THE NUMÉRAIRE PROPERTY AND LONG-TERM GROWTH OPTIMALITY FOR DRAWDOWN-CONSTRAINED INVESTMENTS

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Abstract. We consider the portfolio choice problem for an investor interested in long-run growth optimality while facing drawdown constraints in a general continuous semimartingale model. The paper introduces the numéraire property through the notion of expected relative return and shows that drawdown-constrained strategies with the numéraire property exist and are unique, but may depend on the financial planning horizon. We explicitly characterize the growth-optimal strategy and show that it enjoys the numéraire property within the class of investments satisfying the drawdown constraint, when sampled at the times of its maximum and asymptotically as the time-horizon becomes distant. Finally, it is established that the asymptotically growth-optimal strategy is obtained as limit of numéraire strategies on finite horizons.

Introduction

Discussion. The debate whether an investor with long financial planning horizon should use the growth-optimal strategy, as postulated by Kelly (1956), or not is among the oldest in portfolio selection literature, see MacLean et al. (2011). In particular, opposite sides were assumed by two of the most prominent scholars in the field: while Paul Samuelson fiercely criticized the use of Kelly's strategy (the famous refute Samuelson (1979), in words of one-syllable, is a representative account of his attitude), Harry Markowitz is quite fond of it—see Markowitz (2006). The arguments in favor of Kelly's investment strategy rely on the fact that asymptotic growth should be of prime interest for long-run investment. More recently, this line of argument has seen a revived interest in particular through the so-called benchmark approach, see Platen and Heath (2006). The arguments against it point to the fact that the growth-rate maximization does not take into account investor's risk appetite and is too simplistic. Samuelson, as well as many others including seminal works of Merton (1971), suggested maximizing expected utility instead. While Kelly's strategy itself falls into this category, with the utility function being the logarithmic one, choices of other utility functions result in criteria that can accommodate different preference profiles. The concept of expected utility maximization has lead to an extremely rich field of research. However, although

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quite flexible, this line of reasoning is vulnerable to critique as it involves several often arbitrary choices. For example, it is hard to justify why the investors should think in terms of, and be able to specify, their utility functions. Further, optimal investment decisions of investors maximizing their expected utility typically depend on the choice of time-horizon, which is a rather arbitrary input, particularly so for a long-run investor.[1]

In this work, we explore a potential way to bridge the two opposing sides by considering an investor who is maximizing growth but is also conscious of risk. Our interpretation of "attitude towards risk" is one commonly utilized in practice: the investor has a stop-loss safety trigger to avoid large drawdowns and will effectively only invest in portfolios which never fall below a given fraction of their past maximum. This problem was first considered in a continuous-time framework by Grossman and Zhou (1993), then by Cvitanić and Karatzas (1994) and more recently by Cherny and Obłój (2011). While these contributions focus on maximizing the growth rate of expected utility, here we consider optimality via the numéraire property, meaning that expected relative returns of any other investment with respect to the optimal portfolio over the same time-period have to be non-positive. Such choices can be seen to result in an axiomatic way from numéraire-invariant preferences, as set forth in Kardaras (2010*b*). Working in a general continuous-path semimartingale setup, we establish existence of unique portfolios with the numéraire property over different time-horizons under drawdown-constrained investment, and discuss detailed structural asymptotic optimality properties, including a version of the so-called turnpike theorem.

Drawdown constraints have features appealing to various participants in financial markets and are, thus, often encountered in practice, in either explicit or implicit manner. For an investor in a fund, past performance often serves as a benchmark: a large drawdown—the difference between the past maximum and the current value—would indicate and be perceived as a worsening of performance and a loss. Accordingly, it may be used as a stop-loss trigger. This, in turn, would lead to a flight of capital from the fund, a threatening situation that should be avoided from a managerial perspective. Note that drawdown constraints may also result implicitly from the structure of hedge fund manager's incentives through the high-water mark provision—see Guasoni and Obłój (2011).

Despite their practical importance, there are relatively few theoretical studies of portfolio selection with drawdown constraints. The main obstacle is the inherent difficulty associated with the fact that the constraint is not myopic, but rather depends on the whole history of the process. Apart from the contributions mentioned above, we note that Magdon-Ismail and Atiya (2004) derived results linking the maximum drawdown to the mean return. In Chekhlov et al. (2005), the problem of maximizing expected return subject to a risk constraint expressed in terms of the drawdown was considered and solved numerically in a simple discrete time setting. Finally, in continuous-time

[1] For an attempt to circumvent the horizon-dependence, see recent developments in Musiela and Zariphopoulou (2009).

models drawdown constraints were also recently incorporated into problems of maximizing expected utility of consumption—see Elie (2008) and Elie and Touzi (2008). Options on drawdowns were also explored as instruments to hedge against portfolio losses, see Vecer (2006). Finally, maximization of growth subject to constraints arising from alternative risk measures is discussed in Pirvu and Žitković (2009).

The central building block for creating the portfolio with long-term asymptotic growth under drawdown-constrained investment is the non-constrained numéraire portfolio $\widehat{X}$, a wealth process which has the property that all other investments, in units of $\widehat{X}$, are supermartingales. It is well known that $\widehat{X}$ also maximizes the asymptotic long-term growth-rate and is exactly the investment corresponding to Kelly's criterion, see Bansal and Lehmann (1997). Some recent contributions explored the numéraire property under a constrained investment universe—in particular, Karatzas and Kardaras (2007) showed that with point-wise convex constraints on the proportions invested in each asset, one can retrieve existence and all useful properties of the numéraire portfolio. We contribute to this direction of research by providing a detailed analysis of the numéraire property within the class of investments which satisfy a given linear drawdown constraint, where wealth can never fall below a fraction $\alpha \in [0, 1)$ of its running maximum.

An important tool in our study is the Azéma–Yor transformation which provides an explicit bijection between all wealth processes and wealth processes satisfying a drawdown constraint. This transformation was established in a very general setup in Carraro et al. (2012) and used by Cherny and Obłój (2011) in a utility maximization setting. However, in a specific form, it was already used in Cvitanić and Karatzas (1994). We show here that the Azéma–Yor transform ${}^{\alpha}\widehat{X}$ of $\widehat{X}$ has the numéraire property in an asymptotic sense, as well as when sampled at times of its maximum. The portfolio ${}^{\alpha}\widehat{X}$ can be explicitly produced by investing at each time a fraction of current wealth in the fund represented by $\widehat{X}$ and the remaining fraction in the baseline asset, which is a form of a two-fund separation result. Moreover, we show that ${}^{\alpha}\widehat{X}$ is the only wealth process which enjoys the numéraire property along some increasing sequence of stopping times that tends to infinity. Further, portfolios enjoying the numéraire property for investment with long time-horizons are close (in a very strong sense) to ${}^{\alpha}\widehat{X}$ for initial times.

We stress the fact that the results presented here do not follow from previous literature because of the complex nature of the drawdown constraints. In fact, novel characteristics appear in this setting; for example, portfolios with the numéraire property are no longer myopic, and depend on the financial planning horizon. Interestingly, it is one of our main points that in an asymptotic sense the myopic structure is reinstated. Finally, we emphasize that the findings of this paper are essentially model-independent and, therefore, rather robust.

Structure of the paper. Section 1 contains a description of the financial market and introduces drawdown-constrained investments. In Section 2, the numéraire property of drawdown-constrained investments is explored. Main results are Theorem 2.4, establishing existence and uniqueness of portfolios with the numéraire property for finite time-horizons, and Theorem 2.8, which explicitly describes an investment that has the numéraire property at special stopping times where it achieves its maximum—in particular, this includes its asymptotic numéraire property. More asymptotic optimality properties of the aforementioned investment are explored in Section 3. More precisely, its asymptotic (or long-run) growth-optimality is taken up in Theorem 3.1, and an important result in the spirit of turnpike theorems is given in Theorem 3.7. In Appendix A, proofs of certain technical results are collected. Finally, in Appendix B we present an example in order to shed more light on the conclusion of the turnpike-type Theorem 3.7.

1. Market and Drawdown Constraints

1.1. **Financial market.** On a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a filtration satisfying the usual hypotheses of right-continuity and saturation by $\mathbb{P}$-null sets of $\mathcal{F}$, let $S = (S^1, \ldots, S^d)$ be a d-dimensional semimartingale with a.s. continuous paths. Each S^i, $i \in \{1, \ldots, d\}$, is modeling the random movement of an asset price in the market, discounted by a baseline asset. It is customary to assume that the baseline asset is the (domestic) savings account, but it does not necessarily have to be so. For $i \in \{1, \ldots, d\}$ write $S^i = S^i_0 + B^i + M^i$ for the Doob-Meyer decomposition of S^i into a continuous finite variation process B^i with $B^i_0 = 0$ and a local martingale M^i with $M^i_0 = 0$. For $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d\}$, $[S^i, S^j] = [M^i, M^j]$ denotes the covariation process of S^i and S^j.

Define $\mathcal{X}$ to be the class of all *nonnegative* processes X of the form

$$X = 1 + \int_0^\cdot (H_t, \mathrm{d}S_t) \equiv 1 + \int_0^\cdot \left(\sum_{i=1}^d H^i_t \mathrm{d}S^i_t \right), \tag{1.1}$$

where $H = (H^1, \ldots, H^d)$ is a d-dimensional predictable and S-integrable process. Throughout the paper, $(\cdot, \cdot)$ is used to (sometimes, formally) denote the inner product in $\mathbb{R}^d$. Note that all integrals are understood in the sense of vector stochastic integration.[2] The process X of (1.1) represents the outcome of trading according to the investment strategy H, denominated in units of the baseline asset. In the sequel we are interested in ratios of portfolios; therefore, the initial value X_0 plays no role as long as it is the same for all investment strategies. For convenience we assume $X_0 = 1$ holds for all $X \in \mathcal{X}$.

Definition 1.1. We shall say that there are opportunities for arbitrage of the first kind if there exist $T \in \mathbb{R}_+$ and an $\mathcal{F}_T$-measurable random variable ξ such that:

- $\mathbb{P}[\xi \geq 0] = 1$ and $\mathbb{P}[\xi > 0] > 0$;
- for all $x > 0$ there exists $X \in \mathcal{X}$, which may depend on x, with $\mathbb{P}[x X_T \geq \xi] = 1$.

[2]This somewhat mathematical restriction has proved essential in order to formulate nice versions of the Fundamental Theorem of Asset Pricing, as well as to ensure that optimal wealth processes exist—for example, it is crucial for the validity of Theorem 1.3 below, upon which our whole discussion depends.

The following mild and natural assumption is key to the development of the paper.

Assumption 1.2. In the market described above, the following hold:

(A1) There is no opportunity for arbitrage of the first kind.

(A2) There exists $X \in \mathcal{X}$ such that $\mathbb{P}\left[\lim_{t\to\infty} X_t = \infty\right] = 1$.

Condition (A1) in Assumption 1.2 is a minimal market viability assumption. On the other hand, condition (A2) asks for sufficient market growth in the long run. The following result follows (Kardaras, 2010*a*, Theorem 4) and contains useful equivalent conditions to the ones presented in Assumption 1.2.

