THE GENERALIZED POVERTY INDEX

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ABSTRACT. We introduce the General Poverty Index (GPI), which summarizes most of the known and available poverty indices, in the form

$$GPI = \delta(\frac{A(Q_n,n,Z)}{nB(Q,n)}\sum_{j=1}^{Q_n} w(\mu_1 n+\mu_2 Q_n-\mu_3 j+\mu_4) d\left(\frac{Z-Y_{j,n}}{Z}\right)),$$

where

$$B(Q_n,n)=\sum_{j=1}^{Q} w(j),$$

$A(\cdot)$, $w(\cdot)$, and $d(\cdot)$ are given measurable functions, Q_n is the number of the poor in the sample, Z is the poverty line and $Y_{1,n} \leq Y_{2,n} \leq ... \leq Y_{n,n}$ are the ordered sampled incomes or expenditures of the individuals or households. We show here how the available indices based on the poverty gaps are derived from it. The asymptotic normality is then established and particularized for the usual poverty measures for immediate applications to poor countries data.

1. Introduction

The Economists are interested in monitoring the welfare of the worse-off in one given population. In this capacity, poverty measures are defined and used to compare subgroups and to follow the evolution of the poor with respect to time. A poverty measure is assumed to fulfill a number of axioms since the pioneering work of Sen ([13]). Many authors proposed poverty indices and studied their advantages, like Sen himself, Thon ([17]), Kakwani ([7]), Clark-Hemming-Ulph ([2]), Foster-Greer-Thorbecke ([6]), Ray ([20]), Shorrocks ([15]). Most of the required properties for such indices are stated and described in ([19]) along with a broad survey of the available poverty indices.

Asymptotic theories for theses quantities, when they come from random samplings, have been given in recent years. Dia([5]) used

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point process theory to give asymptotic normality for the Foster-Greer-Thorbecke (FGT) index. Sall and Lo([11]) studied an asymptotic theory for the poverty intensity defined below and further, Sall, Seck and Lo([9]) proved a larger asymptotic normality for a general measure including the Sen, Kakwani, FGT and Shorrocks ones.

Now our aim, here, is to unify the monetary poverty measurements with respect as well to Sen's axiomatic approach as to the asymptotic aspects. We point out that poverty may be studied through aspects other than monetary ones as well. It can be viewed through the capabilities to meet basic needs (food, education, health, clothings, lodgings, etc.). In our monetary frame, the main tools are the poverty indices. We give, here, a general poverty index denoted as the General Poverty Index (GPI), which is aimed to summarize all the known and former ones. Let us make some notation in order to define it.

We consider a population of individuals, each of which having a random income or expenditure Y with distribution function $G(y) = P(Y \leq y)$. An individual is classified as poor whenever his income or expenditure Y fulfills $Y < Z$, where Z is a specified threshold level (the poverty line).

Consider now a random sample $Y_1, Y_2, ...Y_n$ of size n of incomes, with empirical distribution function $G_n(y) = n^{-1}\#\left\{Y_i \leq y : 1 \leq i \leq 1\right\}$. The number of poor individuals within the sub-population is then equal to $Q_n = nG_n(Z)$.

Given these preliminaries, we introduce measurable functions $A(p, q, z)$, $w(t)$, and $d(t)$ of $p, q \in N$, and $z, t \in R$. The meaning of these functions will be discussed later on. Set $B(Q_n, n) = \sum_{i=1}^{q} w(i)$.

Let now $Y_{1,n} \leq Y_{2,n} \leq ... \leq Y_{n,n}$ be the order statistics of the sample $Y_1, Y_2, ...Y_n$ of Y. We consider general poverty indices of the form

(1.1)
$$GPI_n = \frac{A(Q_n, n, , Z)}{nB(Q_n, n)} \sum_{j=1}^{Q_n} w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4)\ d\left(\frac{Z - Y_{j,n}}{Z}\right),$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are constants. By particularizing the functions A and w and by giving fixed values to the $\mu_i' s$, we may find almost all the available indices, as we will do it later on. In the sequel, (1.1) will be called a poverty index (indices in the plural) or simply a poverty measure according to the economists terminology.

The poverty line Z is defined by economics specialists or governmental authorities so that any individual or household with income (say yearly) less than Z is considered as a poor one. The poverty line determination raises very difficult questions as mentioned and shown in ([8]). We suppose here that Z is known, given and justified by the specialists.

