H}_n \neq \emptyset\}$. The elements of $\mathcal{H}$ in increasing order form the sequence $(n_i)_{i\geq 1}$ of all possible orders of Hadamard matrices ($n_1 = 1$, $n_2 = 2$, $n_3 = 4$, $n_4 = 8$, $n_5 = 12, \ldots$). The distance of n from a Hadamard order is

$$\delta(n) := \min_{h\in\mathcal{H}} |n - h|. \tag{1}$$

The primes are denoted by $(p_i)_{i\geq 1}$ with $p_1 = 2, p_2 = 3$, etc. The *prime gap* function $\lambda : \mathbb{R} \to \mathbb{Z}$ is

$$\lambda(x) := \max\,\{p_{i+1} - p_i \mid p_i \leq x\} \cup \{0\}.$$

By analogy, we define the *Hadamard gap* function $\gamma : \mathbb{R} \to \mathbb{Z}$ to be

$$\gamma(x) := \max\,\{n_{i+1} - n_i \mid n_i \leq x\} \cup \{0\}.$$

Finally, β_n denotes the well-known mapping from $\{+1, -1\}$ matrices of order $n > 1$ to $\{0, 1\}$ matrices of order $n - 1$, such that

$$|\det(A)| = 2^{n-1} |\det \beta_n(A)|.$$

3 Unconditional lower bounds on $D(n)$

The connection between the prime gap function λ and the Hadamard gap function γ is given by the following lemma.

Lemma 1. *For $n \geq 8$, we have $\gamma(n) \leq 2\lambda(n/2 - 1)$.*

Proof. If p is an odd prime, then $n = 2(p + 1) \in \mathcal{H}$. This follows from the second Paley construction [26] if $p \equiv 1 \pmod 4$, or from the first Paley construction followed by the Sylvester construction if $p \equiv 3 \pmod 4$. Thus, if p_i, p_{i+1} are consecutive odd primes, then $n_j = 2(p_i + 1) \in \mathcal{H}$, $n_k = 2(p_{i+1} + 1) \in \mathcal{H}$, and $k > j$. The result now follows from the definitions of the two gap functions. $\square$

Remark 1. De Launey and Gordon [22] have shown that the sequence of Hadamard orders (n_i) is asymptotically denser than the sequence of primes. Even if we consider only the Paley and Sylvester constructions and Kronecker products arising from them [1], we can frequently find Hadamard matrices whose orders lie in the interior of the interval $(2(p_i + 1), 2(p_{i+1} + 1))$ defined by a large prime gap. It would be interesting to compute the Hadamard gap function $\gamma(n)$ for $n \le 10^{12}$ say, and compare it with $2\lambda(n/2 - 1)$. On probabilistic grounds [8, 30] we expect $\gamma(n) \ll \lambda(n) \ll (\log n)^2$.

Corollary 1. *For $n \ge 8$, we have $\delta(n) \le \lambda(n/2 - 1)$.*

Proof. By the definition of $\delta(n)$ we have $\delta(n) \le \gamma(n)/2$, so the result follows from Lemma 1. □

Lemma 2 gives an inequality that is often useful.

Lemma 2. *If $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $n > |\alpha| > 0$, then*

$$\frac{(n-\alpha)^{n-\alpha}}{n^n} > \left(\frac{1}{ne}\right)^{\alpha}.$$

Proof. Taking logarithms, and writing $x = \alpha/n$, the inequality reduces to

$$(1 - x)\log(1 - x) + x > 0,$$

or equivalently (since $0 < |x| < 1$)

$$\frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots > 0.$$

This is clear if $x > 0$, and also if $x < 0$ because then the terms alternate in sign and decrease in magnitude. □

Recently Szöllősi [32, Proposition 5.5] established an elegant correspondence between the minors of order n and of order $h - n$ of a Hadamard matrix of order h. His result applies to complex Hadamard matrices, of which $\{+1, -1\}$ Hadamard matrices are a special case. More precisely, if $d+n = h$, $0 < d < h$, then for each minor of order d and value Δ there corresponds a minor of order n and value $\pm h^{h/2-d}\Delta$. Previously, only a few special cases (for small d or n, see for example [9, 21, 29, 31]) were known. We note that Szöllősi's crucial Lemma 5.7 follows easily from Jacobi's determinant identity [5, 14, 18], although Szöllősi gives a different proof.