Theorem 1.3. *Condition (A1) of Assumption 1.2 is equivalent to any of the following:*

(B1) *There exists $\widehat{X} \in \mathcal{X}$ such that $X/\widehat{X}$ is a (nonnegative) local martingale for all $X \in \mathcal{X}$.*

(C1) *There exists a d-dimensional process ρ such that $B^i = \int_0^\cdot \sum_{j=1}^d \rho_t^j \mathrm{d}[S^j, S^i]_t$ holds for each $i \in \{1, \ldots, d\}$. Furthermore, the nonnegative and nondecreasing process*

$$G := \frac{1}{2}\int_0^\cdot (\rho_t, \mathrm{d}[S,S]_t \rho_t) \equiv \frac{1}{2}\int_0^\cdot \sum_{i=1}^d \sum_{j=1}^d \rho_t^i \rho_t^j \mathrm{d}[S^j, S^i]_t \tag{1.2}$$

is such that $\mathbb{P}\left[G_T < \infty\right] = 1$ holds for all $T \in \mathbb{R}_+$.

Under the validity of any of (A1), (B1), (C1), and with the above notation, it holds that

$$\log(\widehat{X}) = G + L, \quad \text{where } L := \int_0^\cdot \sum_{i=1}^d \rho_t^i \mathrm{d}M_t^i \tag{1.3}$$

Furthermore, under the validity of any of the equivalent (A1), (B1), (C1), condition (A2) of Assumption 1.2 is equivalent to any of the following:

(B2) $\mathbb{P}\big[\lim_{t\to\infty} \widehat{X}_t = \infty\big] = 1$.

(C2) $\mathbb{P}\big[G_\infty = \infty\big] = 1$, *where* $G_\infty := \uparrow \lim_{t\to\infty} G_t$.

Proof. The fact that the three conditions (A1), (B1) and (C1) are equivalent, as well as the validity of (1.3), can be found in (Kardaras, 2010*a*, Theorem 4). Now, assume any of the equivalent conditions (A1), (B1) or (C1). Clearly, (B2) implies (A2). On the other hand, suppose that there exists $X \in \mathcal{X}$ such that $\mathbb{P}\left[\lim_{t\to\infty} X_t = \infty\right] = 1$. The nonnegative supermartingale theorem implies that $\lim_{t\to\infty}\big(X_t/\widehat{X}_t\big)$ $\mathbb{P}$-a.s. exists in $\mathbb{R}_+$, which implies that $\mathbb{P}\big[\lim_{t\to\infty} \widehat{X}_t = \infty\big] = 1$ holds as well. Therefore, (A2) implies (B2). Continuing, note that (1.3) implies that

$$[L, L] = \int_0^\cdot (\rho_t, \mathrm{d}[M,M]_t \rho_t) = \int_0^\cdot (\rho_t, \mathrm{d}[S,S]_t \rho_t) = 2G.$$

In view of the celebrated result of Dambis, Dubins and Schwarz (Karatzas and Shreve, 1991, Theorem 3.4.6), there exists a standard Brownian motion β (in a potentially enlarged probability space, and the Brownian motion property of β is with respect to its own natural filtration) such that $L_t = \beta_{2G_t}$ holds for $t \in \mathbb{R}_+$. It follows that $\log(\widehat{X}_t) = G_t + \beta_{2G_t}$ holds for $t \in \mathbb{R}_+$. Therefore, on $\{G_\infty < \infty\}$, $\lim_{t\to\infty} \widehat{X}_t$ a.s. exists and is $\mathbb{R}_+$-valued. On the other hand, the strong law of large numbers for Brownian motion implies that on $\{G_\infty = \infty\}$, $\lim_{t\to\infty} \big(\log\big(\widehat{X}_t\big)/G_t\big) = 1$ a.s. holds, which in turn implies that $\lim_{t\to\infty} \widehat{X}_t = \infty$ a.s. holds. The previous facts imply the a.s. set-equality $\{G_\infty = \infty\} = \{\lim_{t\to\infty} \widehat{X}_t = \infty\}$, which establishes the equivalence of conditions (B2) and (C2) and completes the proof. □

Remark 1.4. In Itô process models, it holds that $B^i = \int_0^\cdot S_t^i b_t^i \mathrm{d}t$ and $M^i = \int_0^\cdot S_t^i \sum_{j=1}^m \sigma_t^{ij} \mathrm{d}W_t^j$ for $i \in \{1,\dots,d\}$, where $b = (b^1,\dots,b^d)$ is the predictable d-dimensional vector of excess rates of return, $(W^1,\dots,W^m)$ is an m-dimensional standard Brownian motion, and we write $c = \sigma\sigma^\top$ for the predictable $d\times d$ matrix-valued process of local covariances. According to Theorem 1.3, condition (A1) of Assumption 1.2 is equivalent to the fact that there exists a d-dimensional process ρ such that $c\rho = b$, in which case we write $\rho = c^\dagger b$ where $c^\dagger$ is the Moore-Penrose pseudo-inverse of c, and that $G := (1/2)\int_0^\cdot (b_t, c_t^\dagger b_t)\mathrm{d}t = (1/2)\int_0^\cdot (\rho_t, c_t\rho_t)\mathrm{d}t$ is an a.s. finitely-valued process. Observe that the process G is half of the integrated squared risk-premium in the market.

Remark 1.5. If the process $\widehat{X}$ in (B1) exists, then it is unique and is said to have the *numéraire* property. It is well known that it solves the log-utility maximization problem on any finite time horizon, and that it achieves optimal asymptotic growth. We shall revisit these properties in a more general setting—see Remark 2.5 and Theorem 3.1.

1.2. **Drawdown constraints.** To each wealth process $X \in \mathcal{X}$, we associate its *running maximum* process X^* defined via $X_t^* := \sup_{u\in[0,t]} X_u$ for $t \in \mathbb{R}_+$. The difference $X^* - X$ between the current wealth and its past maximum is called the *drawdown process*. As we argued in the Introduction, different participants in financial markets may be interested to restrict the universe of their strategies to the ones which do not permit for large drawdowns.

For any $\alpha \in [0,1)$, we write ${}^\alpha\mathcal{X}$ for the class of wealth processes $X \in \mathcal{X}$ such that $X^* - X \le (1-\alpha)X^*$. The $[0,1]$-valued process X/X^* is called the *relative drawdown* process associated to X. Note that $X \in {}^\alpha\mathcal{X}$ if and only if $X/X^* \ge \alpha$ holds identically. It is clear that ${}^\beta\mathcal{X} \subseteq {}^\alpha\mathcal{X}$ for $0 \le \alpha \le \beta < 1$, and that ${}^0\mathcal{X} = \mathcal{X}$. Note that if $X \in \mathcal{X}$ satisfies $X \ge \alpha X^*$ on the stochastic interval $[\![0,T]\!]$, then $(X_{T\wedge t})_{t\in\mathbb{R}_+} \in {}^\alpha\mathcal{X}$; therefore, it is appropriate to use ${}^\alpha\mathcal{X}$ as the set of wealth processes regardless of the investment horizon.

Interestingly, there is a one-to-one correspondence between wealth processes in $\mathcal{X}$ and wealth processes in ${}^\alpha\mathcal{X}$ for any $\alpha \in [0,1)$. The bijection was derived explicitly in terms of the so–called Azéma–Yor processes in (Carraro et al., 2012, Theorem 3.4) and recently exploited in Cherny and Obłój (2011), in a general setting of possibly non-linear drawdown constraints. This elegant machinery simplifies greatly in the case of "linear" drawdown constraints considered here, and we provide explicit arguments, similarly to the pioneering work of Cvitanić and Karatzas (1994). We first discuss how processes in $\mathcal{X}$ generate processes in ${}^\alpha\mathcal{X}$—the converse will be established in the proof of Proposition 1.6 below. For $X \in \mathcal{X}$ and $\alpha \in [0,1)$, define a process ${}^{\alpha}X$ via[3]

$$ {}^{\alpha}X := \alpha(X^*)^{1-\alpha} + (1-\alpha)X(X^*)^{-\alpha}. \tag{1.4} $$

Using the fact that $\int_0^\infty \mathbb{I}_{\{X_t < X_t^*\}} \mathrm{d}X_t^* = 0$ $\mathbb{P}$-a.s. holds, an application of Itô's formula gives

$$ {}^{\alpha}X = 1 + \int_0^{\cdot} (1-\alpha)(X_t^*)^{-\alpha} \mathrm{d}X_t, \tag{1.5} $$

which implies that ${}^{\alpha}X \in \mathcal{X}$. Furthermore, (1.4) gives $\alpha(X^*)^{1-\alpha} \le {}^{\alpha}X \le (X^*)^{1-\alpha}$. Note also that times of maximum of X coincide with times of maximum of ${}^{\alpha}X$ and consequently ${}^{\alpha}X^* = (X^*)^{1-\alpha}$. It follows that

$$ \frac{{}^{\alpha}X}{{}^{\alpha}X^*} = \frac{\alpha(X^*)^{1-\alpha} + (1-\alpha)X(X^*)^{-\alpha}}{(X^*)^{1-\alpha}} = \alpha + (1-\alpha)\frac{X}{X^*} \ge \alpha, \tag{1.6} $$

implying ${}^{\alpha}X \in {}^{\alpha}\mathcal{X}$.

The converse construction is presented in Proposition 1.6 below. Together with (1.4) they provide an extremely convenient representation of the class ${}^{\alpha}\mathcal{X}$ for $\alpha \in [0,1)$.

Proposition 1.6 (Proposition 2.2 of Carraro et al. (2012))**.** *It holds that* ${}^{\alpha}\mathcal{X} = \{{}^{\alpha}X \mid X \in \mathcal{X}\}$.

Proof. Since $\{{}^{\alpha}X \mid X \in \mathcal{X}\} \subseteq {}^{\alpha}\mathcal{X}$ has already been established, we only need to show that ${}^{\alpha}\mathcal{X} \subseteq \{{}^{\alpha}X \mid X \in \mathcal{X}\}$ also holds. Pick any $\chi \in {}^{\alpha}\mathcal{X}$ and define

$$ X := \frac{1}{1-\alpha}(\chi^*)^{\alpha/(1-\alpha)}\chi - \frac{\alpha}{1-\alpha}(\chi^*)^{1/(1-\alpha)} = \frac{1}{1-\alpha}(\chi^*)^{\alpha/(1-\alpha)}\left(\chi - \alpha\chi^*\right). $$

The fact that $\chi/\chi^* \ge \alpha$ implies that $X \ge 0$. Furthermore, and since $\int_0^\infty \mathbb{I}_{\{\chi_t < \chi_t^*\}} \mathrm{d}\chi_t^* = 0$ $\mathbb{P}$-a.s. holds, a use of Itô's formula gives

$$ X = 1 + \int_0^{\cdot} \frac{1}{1-\alpha}(\chi_t^*)^{\alpha/(1-\alpha)} \mathrm{d}\chi_t, $$

which implies that $X \in \mathcal{X}$. Finally using the fact that χ and X have the same times of maximum—which implies, in particular, that $\chi^* = (X^*)^{1-\alpha}$—it is straightforward to check that $\chi = {}^{\alpha}X$. Therefore, ${}^{\alpha}\mathcal{X} \subseteq \{{}^{\alpha}X \mid X \in \mathcal{X}\}$ and the proof of Proposition 1.6 is complete. □