Our unified and global approach will permit various research works, as well in the Statistical Mathematics field as in the Economics one. It happens that poverty indices are also somewhat closely connected with economic growth questions. We should find conditions on the functions and the constants in (1.1) so that any kind of needed requirements are met and that the hypotheses imposed by the asymptotic normality are also fulfilled. This may lead to a class of perfect or almost perfect poverty measures. In this paper, we concentrate on the description of the GPI and on the asymptotic normality theory. Our best achievement is that (1.1), is asymptotically normal for a very broad class of underlying distributions. These results are then specialized for the particular and popular indices.

We then begin to describe all the available indices in the frame of (1.1) in the next section. In section 3, we establish the asymptotic normality. Related application works to poverty databases can be found in [10] for instance.

2. How does the GPI include the poverty indices

We begin by making two remarks. First, for almost all the indices, the function $\delta(\cdot)$ is the identity one

$$\forall(u \geq 0),\ \delta(u) = I_d(u) = u.$$

We only noticed one exception in the Clark-Hemming-Ulph (CHUT) index. Secondly, we may divide the poverty indices into non-weighted and weighted ones. The non weighted measures correspond to those for which the weight is constant and equal to one :

$$w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4) \equiv 1.$$

We begin with them.

2.1. **The non-weighted indices.**

First of all, the Foster-Greer-Thorbecke (FGT) index of parameter [6] defined for $\alpha \geq 0$,

$$FGT_n(\alpha) = \frac{1}{n}\sum_{j=1}^{Q_n}\left(\frac{Z - Y_{j,n}}{Z}\right)^{\alpha}. \tag{2.1}$$

is obtained from the GPI with

$$\delta = I_d, \quad w \equiv 1, \quad d(u) = u^{\alpha}, \quad B(Q_n, n) = Q_n \textit{ and } A(Q_n, n, Z) = Q_n.$$

The Ray index defined by (see [20]), for $\alpha > 0$,

$$R_{R,n} = \frac{g}{nZ} \sum_{i=1}^{Q_n} ((Z - Y_{j,n})/g)^{\alpha} \tag{2.2}$$

where

$$g = \frac{1}{Q_n} \sum_{j=1}^{j=Q_n} (Z - Y_{j,n}) \tag{2.3}$$

is derived from the GPI with

$$\delta = I_d, \quad w \equiv 1, \quad d(u) = u^{\alpha}, \quad B(Q_n, n) = Q_n \ and \ A(Q_n, n, Z) = Q_n (g/Z)^{\alpha - 1}.$$

The coefficient $A(Q_n, n, Z)$ depends here on the income or the expenditure. This is quite an exception among the poverty indices. We may also cite here the Watts index (see [18])

$$P_{W,n} = \frac{1}{n} \sum_{j=1}^{j=Q_n} (\ln Z - lnY_{j,n}).$$

But this may be derived from the FGT one as follows. The income Y is transformed into $\ln Y$ and, consequently, the poverty line is taken as $\ln Z$. It follows that

$$W(Y) = FGT(1, \ln Y)$$

for the poverty line $\ln Z$. The case is similar for the Chakravarty index (see [1]), $0 < \alpha < 1$,

$$P_{Ch} = \frac{1}{n} \sum_{j=1}^{j=Q_n} (1 - (\frac{Y_{j,n}}{Z})^{\alpha}).$$

We may consider it through the FGT class

$$W(Y) = FGT(1, Y^{\alpha})$$

for the poverty line Z^{α}.
Now, we have that the CHU index is clearly of the GPI form with

$$\delta = (u) = u^{1/\alpha}, \quad w \equiv 1, \quad d(u) = u^{\alpha} \quad B(Q_n, n) = Q_n \ and \ A(Q_n, n, Z) = Q_n^{\alpha} / n^{\alpha - 1}.$$

Now let us describe the weighted indices.