Lemma 3. *Suppose* $0 < n < h$ *and* $h \in \mathcal{H}$. *Then* $D(n) \geq 2^{d-1} h^{h/2-d}$, *where* $d = h - n$.

Proof. Let $H \in \mathcal{H}_h$ be a Hadamard matrix of order h. By Szöllősi's theorem, H has a submatrix M of order n with $|\det(M)| = h^{h/2-d} |\det(M')|$, where M' is the corresponding submatrix of order $d = h - n$. At least one such pair (M, M') has a nonsingular M', so has $|\det(M')| \geq 2^{d-1}$. □

Remark 2. We could improve Lemma 3 for large d by using the fact that, from a result of de Launey and Levin [23, proof of Prop. 5.1], there exists M' with $|\det(M')| \geq (d!)^{1/2}$, which is asymptotically larger than the bound $|\det(M')| \geq 2^{d-1}$ that we used in our proof. However, in our application of the lemma, $h \gg d$, so it is the power of h in the bound that is significant.

Lemma 4. *Suppose* $0 < h < n$ *and* $h \in \mathcal{H}$. *Then* $D(n) \geq 2^{n-h} h^{h/2}$.

Proof. The case $h = 1$ is trivial, so suppose that $h > 1$. Let $H \in \mathcal{H}_h$ be a Hadamard matrix of order h, so H has determinant $\pm h^{h/2}$ and the corresponding $\{0, 1\}$ matrix $\beta_h(H)$ has determinant $\pm 2^{1-h} h^{h/2}$. We can construct a $\{0, 1\}$ matrix A of order $n - 1$ and the same determinant as $\beta_h(H)$ by adding a border of $n - h$ rows and columns (all zero except for the diagonal entries). Now construct a $\{+1, -1\}$ matrix $B = \beta_n^{(-1)}(A)$ by applying the standard mapping from $\{0, 1\}$ matrices to $\{+1, -1\}$ matrices. We have $|\det(B)| = 2^{n-1} |\det(A)| = 2^{n-h} h^{h/2}$. □

Lemma 5. *Let* $n \in \mathbb{N}$ *and* $\delta = \delta(n)$ *be defined by* (1). *Then* $n \geq 3\delta$.

Proof. The interval $[2n/3, 4n/3)$ contains a unique power of two, say h. By the Sylvester construction, $h \in \mathcal{H}$. However, $|n - h| \leq n/3$, so $\delta \leq n/3$. □

Theorem 1. *Let* $n \in \mathbb{N}$ *and* $\delta = \delta(n)$ *be defined by* (1). *Then*

$$\frac{D(n)}{n^{n/2}} \geq \left(\frac{4}{ne} \right)^{\delta/2} . \tag{2}$$

Proof. By the definition of $\delta(n)$, there exists a Hadamard matrix H of order $h = n \pm \delta$. If $\delta = 0$ the result is trivial, so suppose $\delta \geq 1$. We consider two cases. First suppose that $h = n + \delta$. Applying Lemma 3, we have

$$D(n) \geq 2^{\delta-1} h^{h/2-\delta} \geq h^{h/2-\delta}.$$

Applying Lemma 2 with $\alpha = -\delta$ gives

$$\frac{D(n)}{n^{n/2}} \geq \frac{h^{h/2-\delta}}{n^{n/2}} = \frac{(n+\delta)^{(n+\delta)/2}}{n^{n/2}}(n+\delta)^{-\delta} \geq \left(\frac{ne}{(n+\delta)^2}\right)^{\delta/2}.$$

By Lemma 5 we have $\delta/n \leq 1/3 < (e/2-1)$, from which it is easy to verify that $ne/(n+\delta)^2 > 4/(ne)$. The inequality (2) follows.