Remark 1.7. One can rewrite equation (1.5) in differential terms as

$$ \frac{\mathrm{d}{}^{\alpha}X_t}{{}^{\alpha}X_t} = \left(\frac{(1-\alpha)(X_t^*)^{-\alpha}X_t}{{}^{\alpha}X_t}\right)\frac{\mathrm{d}X_t}{X_t} = \frac{{}^{\alpha}X_t - \alpha{}^{\alpha}X_t^*}{{}^{\alpha}X_t}\frac{\mathrm{d}X_t}{X_t}, \quad \text{for } t < \inf\{u \in \mathbb{R}_+ \mid X_u = 0\}. $$

The above equation carries an important message: for $X \in \mathcal{X}$, the way that ${}^{\alpha}X$ is built is via investing a proportion

$$ {}^{\alpha}\pi^X := \frac{{}^{\alpha}X - \alpha{}^{\alpha}X^*}{{}^{\alpha}X} = 1 - \frac{\alpha}{{}^{\alpha}X/{}^{\alpha}X^*} = \frac{(1-\alpha)(X/X^*)}{\alpha + (1-\alpha)(X/X^*)} $$

[3] In the notation of Carraro et al. (2012), we have ${}^{\alpha}X = M^{F_\alpha}(X)$ with $F_\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined via $F_\alpha(x) = x^{1-\alpha}$ for $x \in \mathbb{R}_+$; furthermore, Proposition 2.2 therein implies that $X = M^{G_\alpha}({}^{\alpha}X)$ with $G_\alpha = F_\alpha^{-1}$. This last converse construction is presented explicitly in Proposition 1.6.

in the fund represented by X, and the remaining proportion $1 - {}^{\alpha}\pi^X$ in the baseline asset. In particular, when the baseline asset is the domestic savings account, it follows that the Sharpe ratio of X and ${}^{\alpha}X$ is the same. Note that $0 \leq {}^{\alpha}\pi^X \leq 1 - \alpha$ (so that $\alpha \leq 1 - {}^{\alpha}\pi^X \leq 1$). Furthermore, ${}^{\alpha}\pi^X$ depends only on $\alpha \in [0, 1)$ and the relative drawdown X/X^* of X. In fact, the proportion ${}^{\alpha}\pi^X$ invested in the underlying fund represented by X is an increasing function of the relative drawdown X/X^*.

Recall the process $\widehat{X}$ in (B1) in Theorem 1.3. When the above discussion is applied to ${}^{\alpha}\widehat{X}$, defined from $\widehat{X}$ via (1.4), it follows from (Platen and Heath, 2006, Theorem 11.1.3 and Corollary 11.1.4) that ${}^{\alpha}\widehat{X}$ is a locally optimal portfolio, in the sense that it locally maximizes the excess return over all investments with the same volatility. In view of (1.3), the wealth process $\widehat{X}$ is given explicitly in terms of the drift and quadratic covariation process of the multi-dimensional asset-price process. It follows that ${}^{\alpha}\widehat{X}$ for $\alpha \in [0, 1)$ is explicitly specified as well.

Even though the numéraire portfolio $\widehat{X}$ has optimal growth in an asymptotic sense (in this respect, see also Theorem 3.1 later in the text), it is a quite risky investment. In fact, it experiences arbitrarily large flights of capital, as its relative drawdown process $\widehat{X}/\widehat{X}^*$ will become arbitrarily close to zero infinitely often. This is in fact equivalent to the following, seemingly more general statement, showing an oscillatory behavior of the relative drawdown for all wealth processes ${}^{\alpha}\widehat{X}$, $\alpha \in [0, 1)$.

Proposition 1.8. *Under Assumption 1.2, it holds that*

$$\alpha = \liminf_{t \to \infty} \left(\frac{{}^{\alpha}\widehat{X}_t}{{}^{\alpha}\widehat{X}^*_t} \right) < \limsup_{t \to \infty} \left(\frac{{}^{\alpha}\widehat{X}_t}{{}^{\alpha}\widehat{X}^*_t} \right) = 1, \ \ a.s. \quad \forall \alpha \in [0, 1).$$

The proof of Proposition 1.8 is given in Subsection A.1 of Appendix A.

2. The Numéraire Property

2.1. **Expected relative return.** Fix a stopping time T and $X, X' \in \mathcal{X}$, and define the *return of X relative to X' over the period* $[0, T]$ via

$$\mathsf{rr}_T(X|X') \ := \ \limsup_{t \to \infty} \left(\frac{X_{T \wedge t} - X'_{T \wedge t}}{X'_{T \wedge t}} \right) = \limsup_{t \to \infty} \left(\frac{X_{T \wedge t}}{X'_{T \wedge t}} \right) - 1.$$

(The convention $0/0 = 1$ is used throughout.) In other words, $\mathsf{rr}_T(X|X') = (X_T - X'_T)/X'_T$ holds on the event $\{T < \infty\}$, while $\mathsf{rr}_T(X|X') = \limsup_{t \to \infty} ((X_t - X'_t)/X'_t) = \mathsf{rr}_\infty(X|X')$ holds on the event $\{T = \infty\}$. The above definition conveniently covers both cases. Observe that $\mathsf{rr}_T(X|X')$ is a $[-1, \infty]$-valued random variable. Therefore, for any stopping time T and $X, X' \in \mathcal{X}$, the quantity

$$\mathbb{E}\mathsf{rr}_T(X|X') \ := \ \mathbb{E}\left[\mathsf{rr}_T(X|X')\right]$$

is well defined and $[-1, \infty]$-valued. $\mathbb{E}\mathsf{rr}_T(X|X')$ represents the *expected return of X relative to X' over the time-period* $[0, T]$.

The concept of expected relative returns is introduced for purposes of portfolio selection. A first idea that comes to mind is to proclaim that $X' \in \mathcal{X}$ is "strictly better" than $X \in \mathcal{X}$ for investment over the period $[0, T]$ if $\mathbb{E}\mathsf{rr}_T(X'|X) > 0$. However, this is not an appropriate notion: it is easy to construct examples where both $\mathbb{E}\mathsf{rr}_T(X'|X) > 0$ and $\mathbb{E}\mathsf{rr}_T(X|X') > 0$ hold. The reason is that, in general, $\mathsf{rr}_T(X|X') \neq -\mathsf{rr}_T(X'|X)$. In fact, Proposition 2.3 below implies that $\mathsf{rr}_T(X|X') \geq -\mathsf{rr}_T(X'|X)$, with equality holding only on the event $\{\lim_{t\to\infty}(X_{T\wedge t}/X'_{T\wedge t}) = 1\}$. A more appropriate definition would call $X' \in \mathcal{X}$ "strictly better" than $X \in \mathcal{X}$ for investment over the period $[0, T]$ if *both* $\mathbb{E}\mathsf{rr}_T(X'|X) > 0$ and $\mathbb{E}\mathsf{rr}_T(X|X') \leq 0$ hold. In fact, because of the inequality $\mathsf{rr}_T(X|X') \geq -\mathsf{rr}_T(X'|X)$, $\mathbb{E}\mathsf{rr}_T(X|X') \leq 0$ is enough to imply $\mathbb{E}\mathsf{rr}_T(X'|X) \geq 0$, and one has $\mathbb{E}\mathsf{rr}_T(X'|X) > 0$ in the case where $\mathbb{P}\left[\lim_{t\to\infty}(X_{T\wedge t}/X'_{T\wedge t}) = 1\right] < 1$.

The discussion of the previous paragraph can be summarized as follows: while positive expected returns of $X \in \mathcal{X}$ with respect to $X' \in \mathcal{X}$ do not imply that X is a better investment than X', we may regard non-positive expected returns of $X \in \mathcal{X}$ with respect to $X' \in \mathcal{X}$ to indicate that X' is a better investment than X. Given the use of "lim sup" in the equality $\mathsf{rr}_T(X|X') = \limsup_{t\to\infty}((X_t - X'_t)/X'_t)$, valid on $\{T = \infty\}$, it seems particularly justified to regard X' as better than X when $\mathbb{E}\mathsf{rr}_\infty(X|X') \leq 0$ holds, at least in an asymptotic sense. We are led to the following concept.

Definition 2.1. We say that X' has the *numéraire property* in a certain class of wealth processes for investment over the period $[0, T]$ if and only if $\mathbb{E}\mathsf{rr}_T(X|X') \leq 0$ holds for all other X in the same class.

Remark 2.2. In view of results pertaining to the non-constrained case, one may be tempted to define the numéraire portfolio in a certain class of wealth processes by postulating that all other wealth processes in this class are supermartingales in units of the numéraire. However, in the context of drawdown constraints this would be a void concept as portfolios with the numéraire property may depend on the planning horizon—see Example 2.6.

The next result contains some useful information regarding (expected) relative returns. In particular, it implies that the terminal value of an investment with the numéraire property within a certain class of processes for investment over a specified period of time is essentially unique.

Proposition 2.3. *For any stopping time T, any $X \in \mathcal{X}$ and any $X' \in \mathcal{X}$, it holds that*

$$\mathsf{rr}_T(X'|X) \geq -\frac{\mathsf{rr}_T(X|X')}{1 + \mathsf{rr}_T(X|X')} \geq -\mathsf{rr}_T(X|X'),$$

with equality on $\{T < \infty\}$. Furthermore, the following equivalence is valid:

$$\mathbb{E}\mathsf{rr}_T(X'|X) \leq 0 \text{ and } \mathbb{E}\mathsf{rr}_T(X|X') \leq 0 \iff \mathbb{P}\left[\lim_{t\to\infty}\left(\frac{X_{T\wedge t}}{X'_{T\wedge t}}\right) = 1\right] = 1.$$

Proof. To begin with, note that

$$1 + \mathsf{rr}_T(X|X') = \limsup_{t\to\infty} \left(\frac{X_{T\wedge t}}{X'_{T\wedge t}}\right) \geq \left(\limsup_{t\to\infty} \left(\frac{X'_{T\wedge t}}{X_{T\wedge t}}\right)\right)^{-1} = \frac{1}{1 + \mathsf{rr}_T(X'|X)},$$

with equality holding on $\{T < \infty\}$. Continuing, we obtain

$$\mathsf{rr}_T(X|X') + \mathsf{rr}_T(X'|X) \geq \frac{1}{1 + \mathsf{rr}_T(X'|X)} - 1 + \mathsf{rr}_T(X'|X) = \frac{\mathsf{rr}_T(X'|X)^2}{1 + \mathsf{rr}_T(X'|X)}.$$

Upon interchanging the roles of X and X', we also obtain the corresponding inequality $\mathsf{rr}_T(X|X') + \mathsf{rr}_T(X'|X) \geq \mathsf{rr}_T(X|X')^2 / (1 + \mathsf{rr}_T(X|X'))$; therefore,

$$\mathsf{rr}_T(X|X') + \mathsf{rr}_T(X'|X) \geq \frac{\mathsf{rr}_T(X'|X)^2}{1 + \mathsf{rr}_T(X'|X)} \vee \frac{\mathsf{rr}_T(X|X')^2}{1 + \mathsf{rr}_T(X|X')}. \tag{2.1}$$

It immediately follows that $\mathsf{rr}_T(X'|X) + \mathsf{rr}_T(X|X') \geq 0$. Therefore, by (2.1), the conditions $\mathbb{E}\mathsf{rr}_T(X'|X) \leq 0$ and $\mathbb{E}\mathsf{rr}_T(X|X') \leq 0$ are equivalent to $\mathbb{P}\left[\mathsf{rr}_T(X|X') = 0 = \mathsf{rr}_T(X'|X)\right] = 1$, which is in turn equivalent to $\mathbb{P}\left[\lim_{t\to\infty} \left(X_{T\wedge t}/X'_{T\wedge t}\right) = 1\right] = 1$. □

By Proposition 2.3, if $\mathbb{E}\mathsf{rr}_T(X|X') \leq 0$ and $\mathbb{E}\mathsf{rr}_T(X'|X) \leq 0$ both hold, then $X_T = X'_T$ a.s. on $\{T < \infty\}$, while $\lim_{t\to\infty}(X_t/X'_t) = 1$ a.s. on $\{T = \infty\}$, the latter being a version of "asymptotic equivalence" between X and X'.