2.2. **The weighted indices.** First, the Kakwani ([7]) class of poverty measures

$$P_{KAK,n}(k) = \frac{Q_n}{n\Phi_k(Q_n)} \sum_{j=1}^{Q_n} (Q_n - j + 1)^k \left(\frac{Z - Y_{j,n}}{Z} \right), \tag{2.4}$$

where

$$\Phi_k(Q_n) = \sum_{j=1}^{j=Q_n} j^k \ = B(Q_n, n)$$

comes from the GPI with

$$\delta = I_d,\ w(u) \equiv (u),\ d(u) = u,\ \mu_1 = 0,$$

$$\mu_2 = 1,\ \mu_3 = -1,\ \mu_4 = 1 \ \textit{and}\ A(n, Q_n, Z) = Q_n$$

For $k = 1$, $P_{KAK,n}(1)$ is nothing else but the Sen poverty measure

$$P_{Sen} = \frac{2}{n(Q_n + 1)} \sum_{j=1}^{Q_n} (Q_n - j + 1) \left(\frac{Z - Y_{j,n}}{Z} \right). \tag{2.5}$$

As to the Shorrocks ([15]) index

$$P_{SH,n} = \frac{1}{n^2} \sum_{j=1}^{Q_n} (2n - 2j + 1) \left(\frac{Z - Y_{j,n}}{Z} \right), \tag{2.6}$$

it is obtained from the GPI with

$$B(Q_n, n) = Q(2n - Q),\ A(n, Q_n, Z) = Q_n(2n - Q_n)/n$$

and

$$\delta = I_d,\quad w(u) \equiv (u),\quad d(u) = u,\quad \mu_1 = 2,\quad \mu_2 = 0,\quad \mu_3 = 2,\ \mu_3 = 1.$$

Thon ([17]) proposed the following measure

$$P_{Th} = \frac{2}{n(n + 1)} \sum_{j=1}^{Q_n} (n - j + 1) \left(\frac{Z - Y_{j,n}}{Z} \right)$$

which belongs to the GPI family for

$$B(n, Q_n) = Q_n(n - Q_n + 1)/2,\ A(n, Q_n, Z) = Q(n - Q + 1)/(n + 1),$$

and

$$\delta = I_d,\quad w(u) \equiv u,\quad d(u) = u,\quad \mu_1 = 1,\quad \mu_2 = 0,\quad \mu_3 = 1,\quad \mu_3 = 1.$$

Not all the poverty indices are derived from the GPI. What precedes only concerns those based on the poverty gaps

$$(Z - Y_j),\ 1 \leq j \leq Q_n.$$

We mention one of them in the next subsection.

2.3. **An index not derived from the GPI.** The Takayama ([16]) index

$$P_{TA,n} = 1 + \frac{1}{n} - \frac{2}{\mu n^2} \sum_{j=1}^{Q_n} (n-j+1) Y_{j,n},$$

where μ is the empirical mean of the censored income, cannot be derived from the GPI. The main reason is that, it is not based on the poverty gaps $Z - Y_{j,n}$. It violates the monotonicity axiom which states that the poverty measure increases when one poor individual or household becomes richer.

Now we must study the so-called GPI with respect to the axiomatic approach as well as to the asymptotic theory. We focus in this paper to the general theory of asymptotic normality. The interest of this unified approach is based on the fact that we obtain at once the asymptotic behaviors for all the available poverty indices, as particular cases. Indeed, in the next section, we will describe apply the general theorem to the particular usual indices.

3. Asymptotic normality of the GPI

Let us write the GPI in the form

$$GPI_n = \delta(J_n) \tag{3.1}$$

with

$$J_n = \frac{1}{n} \sum_{j=1}^{Q_n} c(n, Q_n, j)\ d\left(\frac{Z - Y_{j,n}}{Z}\right), \tag{3.2}$$

where $c(n, Q_n, j) = A(Q_n, n, , Z) \times w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4)\ /\ B(Q_n, n)$.

Since Y is an income or expenditure variable, its lower endpoint y_0 is not negative. This allows us to study (3.1) via the transform $X = 1/(Y - y_0)$. Throughout this paper, the distribution function of X is

$$F(\cdot) = 1 - G(y_0 + 1/\cdot),$$

whose upper endpoint is then $+\infty$. Hence (3.2) is transformed as

$$J_n = \frac{1}{n} \sum_{j=1}^{q} c(n, q, j)\ d\left(\frac{Z - y_0 - X_{n-j+1,n}^{-1}}{Z}\right). \tag{3.3}$$

We will need conditions on the function d($\cdot$) and on the weight c(n,q,j), as in ([9]). First assume that

$(D1)$ $d(\cdot)$ admits a continuous derivative on $]0,1)$.

$(D2)$ $d'(\frac{z-y_0}{z})$ and $d((z-y_0)/z)$ are finite.

For $A(u) = 1/F^{-1}(1-u)$, we assume that:

$(C1)$ $A(\cdot)$ is differentiable $(0,1)$ (and its derivative is denoted $A'(u) = a(u)$.)