Now suppose that $h = n - \delta$. From Lemma 4 we have $D(n) \geq 2^\delta h^{h/2}$. Using Lemma 2 with $\alpha = \delta$, we have

$$\frac{D(n)}{n^{n/2}} > 2^\delta \left(\frac{1}{ne}\right)^{\delta/2} = \left(\frac{4}{ne}\right)^{\delta/2}.$$

Thus, in all cases we have established the desired lower bound on $D(n)$. □

Remark 3. De Launey and Levin [23, Theorem 3] give (in our notation) the bound $D(n)/n^{n/2} \geq n^{-d/2}$, where the exponent d could be as large as 2δ, so their bound could be worse than ours by a factor $\Omega(n^{\delta/2})$. The reason for the difference is that they always take a Hadamard matrix with order $h > n$, whereas we take $h < n$ and use Lemma 4 if that gives a sharper bound.

Corollary 2. *Let $n \in \mathbb{N}$, $n > 2$, and let $\lambda(n)$ be the prime gap function defined in §2. Then*

$$\frac{D(n)}{n^{n/2}} \geq \left(\frac{4}{ne}\right)^{\lambda(n/2)/2}.$$

Proof. For $n \geq 8$ this follows from Theorem 1, using Corollary 1. It is easy to check that the inequality holds for $2 < n < 8$ by using the known values of $D(n)$ listed in [25]. □

Remark 4. In the literature there are many inequalities for $\lambda(n)$, see for example Hoheisel [16] or Huxley [17]. The best result so far seems to be that of Baker, Harman and Pintz [2], who proved that $\lambda(n) \leq n^{21/40}$ for $n \geq n_0$, where n_0 is a sufficiently large (effectively computable) constant. Assuming the Riemann hypothesis, Cramér [8] proved that $\lambda(n) = O(n^{1/2}\log n)$. "Cramér's conjecture" (made by Shanks [30]) is that $\lambda(n) = O((\log n)^2)$, and numerical computations [24] provide some evidence for this conjecture. For a discussion of other relevant results on prime gaps, see [23, §1].

Corollary 3. *If $n \in N$, then*

$$0 \leq n \log n - 2 \log D(n) = O(n^{21/40} \log n) \quad as \quad n \to \infty.$$

Proof. The result follows from Corollary 2 and the theorem of Baker, Harman and Pintz [2]. □

Remark 5. Corollary 3 improves on Cohn [7, Theorem 13], who showed that $n \log n - 2 \log D(n) = o(n \log n)$, and Clements and Lindström [6], who showed that $n \log n - 2 \log D(n) = O(n)$.

4 Conditional lower bounds on $D(n)$

In this section we assume the Hadamard conjecture and give lower bounds on $D(n)$ that are sharper than the unconditional bounds of §3.

The idea of the proof of Theorem 2 is similar to that of Theorem 1 – we use a Hadamard matrix of slightly smaller or larger order to bound $D(n)$ when $n \not\equiv 0 \pmod 4$. In each case, we choose whichever construction gives the sharper bound. First we make a definition and state two well-known lemmas.

Definition 1. *Let A be a $\{\pm 1\}$ matrix. The* excess *of A is $\sigma(A) := \sum_{i,j} a_{i,j}$. If $n \in \mathcal{H}$, then $\sigma(n) := \max_{H \in \mathcal{H}_n} \sigma(H)$.*

The following lemma is a corollary of [12, Theorem 1], and gives a small improvement on Best's lower bound [4, Theorem 3] $\sigma(h) \geq 2^{-1/2} h^{3/2}$.

Lemma 6. *If $4 \leq h \in \mathcal{H}$, then*

$$\sigma(h) \geq (2/\pi)^{1/2} h^{3/2}.$$

The following lemma is "well-known" – it follows from [28, Theorem 2] and is also mentioned in later works such as [13, pg. 166].