The next result establishes existence of a process with the numéraire property in the class ${}^\alpha\mathcal{X}$ sampled at T for all $\alpha \in [0, 1)$ and finite time-horizon T, and shows that such process is uniquely defined on the stochastic interval $[\![0, T]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid 0 \leq t \leq T(\omega)\}$. (Note that the latter uniqueness property is stronger than plain uniqueness of the terminal value of processes with the numéraire property that is guaranteed by Proposition 2.3.) Theorem 2.8 later will address the possibility of infinite time-horizon.

Theorem 2.4. *Let T be a stopping time with $\mathbb{P}[T < \infty] = 1$. Under condition (A1) of Assumption 1.2, there exists $\widetilde{X} \in {}^\alpha\mathcal{X}$, which may depend on T, such that $\mathbb{E}\mathsf{rr}_T(X|\widetilde{X}) \leq 0$ holds for all $X \in {}^\alpha\mathcal{X}$. Furthermore, $\widetilde{X}$ has the following uniqueness property: for any other process $\widetilde{Z} \in {}^\alpha\mathcal{X}$ such that $\mathbb{E}\mathsf{rr}_T(X|\widetilde{Z}) \leq 0$ holds for all $X \in {}^\alpha\mathcal{X}$, $\widetilde{X} = \widetilde{Z}$ holds a.s. on $[\![0, T]\!]$.*

The proof of Theorem 2.4 is given at Subsection A.2 of Appendix A.

Remark 2.5. In the notation of Theorem 2.4, the log-utility maximization problem at time T is solved by the wealth process ${}^\alpha\widetilde{X}$. Indeed, the inequality $\log(x) \leq x - 1$, valid for all $x \in \mathbb{R}_+$, gives

$$\mathbb{E}\left[\log\left(\frac{X_T}{\widetilde{X}_T}\right)\right] \leq \mathbb{E}\left[\frac{X_T}{\widetilde{X}_T} - 1\right] = \mathbb{E}\mathsf{rr}_T(X|\widetilde{X}) \leq 0,$$

for all $X \in {}^\alpha\mathcal{X}$. This is a version of *relative* expected log-optimality, which turns to actual expected log-optimality as soon as the expected log-maximization problem is well-posed—in this respect, see also (Karatzas and Kardaras, 2007, Subsection 3.7).

In view of Theorem 2.4, the above discussion ensures existence and uniqueness of expected log-utility optimal wealth processes for finite time-horizons in a drawdown-constrained investment framework. To the best of the authors' knowledge, results regarding existence of optimal processes for utility maximization problems involving finite time-horizon and drawdown constraints are absent from the literature.

2.2. **The numéraire property at times of maximum of $\widehat{X}$.** When $\alpha = 0$, the fact that $X/\widehat{X}$ is a nonnegative supermartingale and the optional sampling theorem imply that $\mathbb{E}\mathsf{rr}_T(X|\widehat{X}) \leq 0$ holds for all stopping times T and all $X \in \mathcal{X}$. Therefore, the process $\widehat{X}$ has a "global" (in time) numéraire property. Furthermore, the supermartingale convergence theorem implies that $\lim_{t\to\infty}(X_t/\widehat{X}_t)$ $\mathbb{P}$-a.s. exists for all $X \in \mathcal{X}$; therefore,

$$\mathsf{rr}_\infty(X|\widehat{X}) = \lim_{t\to\infty}\left(\frac{X_t - \widehat{X}_t}{\widehat{X}_t}\right) = \lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right) - 1. \tag{2.2}$$

For finite time-horizons, the situation is more complicated for $\alpha \in (0,1)$. In Theorem 2.8, we shall see that ${}^\alpha\widehat{X}$ has the numéraire property in ${}^\alpha\mathcal{X}$ for certain stopping times (which include the asymptotic case $T = \infty$). However, ${}^\alpha\widehat{X}$ does not have the numéraire property for all finite time-horizons, as the Example 2.6 shows. This fact motivates the statement of Theorem 2.4, where it is hinted that portfolios with the numéraire property may depend on the time-horizon—see Remark 2.2.

Example 2.6. Fix $\alpha \in (0,1)$. Define $T := \inf\{t \in (0,\infty) \mid \widehat{X}_t/\widehat{X}^*_t = \alpha\}$ and observe that Proposition 1.8 implies that $\mathbb{P}[T < \infty] = 1$ holds. With $\widehat{X}^T$ denoting the process $\widehat{X}$ stopped at T, we have $\widehat{X}^T \in {}^\alpha\mathcal{X}$. The numéraire property of $\widehat{X}$ in $\mathcal{X} \supseteq {}^\alpha\mathcal{X}$ implies that $\mathbb{E}\mathsf{rr}_T(X|\widehat{X}^T) \leq 0$ for all $X \in {}^\alpha\mathcal{X}$, resulting in the numéraire property of $\widehat{X}^T$ in ${}^\alpha\mathcal{X}$ over the investment period $[0,T]$. Since $\mathbb{P}[{}^\alpha\widehat{X}_T = \widehat{X}_T] = 0$, it follows that ${}^\alpha\widehat{X}_T$ fails to have the numéraire property in ${}^\alpha\mathcal{X}$ over the investment period $[0,T]$.

Before abandoning this example, note that if one follows the non-constrained numéraire portfolio $\widehat{X}$ up to T, the drawdown constraints will mean that one has to invest all capital in the baseline account from time T onwards. It is clear that this strategy will not be long-run optimal.

We continue with a definition of a class of stopping times which will be important in the sequel.

Definition 2.7. A stopping time τ will be called a *time of maximum of* $\widehat{X}$ if $\widehat{X}_\tau = \widehat{X}^*_\tau$ holds a.s. on the event $\{\tau < \infty\}$.

A couple of remarks are in order. Firstly, from (1.4) one can immediately see that times of maximum of $\widehat{X}$ are also times of maximum of ${}^\alpha\widehat{X}$ for all $\alpha \in [0,1)$. Secondly, the restriction in the definition of a time τ of maximum of $\widehat{X}$ is only enforced on $\{\tau < \infty\}$. Under Assumption 1.2, and in view of Theorem 1.3, one has $\widehat{X}_\tau = \widehat{X}^*_\tau = \infty$ holding a.s. on $\{\tau = \infty\}$. For this reason, $\tau = \infty$ is an important special case of a time of maximum of $\widehat{X}$.

The following theorem, the second main result of this section, establishes in particular the *numéraire property* of ${}^{\alpha}\widehat{X}$ in ${}^{\alpha}\mathcal{X}$ over $[0, \infty]$ or, more generally, over $[0, \tau]$ for any time τ of maximum of $\widehat{X}$.

Theorem 2.8. *Recall that* ${}^{\alpha}\widehat{X} \in {}^{\alpha}\mathcal{X}$ *is defined from* $\widehat{X}$ *via* (1.4). *Under Assumption 1.2, for any* $\alpha \in [0, 1)$ *and* $X \in \mathcal{X}$*, we have:*

(1) $\lim_{t\to\infty}({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t)$ *a.s. exists in* $\mathbb{R}_+$*. Moreover,*

$$\mathsf{rr}_{\infty}({}^{\alpha}X|{}^{\alpha}\widehat{X}) = \left(\lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right)\right)^{1-\alpha} - 1 = \left(1 + \mathsf{rr}_{\infty}(X|\widehat{X})\right)^{1-\alpha} - 1. \tag{2.3}$$

(2) *for* σ *and* τ *two times of maximum of* $\widehat{X}$ *with* $\sigma \le \tau$ *we have*

$$\mathbb{E}\left[\mathsf{rr}_{\tau}({}^{\alpha}X|{}^{\alpha}\widehat{X}) \mid \mathcal{F}_{\sigma}\right] \le \mathsf{rr}_{\sigma}({}^{\alpha}X|{}^{\alpha}\widehat{X}) \quad a.s. \tag{2.4}$$

In particular, letting $\sigma = 0$, $\mathbb{E}\mathsf{rr}_{\tau}(Z|{}^{\alpha}\widehat{X}) \le 0$ *holds for any* $\alpha \in [0, 1)$ *and* $Z \in {}^{\alpha}\mathcal{X}$.

We proceed with several remarks on the implications of Theorem 2.8, the proof of which is given in Subsection A.3 of Appendix A.

Remark 2.9. The existence of the limit in (2.2) is guaranteed by the nonnegative supermartingale convergence theorem. In contrast, proving that $\lim_{t\to\infty}({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t)$ exists a.s. for $X \in \mathcal{X}$ and $\alpha \in (0, 1)$ is more involved, since, in general, the process ${}^{\alpha}X/{}^{\alpha}\widehat{X}$ does not have the supermartingale property. In fact, the existence of the latter limit is proved together with the asymptotic relationship (2.3). Note, however, that an analogue of the supermartingale property is provided by statement (2) of Theorem 2.8. Indeed, (2.4) implies that, when sampled at an increasing sequence of times of maximum of $\widehat{X}$, the process ${}^{\alpha}X/{}^{\alpha}\widehat{X}$ is a supermartingale for all $X \in \mathcal{X}$.

Remark 2.10. Given statement (1) of Theorem 2.8, the fact that $\mathbb{E}\mathsf{rr}_{\infty}({}^{\alpha}X|{}^{\alpha}\widehat{X}) \le 0$ holds for any $X \in \mathcal{X}$ and $\alpha \in [0, 1)$ is a simple consequence of Jensen's inequality. Indeed, for any $X \in \mathcal{X}$,

$$\begin{aligned}
\mathbb{E}\mathsf{rr}_{\infty}({}^{\alpha}X|{}^{\alpha}\widehat{X}) &= \mathbb{E}\left[\left(1 + \mathsf{rr}_{\infty}(X|\widehat{X})\right)^{1-\alpha}\right] - 1 \\
&\le \left(\mathbb{E}\left[1 + \mathsf{rr}_{\infty}(X|\widehat{X})\right]\right)^{1-\alpha} - 1 \\
&= \left(1 + \mathbb{E}\mathsf{rr}_{\infty}(X|\widehat{X})\right)^{1-\alpha} - 1 \ \le \ 0.
\end{aligned}$$

The full proof of statement (2) of Theorem 2.8 is more involved.