$(C2)$ $a(\cdot)$ is continuous on an interval $[a', a'']$ with $0 < a' < a'' < 1$.

$(C3)$ $\exists\, u_0 > 0, \exists\, \eta > -3/2,\ \forall\, u \in (0, u_0)\,, |a(u)| < C_0\ u^{\eta} \exp(\int_u^1 b(t)t^{-1}dt), with\ b(t) \to 0\ as\ t \to 0.$

The condition (C3) means that $a(\cdot)$ bounded by a regularly varying function

$$S(u) = C_0\ u^{\eta} \exp(\int_u^1 b(t)t^{-1}dt)$$

of exponent $\eta > -3/2$. As to the function δ, we need it to be differentiable on $]0, +\infty[$, precisely :

(E) There is $\kappa > 0$ such that $\delta(\cdot)$ is continuously differentiable on $]0, \kappa]$.

We also need some conditions on the weight $c(\cdot)$. In order to state the hypotheses, we introduce further notation. In fact we use in this paper the representations of the studied random variables X_i, $i \geq 1$, by $F^{-1}(1-U_i)$, $i \geq 1$, where U_1, $U_2, ...$ is a sequence of independent random variables uniformly distributed on $(0,1)$. Now let $U_n(\cdot)$ and $V_n(\cdot)$ be the uniform empirical distribution and the empirical quantile function based on $U_i, 1 \leq i \leq n$. We have

$$j \geq 1, \quad \frac{j-1}{n} < s \leq \frac{j}{n} \Longrightarrow \frac{j}{n} = U_n(V_n(s)) \tag{3.4}$$

so that

(3.5)
$$j \geq 1, \quad \frac{j-1}{n} < s \leq \frac{j}{n} \Longrightarrow c(n,q,j) = c(n,q,nU_n(V_n(s)) \equiv L_n(s).$$

Since $U_n(V_n(s)) \to s$, as n$\to \infty$, our condition on the weight $c(\cdot)$ is that the function $L_n(\cdot)$ is uniformly bounded by some constant $D > 0$ and

$$L_n(s) \to L(s), as\ n \to \infty, \tag{3.6}$$

where $L(\cdot)$ is a non-negative C^1-function on $(0,1)$.

We further require that, as $n \to \infty$,

$$\sup_{0 \leq s \leq 1} \left| \sqrt{n}(L_n(s) - L(s)) - \gamma(s)\sqrt{n}(G_n(Z) - G(Z)) \right| = o_p(1) \tag{3.7}$$

for some function $\gamma(\cdot)$. Let us finally put

$$m(s) = L(s)\ d\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right).$$

We are now able to give our general theorem for the GPI.

Theorem 1. *Suppose that (C1-2-3), (D1-2) and (3.7) hold and let*

$$\mu = \int_0^{G(Z)} \gamma(s) d\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)\ ds.$$

and

$$D = \int_0^{G(Z)} L(s)\ d\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right) ds,$$

Then

$$\sqrt{n}(J_n - D) \to \mathcal{N}(0, \vartheta^2)$$

with

$$\vartheta^2 = \theta^2 + (m(G(Z)) + \mu)^2 G(Z)(1 - G(Z)) + \frac{2(m(G(Z)) + \mu)}{Z} \int_0^{G(Z)} sL(s)h(s)ds$$

and with

$$\theta^2 = Z^{-2} \int_0^{G(Z)} \int_0^{G(Z)} L(u)\ L(v)\ h(u)h(v)(u \wedge v - uv)\ du\ dv$$

where

$$h(s) = a(s)\ d'(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}).$$

If furthermore (E) holds and $D \in]0, \kappa[$, then

$$\sqrt{n}(GPI_n - \delta(D)) \to \mathcal{N}(0, \vartheta^2 \delta'(D)^2)$$

The interest of this paper resides on the particular applications of the theorem for the known indices. Before this, we give the guidelines of the proof.