Lemma 7. *If $h \in \mathcal{H}$, then*

$$D(h+1) \geq h^{h/2} \left(1 + \frac{\sigma(h)}{h}\right).$$

Theorem 2. *Assume the Hadamard conjecture. For $n \in \mathbb{N}$, $n > 2$, we have*

$$D(n) \ge \begin{cases} \left(\frac{2}{\pi e}\right)^{1/2} n^{n/2} & \text{if } n \equiv 1 \pmod 4, \\ \left(\frac{8}{\pi e^2 n}\right)^{1/2} n^{n/2} & \text{if } n \equiv 2 \pmod 4, \\ (n+1)^{(n-1)/2} \quad \sim \left(\frac{e}{n}\right)^{1/2} n^{n/2} & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3}$$

Proof. Suppose that $4 \le h \equiv 0 \pmod 4$. We are assuming the Hadamard conjecture, so $h \in \mathcal{H}$. Thus, combining the inequalities of Lemma 6 and Lemma 7, we have

$$D(h+1) \ge h^{h/2}(1 + (2h/\pi)^{1/2})\,. \tag{4}$$

Let A be a $\{\pm 1\}$ matrix of order $h+1$ with determinant at least the right side of (4). By the argument used in the proof of Lemma 4, we can construct a $\{\pm 1\}$ matrix of order $h+2$ with determinant at least $2h^{h/2}(1 + (2h/\pi)^{1/2})$ by adjoining a row and column to A. Thus

$$D(h+2) \ge 2h^{h/2}(1 + (2h/\pi)^{1/2})\,. \tag{5}$$

To prove the first inequality in (3), put $h = n-1$ in (4) and use Lemma 2 with $\alpha = 1$. Thus, for $1 < n \equiv 1 \pmod 4$,

$$D(n)/n^{n/2} \ge \left(\frac{2}{\pi e}\right)^{1/2} \left(\left(1 - \frac{1}{n}\right)^{1/2} + \left(\frac{\pi}{2n}\right)^{1/2}\right) > \left(\frac{2}{\pi e}\right)^{1/2}.$$

To prove the second inequality in (3), put $h = n - 2$ in (5) and use Lemma 2 with $\alpha = 2$. Thus, for $2 < n \equiv 2 \pmod 4$,

$$D(n)/n^{n/2} \ge \left(\frac{8}{\pi e^2 n}\right)^{1/2} \left(\left(1 - \frac{2}{n}\right)^{1/2} + \left(\frac{\pi}{2n}\right)^{1/2}\right) > \left(\frac{8}{\pi e^2 n}\right)^{1/2}.$$

Finally, if $n \equiv 3 \pmod 4$, then a Hadamard matrix of order $n+1$ exists. From Lemma 3 with $h = n+1$ we have $D(n) \ge (n+1)^{(n-1)/2}$. $\square$

Corollary 4. *Assume the Hadamard conjecture. If $n \ge 1$ then*

$$D(n)/n^{n/2} \ge 1/\sqrt{3n}\,.$$

Proof. For $n > 2$ this follows from Theorem 2, since $\pi e^2 < 24$. The result is also true if $n \in \{1, 2\}$, as then $D(n)/n^{n/2} = 1$. $\square$

Remark 6. The inequality (4) is within a factor $\sqrt{\pi}$ of the Barba bound $(2h+1)^{1/2}h^{h/2}$.

Remark 7. Corollary 4 sharpens a result of Koukouvinos, Mitrouli and Seberry [20, Theorem 2], also given in [23], that $D(n)/n^{n/2} \geq cn^{-3/2}$.

Remark 8. If $n \equiv 2 \pmod 8$, we get a lower bound $D(n)/n^{n/2} \geq 2/(\pi e)$ by using the Sylvester construction on a matrix of order $n/2 \equiv 1 \pmod 4$. Thus, the remaining cases in which there is a ratio of order $n^{1/2}$ between the upper and lower bounds are $(n \bmod 8) \in \{3, 6, 7\}$.

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