Remark 2.11. The fact that $\mathsf{rr}_{\infty}({}^{\alpha}X|{}^{\alpha}\widehat{X}) = \left(1 + \mathsf{rr}_{\infty}(X|\widehat{X})\right)^{1-\alpha} - 1$ holds for all $\alpha \in [0, 1)$ can be easily seen to imply that $\left|\mathsf{rr}_{\infty}({}^{\beta}X|{}^{\beta}\widehat{X})\right| \le \left|\mathsf{rr}_{\infty}({}^{\alpha}X|{}^{\alpha}\widehat{X})\right|$ holds whenever $0 \le \alpha \le \beta < 1$. In other words, using the same generating wealth process X and enforcing harsher drawdown constraints reduces the (asymptotic) difference in the performance of the drawdown-constrained process ${}^{\alpha}X$ against the long-run optimum ${}^{\alpha}\widehat{X}$.

Remark 2.12. Let us consider the hitting times of $\widehat{X}$, parameterized on the logarithmic scale:

$$\tau_\ell := \inf\left\{t \in \mathbb{R}_+ \mid \widehat{X}_t = \exp(\ell)\right\}, \quad \ell \in \mathbb{R}_+. \tag{2.5}$$

Note that τ_ℓ is a time of maximum of $\widehat{X}$. Since times of maximum of $\widehat{X}$ coincide with times of maximum of ${}^\alpha\widehat{X}$ for $\alpha \in [0,1)$, $\tau_\ell = \inf\left\{t \in \mathbb{R}_+ \mid {}^\alpha\widehat{X}_t = \exp((1-\alpha)\ell)\right\}$ holds for all for $\alpha \in [0,1)$. According to Assumption 1.2, $\mathbb{P}[\tau_\ell < \infty] = 1$ holds for all $\ell \in \mathbb{R}_+$.

By Remark 2.5, the log-utility maximization problem at time τ_ℓ for the class ${}^\alpha\mathcal{X}$ is solved by the wealth process ${}^\alpha\widehat{X}$. Moreover, assume that $U : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ is any increasing and concave function such that $U(x) > -\infty$ for all $x \in (0, \infty)$. Jensen's inequality implies that

$$\mathbb{E}\left[U({}^\alpha X_{\tau_\ell})\right] \leq U(\mathbb{E}\left[{}^\alpha X_{\tau_\ell}\right]) \leq U(\exp((1-\alpha)\ell)) = \mathbb{E}\left[U({}^\alpha\widehat{X}_{\tau_\ell})\right], \quad \text{for all } X \in \mathcal{X}.$$

It follows that *any* (and not only the logarithmic) utility maximization problem at time τ_ℓ for the class ${}^\alpha\mathcal{X}$ is solved by the wealth process ${}^\alpha\widehat{X}$.

3. More on Asymptotic Optimality

3.1. **Maximization of long-term growth.** The next theorem is concerned with the asymptotic growth-optimality property of ${}^\alpha\widehat{X}$ in ${}^\alpha\mathcal{X}$ for $\alpha \in [0,1)$. It extends the result of (Cvitanić and Karatzas, 1994, Section 7) to a more general setting and with a simpler proof. In the subsequent subsection we continue with a considerably finer analysis relating the finite-time and asymptotic optimality of ${}^\alpha\widehat{X}$ in ${}^\alpha\mathcal{X}$.

Theorem 3.1. *Recall the market-growth process G from* (1.2). *Under Assumption 1.2, for any $X \in {}^\alpha X$ we a.s. have that*

$$\limsup_{t\to\infty}\left(\frac{1}{G_t}\log(X_t)\right) \leq 1-\alpha = \lim_{t\to\infty}\left(\frac{1}{G_t}\log\left({}^\alpha\widehat{X}_t\right)\right). \tag{3.1}$$

Proof. In the proof of Theorem 1.3 it was established that $\lim_{t\to\infty}\left(\log\left(\widehat{X}_t\right)/G_t\right) = 1$ holds on $\{G_\infty = \infty\}$. In view of Theorem 1.3, condition (A2) of Assumption 1.2 is equivalent to $\mathbb{P}[G_\infty = \infty] = 1$. Therefore, a.s.,

$$\lim_{t\to\infty}\left(\frac{1}{G_t}\log(\widehat{X}_t)\right) = 1.$$

Observe that by concavity of the function $\mathbb{R}_+ \ni x \mapsto x^{1-\alpha}$, ${}^\alpha\widehat{X} \geq \widehat{X}^{1-\alpha}$ holds. Combining this with the above yields, a.s.,

$$\liminf_{t\to\infty}\left(\frac{1}{G_t}\log({}^\alpha\widehat{X}_t)\right) \geq 1-\alpha.$$

On the other hand, since G is nondecreasing and ${}^{\alpha}\widehat{X}$ achieves maximum values at the times $(\tau_\ell)_{\ell \in \mathbb{R}_+}$ of (2.5), it holds a.s. that

$$\begin{aligned}\limsup_{t\to\infty}\left(\frac{1}{G_t}\log({}^{\alpha}\widehat{X}_t)\right) &= \limsup_{\ell\to\infty}\left(\frac{1}{G_{\tau_\ell}}\log({}^{\alpha}\widehat{X}_{\tau_\ell})\right)\\ &= (1-\alpha)\limsup_{\ell\to\infty}\left(\frac{\ell}{G_{\tau_\ell}}\right)\\ &= (1-\alpha)\limsup_{\ell\to\infty}\left(\frac{1}{G_{\tau_\ell}}\log(\widehat{X}_{\tau_\ell})\right) = 1-\alpha.\end{aligned}$$

It follows that, a.s.,

$$\lim_{t\to\infty}\left(\frac{1}{G_t}\log({}^{\alpha}\widehat{X}_t)\right) = 1-\alpha.$$

Fix $X \in {}^{\alpha}\mathcal{X}$. The full result of Theorem 3.1 now follows immediately upon noticing that, a.s.,

$$\limsup_{t\to\infty}\left(\frac{1}{G_t}\log\left(\frac{X_t}{{}^{\alpha}\widehat{X}_t}\right)\right) \le 0,$$

which is valid in view of the facts that $\mathbb{P}[G_\infty = \infty] = 1$ and $\mathbb{P}\big[\mathsf{rr}_\infty(X | {}^{\alpha}\widehat{X}) < \infty\big] = 1$, the latter following from the inequality $\mathbb{E}\mathsf{rr}_\infty(X | {}^{\alpha}\widehat{X}) \le 0$, which was established in Theorem 2.8. □

Remark 3.2. Fix $\alpha \in [0,1)$. In the setting of Theorem 3.1, any ${}^{\alpha}X \in {}^{\alpha}\mathcal{X}$ such that, a.s.,

$$\lim_{t\to\infty}\left(\frac{1}{G_t}\log\left(\frac{{}^{\alpha}X_t}{{}^{\alpha}\widehat{X}_t}\right)\right) = 0$$

also enjoys the asymptotic growth-optimality property in the sense of achieving equality in (3.1). As a simple example, let $X \in \mathcal{X}$, $\kappa \in (0,1)$ and $\widetilde{X} := \kappa\widehat{X} + (1-\kappa)X$. Then $\widetilde{X}^* \ge \kappa\widehat{X}^*$ so that ${}^{\alpha}\widetilde{X} \ge \alpha{}^{\alpha}\widetilde{X}^* \ge \alpha(\kappa\widehat{X}^*)^{1-\alpha} = \big(\alpha\kappa^{1-\alpha}\big)\,{}^{\alpha}\widehat{X}^*$ and, consequently, ${}^{\alpha}\widetilde{X}$ enjoys the asymptotic growth optimality. In contrast, the asymptotic numéraire property is much stronger. Combining Theorem 2.8 and Proposition 2.3, it follows that if ${}^{\alpha}X \in {}^{\alpha}\mathcal{X}$ is to have the asymptotic numéraire property, then the much stronger "asymptotic equivalence" condition $\lim_{t\to\infty}\big({}^{\alpha}X_t / {}^{\alpha}\widehat{X}_t\big) = 1$ has to be a.s. valid. We shall see below that this in fact implies ${}^{\alpha}X = {}^{\alpha}\widehat{X}$.

3.2. **Optimality through sequences of stopping times converging to infinity.** By Theorem 2.8, $\mathbb{E}\mathsf{rr}_\infty({}^{\alpha}X | {}^{\alpha}\widehat{X}) \le 0$ holds for all $X \in \mathcal{X}$, a result which can be interpreted as long-run numéraire optimality property of ${}^{\alpha}\widehat{X}$ in ${}^{\alpha}\mathcal{X}$. However, in effect, this result assumes that the investment time-horizon is actually equal to infinity. On both theoretical and practical levels, one may be rather interested in considering a sequences of stopping times $(T_n)_{n\in\mathbb{N}}$ that converge to infinity and examine the behavior of optimal wealth processes (in the numéraire sense) in the limit. We present two results in this direction. Proposition 3.3 establishes that the only process in ${}^{\alpha}\mathcal{X}$ possessing the numéraire property along an increasing sequence of stopping times tending to infinity is ${}^{\alpha}\widehat{X}$. The second result, Theorem 3.7, is more delicate than Proposition 3.3, and may be regarded as a version of so-called *turnpike theorems*, an appellation coined in Leland (1972). While the traditional formulation of turnpike theorems involves two investors with long financial planning horizon and similar preferences for large levels of wealth, Theorem 3.7 compares a portfolio having the numéraire property for a long, but finite, time-horizon with the corresponding portfolio having the asymptotic numéraire property. Loosely speaking, Theorem 3.7 states that, when the time horizon T is long, the process ${}^{\alpha}\widetilde{X}$ that has the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T]$ will be very close initially (in time) to ${}^{\alpha}\widehat{X}$ in a very strong sense.

Proposition 3.3. *Under the validity of Assumption 1.2, suppose that there exist $X \in \mathcal{X}$ and a sequence of, possibly infinite valued, stopping times $(T_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \mathbb{P}[T_n > t] = 1$ holding for all $t \in \mathbb{R}_+$, such that $\liminf_{n\to\infty} \mathbb{E}\mathsf{rr}_{T_n}({}^{\alpha}\widehat{X}|{}^{\alpha}X) \leq 0$. Then, ${}^{\alpha}X = {}^{\alpha}\widehat{X}$.*

Proof. Upon passing to a subsequence of $(T_n)_{n\in\mathbb{N}}$ if necessary, we may assume without loss of generality that $\mathbb{P}[\lim_{n\to\infty} T_n = \infty] = 1$. Then, by Theorem 2.8, $\lim_{t\to\infty} \left({}^{\alpha}\widehat{X}_t/{}^{\alpha}X_t\right)$ exists a.s. in $(0, \infty]$ and a use of Fatou's lemma gives

$$\mathbb{E}\left[\lim_{t\to\infty}\left(\frac{{}^{\alpha}\widehat{X}_t}{{}^{\alpha}X_t}\right)\right] = \mathbb{E}\left[\liminf_{n\to\infty}\left(\frac{{}^{\alpha}\widehat{X}_{T_n}}{{}^{\alpha}X_{T_n}}\right)\right] \leq \liminf_{n\to\infty}\left(\mathbb{E}\left[\frac{{}^{\alpha}\widehat{X}_{T_n}}{{}^{\alpha}X_{T_n}}\right]\right) = 1 + \liminf_{n\to\infty} \mathbb{E}\mathsf{rr}_{T_n}({}^{\alpha}\widehat{X}|{}^{\alpha}X) \leq 1.$$

Since we have both $\mathbb{E}\big[\lim_{t\to\infty}\left({}^{\alpha}\widehat{X}_t/{}^{\alpha}X_t\right)\big] \leq 1$ and $\mathbb{E}\big[\lim_{t\to\infty}\left({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t\right)\big] \leq 1$ holding, Jensen's inequality implies that $\lim_{t\to\infty}\left({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t\right) = 1$ a.s. holds. By Theorem 2.8, $\lim_{t\to\infty}\left(X_t/\widehat{X}_t\right) = 1$ a.s. holds. This fact, combined with the conditional form of Fatou's lemma and the supermartingale property of $X/\widehat{X}$ gives $X_t/\widehat{X}_t \geq 1$ a.s. for each $t \in \mathbb{R}_+$. Combined with $\mathbb{E}\big[X_t/\widehat{X}_t\big] \leq 1$, this gives $\widehat{X}_t = X_t$ a.s. for all $t \in \mathbb{R}_+$. The path-continuity of the process $X/\widehat{X}$ implies that $X = \widehat{X}$, i.e., that ${}^{\alpha}X = {}^{\alpha}\widehat{X}$. □

Remark 3.4. Following the reasoning of the proof of Proposition 3.3, one can also show that if τ is a time of maximum of $\widehat{X}$ and $\mathbb{E}\mathsf{rr}_{\tau}({}^{\alpha}\widehat{X}|{}^{\alpha}X) \leq 0$ holds for some $X \in \mathcal{X}$, then ${}^{\alpha}X = {}^{\alpha}\widehat{X}$ holds identically on the stochastic interval $[\![0, \tau]\!]$.