4. PROOFS OF THE RESULTS

All our results will be derived from the lemma below. But, first we place ourselves on a probability space where one version of the so-called Hungarian constructions holds. Namely, M. Csörgő and al. (see [4]) have constructed a probability space holding a sequence of independent uniform random variables $U_1,\ U_2,\ ...$ and a sequence of Brownian bridges $B_1, B_2, ...$ such that for each $0 < \nu < 1/2$, *as* $n \to \infty$,

$$\sup_{1/n \leq s \leq 1-1/n} \frac{|\beta_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}) \tag{4.1}$$

and for each $0 < \nu < 1/4$

$$\sup_{1/n \leq s \leq 1-1/n} \frac{|B_n(s) - \alpha_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}), \tag{4.2}$$

where $\{\alpha_n(s) = \sqrt{n}\,(U_n(s) - s))\,, 0 \leqslant s \leqslant 1\}$ is the uniform empirical process and $\{\beta_n(s) = \sqrt{n}\,(s - V_n(s))\,, 0 \leqslant s \leqslant 1\}$ is the uniform quantile process. (See also [3] for a shorter and more direct proof, and [12] for dual version, in the sens that, 4.1 holds for $0 < \nu < 1/2$ and 4.2 for $0 < \nu < 1/4$ in [3], while 4.1 is established for $0 < \nu < 1/4$ and 4.2 for $0 < \nu < 1/2$ in [12]). Throughout ν will be fixed with $0 < \nu < 1/4$. Now we are able to give the lemma.

Lemma 1. *Suppose that (C1-2-3) and (D1-2) hold and*

$$\sup_{0 \leq s \leq 1} \sqrt{n}\,|L_n(s) - L(s)| = O_P(1) \text{ as } n \to \infty. \tag{4.3}$$

Let

$$D = \int_0^{G(Z)} L(s)\ d\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right) ds.$$

Then we have the expansion

$$\sqrt{n}(J_n - D) = N_n(1) + N_n(2)$$

$$+ \int_{1/n}^{G(Z)} \sqrt{n}(L_n(s) - L(s)) d\left(\frac{Z - y_0 - 1/F^{-1}(1-s))}{Z}\right) ds + o_P(1)$$

with

$$N_n(1)=\frac{1}{Z}\int_{1/n}^{G(Z)}L(s)B_n(s)h(s)ds \tag{4.4}$$

and

$$N_n(2)=m(G(Z))B_n(G(Z)) \tag{4.5}$$

for

$$m(s)=L(s)\ d\left(\frac{Z-y_0-1/F^{-1}(1-s)}{Z}\right).$$

Proof. This expansion is formulae (4.14) in ([9]). Then, we have the expansion

$$\sqrt{n}(J_n-C_n)=\frac{1}{Z}\int_{1/n}^{G(Z)}L(s)B_n(s)h(s)\ ds+n^{-1/2}L_n(1/n)\ d\left(\frac{Z-y_0-1/F^{-1}(1-U_{1,n})}{Z}\right)$$

$$+\int_{1/n}^{G_n(Z)}\sqrt{n}(L_n(s)-L(s))\ d\left(\frac{Z-y_0-1/F^{-1}(1-V_n(s))}{Z}\right)ds$$

$$+\frac{1}{Z}\int_{G(Z)}^{G_n(Z)}L(s)B_n(s)h(s)ds+\frac{1}{Z}\int_{1/n}^{G_n(Z)}L_n(s)B_n(s)\ (h(\zeta_n(s))-h(s))\ ds$$

$$+\frac{1}{Z}\int_{1/n}^{G_n(Z)}L_n(s)\ (\beta_n(s)-B_n(s))\ h(\zeta_n(s))\ ds$$

It is proved in ([9]) that

$$\sqrt{n}(J_n-C_n)=N_n(1)+N_n(2)$$

$$+\int_{1/n}^{G_n(Z)}\sqrt{n}(L_n(s)-L(s))\ d\left(\frac{Z-y_0-1/F^{-1}(1-V_n(s))}{Z}\right)ds+o_P(1).$$

This gives

$$\sqrt{n}(J_n-C_n)=N_n(1)+N_n(2)$$

$$+\int_{1/n}^{G(Z)}\sqrt{n}(L_n(s)-L(s))d\left(\frac{Z-y_0-1/F^{-1}(1-s))}{Z}\right)ds$$

$$+\int_{Gn(Z)}^{G(Z)}\sqrt{n}(L_n(s)-L(s))\ d\left(\frac{Z-y_0-1/F^{-1}(1-V_n(s))}{Z}\right)ds+o_P(1)$$

The condition (4.3) leads to the result. □

We are now able to prove the Theorem.

Proof. Let(Ω,Σ,P) be the probability space on which (4.1) and (4.2) hold. The Lemma together with (4.3), (3.7) and (4.5), imply

$$\sqrt{n}(J_n - D) = N_n(1) + N_n(3) + o_P(1),$$

where $N_n(1)$ is defined in (4.4) and

(4.6)

$$N_n(3) = (m(G(Z)+\mu)\alpha_n(G(Z))+o_P(1) = (m(G(Z)+\mu)B_n(G(Z))+o_P(1).$$

The vector $(N_n(1), N_n(3))$ is Gaussian and

(4.7)

$$cov(N_n(1), N_n(3)) = \frac{m(G(Z)) + \mu}{Z} E \int_{1/n}^{G(Z)} L(s)h(s)B_n(G(Z))B_n(s)ds$$

$$= \frac{m(G(Z)) + \mu}{Z} \int_{1/n}^{G(Z)} s\ L(s)\ h(s)\ ds.$$

Then $\sqrt{n}(J_n - D)$ is a linear transform $N_n(1) + N_n(3)$ of the Gaussian vector $(N_n(1), N_n(3))$, plus an $o_P(1)$ term. The variance of this Gaussian term is easily computed through (4.7) and the conclusion follows, that is $\sqrt{n}(J_n - D)$ is asymptotically a centered Gaussian random variable with variance (4.7). □

5. Asymptotic normality of particular indices

5.1. **The FGT-like class.** This concerns the indices of the form

$$FGT(\alpha) = \frac{1}{n}\sum_{j=1}^{Q_n} d\left(\frac{Z - Y_{j,n}}{Z}\right).$$

We have here

$$L_n = 1$$

so that

$$\gamma = 0$$

Then

$$\sqrt{n}(J_n - D) \to \mathcal{N}(0, \vartheta^2)$$

with

$$\vartheta^2 = \theta^2 + m(G(Z)^2 G(Z)(1 - G(Z)) + \frac{2m(G(Z))}{Z} \int_0^{G(Z)} sh(s)ds$$

and

$$D = \int_0^{G(Z)} \left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds.$$

We should remark that the conditions $(D1 - D2)$ hold for $d(u) = u^{\alpha}, \alpha \geq 0$.

5.1.1. *The statistics nearby the FGT-class.* This concerns the statistics of the form

$$J_n = \delta\left(\frac{A(Q_n, n)}{n}\sum_{j=1}^{Q_n} d\left(\frac{Z - Y_{j,n}}{Z}\right)\right),$$

where we have a random weight not depending on the rank's statistic. We will have two sub-cases.

5.1.2. *The case of CHU's index.* Recall

$$CHU_n(\alpha) = \frac{Q_n}{nZ}\left\{\frac{1}{Q_n}\sum_{j=1}^{Q_n}(Z - Y_{j,n})^{\alpha}\right\}^{1/\alpha}$$

$$= \left\{\frac{1}{n}\frac{Q_n^{\alpha-1}}{n^{\alpha-1}}\sum_{j=1}^{Q_n}(\frac{Z - Y_{j,n}}{Z})^{\alpha}\right\}^{1/\alpha} = \delta(J_n)$$

We easily get,

$$\sqrt{n}((q/n)^{\alpha-1} - G(Z)^{\alpha-1}) = (\alpha - 1)G(Z)^{\alpha-2}\sqrt{n}(G_n(Z) - G(Z)) + o_p(1)$$
$$= (\alpha - 1)G(Z)^{\alpha-2}B_n(G(Z)) + o_p(1).$$

By putting

$$C_n = FGT(\alpha) = \frac{1}{n}\sum_{j=1}^{Q_n}(\frac{Z - Y_{j,n}}{Z})^{\alpha}$$

and

$$C = \int_0^{G(Z)}\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds, \tag{5.1}$$

we have, by the general theorem

$$\sqrt{n}(C_n - C) = N_n(1) + N_n(2) + o_p(1)$$

with $L = 1$. By combining these formulae, we get

$$\sqrt{n}(J_n - G(Z)^{\alpha-1}C) \to N(0, \zeta^2)$$

with

$$\zeta^2 = \theta^2 + H(1 - G(Z))\int_0^{G(Z)} s\ a(s)ds + H^2G(Z)(1 - G(Z))/2$$

where,

$$H = C(\alpha - 1) + G(Z)m(G(Z))G(Z)^{\alpha-2}.$$

Finally, we get

$$\sqrt{n}(CHU_n(\alpha) - \delta(G(Z)^{\alpha-1}C) \to N(0, (\zeta\delta'(G(Z)^{\alpha-1}C)^2),$$

where

$$\delta'(G(Z)^{\alpha-1}C)^2 = G(Z)^{-(\alpha-1)^2/\alpha}C^{(1-\alpha)/\alpha}.$$