In order to state Theorem 3.7, we define a strong notion of convergence in the space of semimartingales, introduced in Emery (1979).

Definition 3.5. For a stopping time T, we say that a sequence $(\xi^n)_{n\in\mathbb{N}}$ of semimartingales converges over $[0, T]$ in the *Emery topology* to another semimartingale ξ, and write $\mathcal{S}_T\text{-}\lim_{n\to\infty} \xi^n = \xi$, if

$$\lim_{n\to\infty} \sup_{\eta\in\mathcal{P}_1} \mathbb{P}\left[\sup_{t\in[0,T]}\left|\eta_0(\xi^n_0 - \xi_0) + \int_0^t \eta_s \mathrm{d}\xi^n_s - \int_0^t \eta_s \mathrm{d}\xi_s\right| > \epsilon\right] = 0 \tag{3.2}$$

holds for all $\epsilon > 0$, where $\mathcal{P}_1$ denotes the set of all predictable processes η with $\sup_{t\in\mathbb{R}_+} |\eta_t| \leq 1$. Furthermore, we say that the sequence $(\xi^n)_{n\in\mathbb{N}}$ of semimartingales converges *locally in the Emery topology* to another semimartingale ξ, and write $\mathcal{S}_{\mathsf{loc}}\text{-}\lim_{n\to\infty} \xi^n = \xi$, if $\mathcal{S}_T\text{-}\lim_{n\to\infty} \xi^n = \xi$ holds for all a.s. finitely-valued stopping times T.

Remark 3.6. In the setting of Definition 3.5, assume that $(\xi^n)_{n\in\mathbb{N}}$ converges locally in the Emery topology to ξ. By taking $\eta \equiv 1$ in (3.2), we see that

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} |\xi^n_t - \xi_t| > \epsilon\right] = 0$$

holds for all $\epsilon > 0$ and all a.s. finitely-valued stopping times T. In other words, the sequence $(\xi^n)_{n\in\mathbb{N}}$ converges *in probability, uniformly on compacts*, to ξ.

Theorem 3.7. *Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times such that $\lim_{n\to\infty} \mathbb{P}[T_n > t] = 1$ holds for all $t \in \mathbb{R}_+$. For each $n \in \mathbb{N}$, let ${}^{\alpha}\widetilde{X}^n \in {}^{\alpha}\mathcal{X}$ have the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the period $[0, T_n]$. Under Assumption 1.2, it holds that $\mathcal{S}_{\mathsf{loc}}\text{-}\lim_{n\to\infty} {}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$.*

The proof of Theorem 3.7 is given in Subsection A.4 of Appendix A.

Remark 3.8. In the setting of Theorem 3.7, the fact that $\mathcal{S}_{\mathsf{loc}}\text{-}\lim_{n\to\infty} {}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$ implies by (Kardaras, 2012, Proposition 2.9) that $\lim_{n\to\infty} \mathbb{P}\big[[{}^{\alpha}\widetilde{X}^n - {}^{\alpha}\widehat{X}, {}^{\alpha}\widetilde{X}^n - {}^{\alpha}\widehat{X}]_T > \epsilon \big] = 0$ holds for all a.s. finitely-valued stopping times T and $\epsilon > 0$. Writing ${}^{\alpha}\widehat{X} = 1 + \int_0^{\cdot} \langle {}^{\alpha}\widehat{H}_t, \mathrm{d}S_t\rangle$ and ${}^{\alpha}\widetilde{X}^n = 1 + \int_0^{\cdot} \langle {}^{\alpha}\widetilde{H}^n_t, \mathrm{d}S_t\rangle$ for all $n \in \mathbb{N}$ for appropriate d-dimensional strategies ${}^{\alpha}\widehat{H}$ and $({}^{\alpha}\widetilde{H}^n)_{n\in\mathbb{N}}$, we obtain

$$\lim_{n\to\infty} \mathbb{P}\left[\int_0^T \left\langle {}^{\alpha}\widetilde{H}^n_t - {}^{\alpha}\widehat{H}_t,\, \mathrm{d}[S,S]_t\big({}^{\alpha}\widetilde{H}^n_t - {}^{\alpha}\widehat{H}_t\big)\right\rangle > \epsilon\right] = 0,$$

for all a.s. finitely-valued stopping times T and $\epsilon > 0$. The previous relation implies that it is not only wealth that converges to the limiting one in each finite time-interval—the corresponding employed strategy does so as well.

Remark 3.9. In the setting of Theorem 3.7, the conclusion is that convergence of ${}^{\alpha}\widetilde{X}^n$ to ${}^{\alpha}\widehat{X}$ holds over finite time-intervals that do *not* depend on $n \in \mathbb{N}$. One can ask whether the whole wealth process ${}^{\alpha}\widetilde{X}^n$ is close to ${}^{\alpha}\widehat{X}$ over the stochastic interval $[\![0, T_n]\!]$ for each $n \in \mathbb{N}$. This is not true in general; in Appendix B we present an example, valid under *all* models for which Assumption 1.2 holds, where the ratio ${}^{\alpha}\widetilde{X}^n_{T_n} / {}^{\alpha}\widehat{X}_{T_n}$ as $n \to \infty$ oscillates between $1/(2-\alpha)$ and ∞. Note that the example only covers cases where $\alpha \in (0,1)$; if $\alpha = 0$, ${}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$ always holds for all $n \in \mathbb{N}$.

4. Conclusions

The numéraire portfolio $\widehat{X}$, in the global sense of condition (B1) in Theorem 1.3, exists and is unique in a very general modeling setup. The numéraire property is a strong one and implies that $\widehat{X}$ maximizes the growth rate as well as the expected logarithmic utility, see Remarks 1.5, 2.5 and Theorem 3.1. An increasing number of experts on portfolio allocation believe that it makes it a natural choice for a long-run investor. On the other hand, one has to admit that it offers little flexibility to control for investor's risk appetites. The literature usually points to expected utility (including "quadratic utility" reflecting in some sense Markowitz's mean-variance approach) as a more flexible framework.

The inherent problem with utility maximization is that its solution typically depends on the (arbitrary) choice of a time horizon. In this paper we suggest a possible way out of this unsatisfactory situation: we propose maximizing the growth rate within a restricted class of investment strategies. We only consider wealth processes X that exhibit constrained drawdown $X \geq \alpha X^*$, with $\alpha \in [0, 1)$ quantifying the investor's attitude against risk. The path dependent, non-myopic, nature of the drawdown constraint renders certain features of traditional asset-allocation theory invalid. On the other hand, one obtains a model-independent, practical and robust approach.

This paper presented a rather complete investigation of the numéraire property in a drawdown-constrained context. First, we give a new definition based on the expected relative return, which extends the numéraire property from a global setting $\mathcal{X}$ to any subset of investment strategies. We showed that for each time horizon, there exists a (essentially unique) portfolio with the numéraire property within the class ${}^{\alpha}\mathcal{X}$. Moreover, as horizon became distant, these are close, in a very strong sense on any fixed time interval to ${}^{\alpha}\widehat{X}$, which is the unique portfolio with the asymptotic numéraire property. It is defined through an explicit Azéma–Yor transformation (1.4) from the global numéraire portfolio $\widehat{X}$ and has a natural investment interpretation, see Remark 1.7. Furthermore, ${}^{\alpha}\widehat{X}$ has the numéraire property also along an increasing sequence of stopping times: times of maximum of $\widehat{X}$. However, contrary to the unconstrained case, it does not enjoy the numéraire property for all times. This is an important novel feature beyond previous studies.

Appendix A. Proofs of Certain Results

During the course of Appendix A, the validity of Assumption 1.2 is always in force. The only exception is Subsection A.2, where only condition (A1) of Assumption 1.2 is required.

A.1. **Proof of Proposition 1.8.** Since ${}^{\alpha}\widehat{X}/{}^{\alpha}\widehat{X}^* = \alpha + (1-\alpha)(\widehat{X}/\widehat{X}^*)$ holds in view of (1.6), we only need to establish that $0 = \liminf_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) < \limsup_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) = 1$. The fact that $\limsup_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) = 1$ follows directly from $\lim_{t\to\infty}\widehat{X}_t = \infty$. On the other hand, the fact that $\liminf_{t\to\infty}(\widehat{X}_t/\widehat{X}_t^*) = 0$ follows immediately from the next result (which is stated separately as it is also used on another occasion) and the martingale version of the Borel-Cantelli lemma.

Lemma A.1. *Let σ be a stopping time with $\mathbb{P}[\sigma < \infty] = 1$. For $\alpha \in (0, 1)$ define the stopping time $T := \inf\left\{t \in (\sigma, \infty) \mid \widehat{X}_t/\widehat{X}_t^* \leq \alpha\right\}$. Then $\mathbb{P}[T < \infty] = 1$.*

Proof. Recall that $\lim_{t\to\infty}\widehat{X}_t = \infty$ holds by Theorem 1.3. Using the result of Dambis, Dubins and Schwarz (Karatzas and Shreve, 1991, Theorem 3.4.6) and a time-change argument, (1.3) implies that we can assume without loss of generality that $\widehat{X}$ satisfies $\widehat{X}_t = \exp(t/2 + \beta_t)$ for $t \in \mathbb{R}_+$, where β is a standard Brownian motion. Furthermore, using again the fact that $\lim_{t\to\infty}\widehat{X}_t = \infty$, we may assume without loss of generality that σ is a time of maximum of $\widehat{X}$. Then, the independent increments property of Brownian motion implies that we can additionally assume without loss of generality that $\sigma = 0$. Set $\sigma_0 = 0$ and, via induction, for each $n \in \mathbb{N}$ set

$$\sigma_n := \inf\left\{ t \in (\sigma_{n-1}, \infty) \mid \widehat{X}_t = \mathrm{e}\widehat{X}_{\sigma_{n-1}} \right\}, \text{ and } T_n = \inf\left\{ t \in (\sigma_{n-1}, \infty) \mid \widehat{X}_t / \widehat{X}^*_t = \alpha \right\}.$$

With $T = T_1$, we wish to show that $\mathbb{P}[T < \infty] = 1$. For each $n \in \mathbb{N}$ define the event $A_n := \{T_n < \sigma_n\}$. Note that $\mathbb{P}\left[A_n \mid \mathcal{F}_{\sigma_{n-1}}\right] = \mathbb{P}[A_1]$ holds for all $n \in \mathbb{N}$ in view of the regenerating property of Brownian motion and the fact that each σ_{n-1}, $n \in \mathbb{N}$, is a time of maximum of $\widehat{X}$. Since $\limsup_{n\to\infty} A_n \subseteq \{T < \infty\}$, the martingale version of the Borel-Cantelli lemma implies that $\mathbb{P}[T < \infty] = 1$ will be established as long as we can show that $\mathbb{P}[T_1 < \sigma_1] = \mathbb{P}[A_1] > 0$.