5.1.3. *The case of Ray's index.* Recall

$$P_{R,n}(\alpha) = \frac{g}{nZ}\sum_{j=1}^{Q_n}((Z - Y_{j,n})/g)^{\alpha} \tag{5.2}$$

where

$$g = \frac{1}{q}\sum_{j=1}^{j=Q_n}(Z - Y_{j,n}). \tag{5.3}$$

We have

$$J_n = g^{\alpha-1} \times C_n$$

with

$$C_n = FGT_n(\alpha)$$

and

$$C(\alpha) = \int_0^{G(Z)} \left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds.$$

We use the notation for the CHU index and we also get (5.1). But

$$g = \frac{Zn}{Q_n} \times \frac{1}{n}\sum_{j=1}^{j=Q_n}(\frac{Z - Y_{j,n}}{Z}) \equiv \frac{Zn}{Q_n}K_n.$$

We also have

$$\sqrt{n}(K_n - K) = \frac{1}{Z}\int_{1/n}^{G(Z)} B_n(s)a(s)ds + m_1(G(Z))B_n(G(Z)) + o_p(1)$$

with

$$K = C(1) = \int_0^{G(Z)} \frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}ds$$

and

$$\begin{aligned}\sqrt{n}(\frac{Zn}{q} - ZG(Z)^{-1}) &= Z\sqrt{N}(G(Z) - G_n(Z))G(Z)^{-2} + o_p(1)\\ &= -Z\sqrt{n}B_n(G_n(Z))G(Z)^{-2} + o_p(1).\end{aligned}$$

By combining all that precedes, we arrive at

$$\begin{aligned}\sqrt{n}(g - KZG(Z)^{-1}) &= (m_1(G)ZG(Z)^{-1} - KZG(Z)^{-2})B_n(G(Z))\\ &+\frac{1}{G(Z)}\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1)\end{aligned}$$

and

$$\sqrt{n}(g^{\alpha-1} - (KZ/G(Z))^{\alpha-1}) = (\alpha-1)(KZ/G(Z))^{\alpha-2}$$

$$= (\alpha-1)(KZ/G(Z))^{\alpha-2}\ (m_1(G)ZG(Z)^{-1} - KZG(Z)^{-2})B_n(G(Z))$$

$$+\frac{(\alpha-1)(KZ/G(Z))^{\alpha-2}}{G(Z)}\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1)$$

$$= R_1 B_n(G(Z)) + R_2\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1).$$

Finally

$$\sqrt{n}(R_n - (KZ/G(Z))^{\alpha-1})C) =$$

$$\frac{(KZ/G(Z))^{\alpha-1})}{Z}\int_{1/n}^{G(Z)} B_n(s)h(s)ds + (KZ/G(Z))^{\alpha-1})\ m_\alpha(G(Z))B_n(G(Z))$$

$$+CR_1B_n(G(Z)) + CR_2\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1)$$

$$= \int_{1/n}^{G(Z)} B_n(s)\psi(s)ds + \left\{(KZ/G(Z))^{\alpha-1})\ m_\alpha(G(Z)) + CR_1\right\} B_n(G(Z)) + o_P(1)$$

$$= \int_{1/n}^{G(Z)} B_n(s)a(s)ds + A_2B_n(G(Z)) + o_P(1),$$

with

$$\psi(s) = a(s)\left\{C(\alpha)R_2 + (KZ/G(Z))^{\alpha-1}Z^{-1}d'(Z^{-1}(Z - y_0 - 1/F^{-1}(1-s)))\right\}.$$

Notice that $\int_{1/n}^{G(Z)} B_n(s)h(s)ds + A_1B_n(G(Z))$ is a normal centered random variable with variance

$$\xi^2 = \int_0^{G(Z)}\int_0^{G(Z)} \psi(u)\psi(v)(u\wedge v - uv)\ du\ dv$$
$$+ A_1^2G(Z)(1-G(Z)) + 2A_1(1-G(Z))\int_0^{G(Z)} s\ \psi(s)ds.$$

We conclude that

$$\sqrt{n}(P_{R,n}(\alpha) - (KZ/G(Z))^{\alpha-1})C) \rightarrow_d N(0,\xi^2)$$

with

$$m_\alpha(u) = (Z^{-1}(Z - y_0 - 1/F^{-1}(1-s))^\alpha,$$
$$R_1 = (\alpha-1)(KZ/G(Z))^{\alpha-2}\ (m_1(G)ZG(Z)^{-1} - KZG(Z)^{-2}),$$
$$R_2 = (\alpha-1)(KZ/G(Z))^{\alpha-2}G(Z)^{-1},$$

and

$$A_1 = (KZ/G(Z))^{\alpha-1})\ m_\alpha(G(Z)) + CR_1$$

5.2. **The Shorrocks-like indices.** This concerns the Thon and Shorrocks measures. They both have a similar asymptotic behavior.