Since $\int_0^\infty \mathbb{I}_{\{\widehat{X}_t < \widehat{X}^*_t\}} \mathrm{d}\widehat{X}^*_t = 0$ a.s. holds, Itô's formula implies that

$$\frac{\widehat{X}^*}{\widehat{X}} = 1 + \int_0^\cdot \widehat{X}^*_t \mathrm{d}\left(\frac{1}{\widehat{X}_t}\right) + \log(\widehat{X}^*).$$

Both processes $\widehat{X}^*/\widehat{X}$ and $\log(\widehat{X}^*)$ are bounded on the stochastic interval $[\![0, \sigma_1 \wedge T_1]\!]$—therefore, since $\mathbb{P}[\sigma_1 < \infty] = 1$ and $\int_0^\cdot \widehat{X}^*_t \mathrm{d}\big(1/\widehat{X}_t\big)$ is a local martingale (by Assumption 1.2 and the fact that $1 \in \mathcal{X}$), a localization argument gives

$$\mathbb{P}[\sigma_1 \le T_1] + \frac{1}{\alpha}\mathbb{P}[T_1 < \sigma_1] = \mathbb{E}\left[\frac{\widehat{X}^*_{\sigma_1 \wedge T_1}}{\widehat{X}_{\sigma_1 \wedge T_1}}\right] = 1 + \mathbb{E}\left[\log(\widehat{X}^*_{\sigma_1 \wedge T_1})\right] \ge 1 + \mathbb{P}[\sigma_1 \le T_1],$$

which gives $\mathbb{P}[T_1 < \sigma_1] \ge \alpha > 0$ and completes the proof of Lemma A.1. □

A.2. **Proof of Theorem 2.4.** For the purposes of Subsection A.2, only condition (A1) of Assumption 1.2 is in force. Fix an a.s. finitely-valued stopping time T throughout. As the result of Theorem 2.4 for the case $\alpha = 0$ is known, we tacitly assume that $\alpha \in (0, 1)$ throughout.

A.2.1. *Existence.* We shall first prove existence of a process with the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the period $[0, T]$. As T is a.s. finitely-valued, without loss of generality we shall assume that all processes that appear below are constant after time T, and their value after time T is equal to their value at time T. In particular, the limiting value of a process for time tending to infinity exists and is equal to its value at time T.

Define $\mathcal{X}^\circ$ as the class of all nonnegative càdlàg processes Y with $Y_0 \le 1$ and with the property that YX is a supermartingale for all $X \in \mathcal{X}$. Note that $(1/\widehat{X}) \in \mathcal{X}^\circ$. In a similar way, define $\mathcal{X}^{\circ\circ}$ as the class of all nonnegative càdlàg processes χ with $\chi_0 \le 1$ and with the property that that $Y\chi$ is a supermartingale for all $Y \in \mathcal{X}^\circ$. It is clear that $\mathcal{X} \subseteq \mathcal{X}^{\circ\circ}$. The next result reveals the exact structure of $\mathcal{X}^{\circ\circ}$.

Theorem A.2 (Optional Decomposition Theorem Föllmer and Kramkov (1997), Stricker and Yan (1998))**.** *The class $\mathcal{X}^{\circ\circ}$ consists exactly of all processes χ of the form $\chi = X(1 - A)$, where $X \in \mathcal{X}$ and A is an adapted, nonnegative and nondecreasing càdlàg process with $0 \le A \le 1$.*

The result that follows enables one to construct a process that will be a candidate to have the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T]$.

Lemma A.3. *For any $\alpha \in [0,1)$ and $t \in [0,\infty]$, the set $\{Z_t \mid Z \in {}^\alpha\mathcal{X}\}$ is convex and bounded in $\mathbb{P}$-measure, i.e.,* $\lim_{K\to\infty} \sup_{Z\in{}^\alpha\mathcal{X}} \mathbb{P}[Z_t > K] = 0$.

Proof. Fix $\alpha \in [0,1)$. Let $\lambda \in [0,1]$ and pick processes $X \in \mathcal{X}$ and $X' \in \mathcal{X}$. Since $\mathcal{X}$ is convex, $((1-\lambda)^\alpha X + \lambda^\alpha X') \in \mathcal{X}$. Furthermore, since

$$\alpha((1-\lambda)^\alpha X + \lambda^\alpha X')^* \leq (1-\lambda)\alpha^\alpha X^* + \lambda\alpha(^\alpha X')^* \leq \alpha((1-\lambda)^\alpha X + \lambda^\alpha X'),$$

we obtain $((1-\lambda)^\alpha X + \lambda^\alpha X') \in {}^\alpha\mathcal{X}$, which shows that ${}^\alpha\mathcal{X}$ is convex for all $\alpha \in [0,1)$.

Furthermore, it holds that $\sup_{X\in\mathcal{X}} \mathbb{E}\big[X_\infty/\widehat{X}_\infty\big] \leq 1$ and, using Markov's inequality, we see that $\{X_\infty/\widehat{X}_\infty \mid X \in \mathcal{X}\}$ is bounded in $\mathbb{P}$-measure. Since $\mathbb{P}[\widehat{X}_\infty > 0] = 1$, the set $\{X_\infty \mid X \in \mathcal{X}\}$ is bounded in $\mathbb{P}$-measure; the same is then true for $\{^\alpha X_t \mid X \in \mathcal{X}\} \subseteq \{X_t \mid X \in \mathcal{X}\} \subseteq \{X_\infty \mid X \in \mathcal{X}\}$ for any value of $t \in [0,\infty]$. □

In the sequel, fix $\alpha \in [0,1)$. In view of Lemma A.3 and (Kardaras, 2010*b*, Theorem 1.1(4)), there exists a random variable $\check{\chi}_\infty$ in the closure in $\mathbb{P}$-measure of $\{X_\infty \mid X \in {}^\alpha\mathcal{X}\}$ such that $\mathbb{E}[X_\infty/\check{\chi}_\infty] \leq 1$ holds for all $X \in {}^\alpha\mathcal{X}$. Define the countable set $\mathbb{T} = \{k/2^m \mid k \in \mathbb{N}, m \in \mathbb{N}\}$. A repeated application of (Delbaen and Schachermayer, 1994, Lemma A1.1) combined with Lemma A.3 and a diagonalization argument implies that one can find an ${}^\alpha\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$ such that $\check{\chi}_\infty = \lim_{n\to\infty} X^n_\infty$ and $\lim_{n\to\infty} X^n_t$ a.s. exists simultaneously for all $t \in \mathbb{T}$. Define then $\check{\chi}_t = \lim_{n\to\infty} X^n_t$ for all $t \in \mathbb{T}$. Since T is a.s. finitely-valued and all processes are constant after T, it is straightforward that $\check{\chi}_\infty = \lim_{t\to\infty} \check{\chi}_t$ a.s.

Since $\mathbb{E}[Y_t X^n_t \mid \mathcal{F}_s] \leq Y_s X^n_s$ holds for all $n \in \mathbb{N}$, $Y \in \mathcal{X}^\circ$, $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0,t]$, the conditional version of Fatou's lemma gives that $\mathbb{E}[Y_t \check{\chi}_t \mid \mathcal{F}_s] \leq Y_s \check{\chi}_s$ holds for all $Y \in \mathcal{X}^\circ$, $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0,t]$. In particular, with $\widehat{Y} := 1/\widehat{X} \in \mathcal{X}^\circ$, the process $(\widehat{Y}_t \check{\chi}_t)_{t\in\mathbb{T}}$ is a supermartingale in the corresponding stochastic basis with time-index $\mathbb{T}$. Since $\mathbb{P}[\inf_{s\in[0,t]} \widehat{Y}_s > 0] = 1$ holds for all $t \in \mathbb{R}_+$, the supermartingale convergence theorem implies that there exists a nonnegative càdlàg process χ such that[4] $\chi_s = \lim_{\mathbb{T}\ni t\downarrow\downarrow s} \check{\chi}_t$ holds for all $s \in \mathbb{R}_+$. (The notation "$\lim_{\mathbb{T}\ni t\downarrow\downarrow s}$" denotes limit along times $t \in \mathbb{T}$ that are *strictly* greater than $s \in \mathbb{R}_+$ and converge to s.) The fact that $\mathbb{E}[Y_t \check{\chi}_t \mid \mathcal{F}_s] \leq Y_s \check{\chi}_s$ holds for all $Y \in \mathcal{X}^\circ$, $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0,t]$, right-continuity of the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and the conditional version of Fatou's lemma give that $\mathbb{E}[Y_t \chi_t \mid \mathcal{F}_s] \leq Y_s \chi_s$ holds for all $Y \in \mathcal{X}^\circ$, $t \in \mathbb{R}_+$ and $s \in [0,t]$. Therefore, $\chi \in \mathcal{X}^{\circ\circ}$. Of course, $\chi_\infty = \check{\chi}_\infty = \lim_{t\to\infty} \chi_t$ a.s. holds. In view of Theorem A.2, it holds that $\chi = \widetilde{X}(1-A)$, where $\widetilde{X} \in \mathcal{X}$ and A is an adapted, nonnegative and nondecreasing càdlàg process with $0 \leq A \leq 1$. Furthermore, note that $\mathbb{E}[X_\infty/\chi_\infty] \leq 1$ holds for all $X \in {}^\alpha X$.

Continuing, we shall show that $A \equiv 0$ and $\chi(= \widetilde{X}) \in {}^\alpha\mathcal{X}$. If $(X^n)_{n\in\mathbb{N}}$ is the ${}^\alpha\mathcal{X}$-valued sequence such that $\check{\chi}_t = \lim_{n\to\infty} X^n_t$ holds a.s. simultaneously for all $t \in \mathbb{T}$, we have that $X^n_t \geq \alpha X^n_s$ a.s.