For Shorrocks's index, we have

$$P_{SH,n} = \frac{1}{n^2}\sum_{j=1}^{Q_n}(2n-2j+1)\left(\frac{Z-Y_{j,n}}{Z}\right).$$

But

$$(5.4)\quad j \geq 1,\ \frac{j-1}{n} < s \leq \frac{j}{n} \Longrightarrow L_n(s) = c(n,q,j) = (2-2*j/n+1/n)$$
$$\to L(s) = 2(1-s),$$

and,

$$\sqrt{n}(L_n(s)-L(s)) = -2*\sqrt{n}(U_n(V_n(s))-s)+1/\sqrt{n},$$

By ([14]), p.151,

$$\sqrt{n}\sup_{0\leq s\leq 1}|L_n(s)-L(s)| \leq 3/\sqrt{n}.$$

and then

$$\gamma \equiv 0,\ h(\cdot) = a(\cdot)$$

For the Thon Statistic,

$$P_{T,n} = \frac{2}{n(n+1)}\sum_{j=1}^{Q_n}(n-j+1)\left(\frac{Z-Y_{j,n}}{Z}\right),$$

we also have

$$L(s) = 2(1-s),\ \ \gamma \equiv 0,\ \ a(\cdot) = h(s).$$

In both cases, for $P_n = P_{SH,n}$ or $j_n = P_{T,n}$, we get

$$\sqrt{n}(P_n - D) \to \mathcal{N}(0,\vartheta^2)$$

with

$$D = 2\int_0^{G(Z)}(1-s)\ \left(\frac{Z-y_0-1/F^{-1}(1-s)}{Z}\right)ds,$$

$$\vartheta^2 = \theta^2 + m(G(Z)G(Z)(1-G(Z)) + \frac{4m(G(Z))}{Z}\int_0^{G(Z)} s(1-s)a(s)ds$$

and with

$$\theta^2 = 4Z^{-2}\int_0^{G(Z)}\int_0^{G(Z)}(1-u)(1-v)\ a(u)a(v)(u\wedge v-uv)\ du\ dv.$$

5.3. **The Kakwani-class.** The Kakwani class

$$P_{KAK,n} = \frac{Q_n}{n\Phi_k(Q_n)} \sum_{j=1}^{Q_n} (Q_n - j + 1)^k \left(\frac{Z - Y_{j,n}}{Z} \right),$$

is introduced with a positive integer. We consider here that k is merely a non-negative real number. It is proved in ([**?**]) that

$$L(s) = (k+1)(1 - s/G(Z))^k$$

and that

$$\gamma(s) = k(k+1)(1 - s/G(Z))^{k-1}(s/G(Z)^2).$$

We remark that $m(G(Z)) = 0$. Then our result is particularized as

$$\sqrt{n}(P_{KAK,n}(k) - D) \rightarrow \mathcal{N}(0, \vartheta^2)$$

with

$$\vartheta^2 = \theta^2 + \mu^2 G(Z)(1 - G(Z)) + \frac{2\mu}{Z}(1 - G(Z)) \int_0^{G(Z)} sL(s)h(s)ds \tag{5.5}$$

and with

$$\theta^2 = Z^{-2} \int_0^{G(Z)} \int_0^{G(Z)} L(u)\ L(v)\ h(u)h(v)(u \wedge v - uv)\ du\ dv.$$

for a fixed real number $k \geq 1$.

We have now finished the poverty indices' review. Some of these results have been simulated and applied in particular issues with the Senegalese Data.

6. Conclusion

The GPI includes most of the poverty indices. We have established here their asymptotic normality with immediate applications to poor countries data for finding accurate confidence intervals of the real poverty measurement. In coming papers, a special study will be devoted to the Takayama statistic. The GPI is to be thoroughly visited through the poverty axiomatic approach as well.

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