[4]Note that χ is indeed the limit of $(X^n)_{n\in\mathbb{N}}$ in the "Fatou" sense. Fatou-convergence has proved to be extremely useful in the theory of Mathematical Finance; for example, see Föllmer and Kramkov (1997), Kramkov and Schachermayer (1999) and Žitković (2002).

holds for all $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$. By passing to the limit, and using the fact that $\mathbb{T}$ is countable, we obtain that $\check{\chi}_t \geq \alpha \check{\chi}_s$ holds a.s. simultaneously for all $t \in \mathbb{T}$ and $s \in \mathbb{T} \cap [0, t]$. Therefore, $\chi_t \geq \alpha \chi_s$ holds a.s. simultaneously for all $t \in \mathbb{R}_+$ and $s \in [0, t]$. Then,

$$\widetilde{X}_t = \frac{\chi_t}{1 - A_t} \geq \frac{\chi_t}{1 - A_s} \geq \alpha \frac{\chi_s}{1 - A_s} = \alpha \widetilde{X}_s$$

holds a.s. simultaneously for all $t \in \mathbb{R}_+$ and $s \in [0, t]$. It follows that $\widetilde{X} \in {}^{\alpha}\mathcal{X}$. This implies in particular that $\mathbb{E}[\widetilde{X}_\infty / \chi_\infty] \leq 1$ has to hold; since $\widetilde{X}_\infty / \chi_\infty = 1/(1 - A_\infty) \geq 1$, we obtain $\mathbb{P}[A_\infty = 0] = 1$, i.e., $A \equiv 0$. Therefore, $\chi = \widetilde{X}$ and $\mathbb{E}[X_\infty / \widetilde{X}_\infty] \leq 1$ holds for all $X \in {}^{\alpha}X$, which concludes the proof of existence of a wealth process that possesses the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over $[0, T]$.

A.2.2. *Uniqueness.* We proceed in establishing uniqueness of a process with the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over the period $[0, T]$. We start by stating and proving a result that will be used again later.

Lemma A.4. *Let $Z \in {}^{\alpha}X$, and let σ be a stopping time such that $Z_\sigma = Z^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$. Fix $X \in {}^{\alpha}\mathcal{X}$ and $A \in \mathcal{F}_\sigma$ and define a new process[5] $\xi := Z \mathbb{I}_{[\![0,\sigma[\![} + \big(Z \mathbb{I}_{\Omega \setminus A} + (Z_\sigma / X_\sigma) X \mathbb{I}_A\big) \mathbb{I}_{[\![\sigma,\infty[\![}$. Then, $\xi \in {}^{\alpha}\mathcal{X}$.*

Proof. It is straightforward to check that $\xi \in \mathcal{X}$. To see that $\xi \in {}^{\alpha}\mathcal{X}$, note that $\xi/\xi^* = Z/Z^* \geq \alpha$ holds on $[\![0, \sigma[\![\cup ([\![\sigma, \infty[\![\cap (\Omega \setminus A))$, while, using the fact that $\xi^*_\sigma = Z^*_\sigma = Z_\sigma$ holds a.s.on $\{\sigma < \infty\}$,

$$\frac{\xi}{\xi^*} = \frac{X}{\sup_{t \in [\sigma, \cdot]} X_t} \geq \frac{X}{X^*} \geq \alpha, \quad \text{holds on } [\![\sigma, \infty[\![\cap A.$$

The result immediately follows. □

Remark A.5. As can be seen via the use of simple counter-examples, if one drops the assumption that σ is a time of maximum of Z in the statement of Lemma A.4, the resulting process ξ may fail to satisfy the drawdown constraints. This is in direct contrast with the non-constrained case $\alpha = 0$, where any stopping time σ will result in ξ being an element of $\mathcal{X}$. It is exactly this fact, a consequence of the non-myopic structure of the drawdown constraints, which results in portfolios with the numéraire property that depend on the investment horizon.

Lemma A.6. *Let $Z \in {}^{\alpha}\mathcal{X}$ be such that $\mathbb{E}\mathsf{rr}_T(X|Z) \leq 0$ holds for all $X \in {}^{\alpha}X$, and suppose that σ is a stopping time such that $Z_\sigma = Z^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$. Then,*

$$\mathbb{E}\left[\frac{X_T}{Z_T} \,\middle|\, \mathcal{F}_{T \wedge \sigma}\right] \leq \frac{X_{T \wedge \sigma}}{Z_{T \wedge \sigma}} \text{ holds a.s. for all } X \in {}^{\alpha}\mathcal{X}.$$

[5]Note that, since we tacitly assume that $\alpha \in (0, 1)$, $X_\sigma > 0$ a.s. holds on $\{\sigma < \infty\}$. Therefore, the process ξ is well-defined.

Proof. Fix $X \in {}^{\alpha}\mathcal{X}$ and $A \in \mathcal{F}_\sigma$. Define the process $\xi := Z\mathbb{I}_{[\![0,\sigma[\![} + \big(Z\mathbb{I}_{\Omega \setminus A} + (Z_\sigma / X_\sigma)\, X \mathbb{I}_A\big)\mathbb{I}_{[\![\sigma,\infty[\![}$; by Lemma A.4, $\xi \in {}^{\alpha}\mathcal{X}$. Furthermore, it is straightforward to check that

$$\frac{\xi_T}{Z_T} = \mathbb{I}_{\Omega \setminus (A \cap \{\sigma \le T\})} + \left(\frac{X_T}{Z_T}\frac{Z_\sigma}{X_\sigma}\right)\mathbb{I}_{A \cap \{\sigma \le T\}}.$$

Therefore, the fact that $\mathbb{E}\mathsf{rr}_T(\xi | Z) \le 0$ holds implies

$$\mathbb{E}\left[\frac{X_T}{Z_T}\frac{Z_\sigma}{X_\sigma}\mathbb{I}_{A \cap \{\sigma \le T\}}\right] \le \mathbb{P}[A \cap \{\sigma \le T\}].$$

As the previous is true for all $A \in \mathcal{F}_\sigma$, we obtain that $\mathbb{E}\left[X_T/Z_T \mid \mathcal{F}_\sigma\right] \le X_\sigma/Z_\sigma$ holds a.s. on $\{\sigma \le T\}$ for all $X \in {}^{\alpha}\mathcal{X}$. Combined with the fact that $\mathbb{E}\left[X_T/Z_T \mid \mathcal{F}_T\right] = X_T/Z_T$ trivially holds a.s. on $\{\sigma > T\}$ for all $X \in {}^{\alpha}\mathcal{X}$, we obtain the result. □

We now proceed to the actual proof of uniqueness. Assume that both $\widetilde{Z} \in {}^{\alpha}\mathcal{X}$ and $\widetilde{X} \in {}^{\alpha}\mathcal{X}$ have the numéraire property in ${}^{\alpha}\mathcal{X}$ for investment over $[0,T]$. Since $\mathbb{P}\left[T < \infty\right] = 1$, Proposition 2.3 implies that $\mathbb{P}\big[\widetilde{X}_T = \widetilde{Z}_T\big] = 1$. We shall show below that $\widetilde{Z} \le \widetilde{X}$ holds on $[\![0,T]\!]$. Interchanging the roles of $\widetilde{X}$ and $\widetilde{Z}$, it will also follow that $\widetilde{X} \le \widetilde{Z}$ holds on $[\![0,T]\!]$, which will establish that $\widetilde{X} = \widetilde{Z}$ holds on $[\![0,T]\!]$ and will complete the proof of Theorem 2.4.

Since $\mathbb{P}\big[\widetilde{X}_T = \widetilde{Z}_T\big] = 1$ and $\mathbb{E}\mathsf{rr}_T(X|\widetilde{Z}) \le 0$ holds for all $X \in {}^{\alpha}\mathcal{X}$, Lemma A.6 above implies that $1 = \mathbb{E}\big[\widetilde{X}_T/\widetilde{Z}_T \mid \mathcal{F}_{T\wedge\sigma}\big] \le \widetilde{X}_{T\wedge\sigma}/\widetilde{Z}_{T\wedge\sigma}$ a.s. holds whenever σ is a stopping time such that $\widetilde{Z}_\sigma = \widetilde{Z}^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$. The fact that $\widetilde{Z}_{T\wedge\sigma} \le \widetilde{X}_{T\wedge\sigma}$ a.s. holds whenever σ is a stopping time such that $\widetilde{Z}_\sigma = \widetilde{Z}^*_\sigma$ a.s. holds on $\{\sigma < \infty\}$ implies in a straightforward way that $\widetilde{Z}^* \le \widetilde{X}^*$ holds on $[\![0,T]\!]$.

We now claim that $\mathbb{P}\big[\widetilde{Z}_T = \widetilde{X}_T\big] = 1$ combined with $\widetilde{Z}^* \le \widetilde{X}^*$ holding on $[\![0,T]\!]$ imply that $\widetilde{Z} \le \widetilde{X}$ on $[\![0,T]\!]$, which will complete the proof. To see the last claim, for $\epsilon > 0$ define the stopping time

$$T_\epsilon := \inf\left\{t \in \mathbb{R}_+ \mid \widetilde{Z}_t > (1+\epsilon)\widetilde{X}_t\right\}.$$

We shall show that $\mathbb{P}\left[T_\epsilon < T\right] = 0$; as this will hold for all $\epsilon > 0$, it will follow that $\widetilde{Z} \le \widetilde{X}$ holds on $[\![0,T]\!]$. Define a new process $\widetilde{X}^\epsilon$ via

$$\widetilde{X}^\epsilon = \widetilde{Z}\mathbb{I}_{[\![0,T_\epsilon[\![} + \left(\frac{\widetilde{Z}_{T^\epsilon}}{\widetilde{X}_{T^\epsilon}}\right)\widetilde{X}\mathbb{I}_{[\![T^\epsilon,\infty[\![} = \widetilde{Z}\mathbb{I}_{[\![0,T_\epsilon[\![} + (1+\epsilon)\,\widetilde{X}\mathbb{I}_{[\![T^\epsilon,\infty[\![}.$$

We first show that $\widetilde{X}^\epsilon \in {}^{\alpha}\mathcal{X}$. The fact that $\widetilde{X} \in \mathcal{X}$ is obvious. Note also that $\widetilde{X}^\epsilon \ge \alpha(\widetilde{X}^\epsilon)^*$ clearly holds on $[\![0,T^\epsilon[\![$, since $\widetilde{Z} \in {}^{\alpha}X$. On the other hand,

$$(\widetilde{X}^\epsilon)^*_t = (\widetilde{Z}_{T^\epsilon})^* \vee \sup_{s\in[T^\epsilon,t]}\Big((1+\epsilon)\widetilde{X}_t\Big) \le (1+\epsilon)\widetilde{X}^*_t \text{ holds for } t \ge T^\epsilon,$$

the latter inequality holding in view of the fact that $\widetilde{Z}^* \le \widetilde{X}^*$. Therefore, for $t \ge T^\epsilon$ it holds that $\widetilde{X}^\epsilon_t = (1+\epsilon)\widetilde{X}_t \ge (1+\epsilon)\alpha\widetilde{X}^*_t \ge \alpha(\widetilde{X}^\epsilon)^*_t$. It follows that $\widetilde{X}^\epsilon \ge \alpha(\widetilde{X}^\epsilon)^*$ also holds on $[\![T^\epsilon,\infty[\![$, which shows that $\widetilde{X}^\epsilon \in {}^\alpha\mathcal{X}$. Note that

$$\widetilde{X}^\epsilon_T = \widetilde{Z}_T \mathbb{I}_{\{T < T^\epsilon\}} + (1+\epsilon)\widetilde{X}_T \mathbb{I}_{\{T^\epsilon \leq T\}} = \widetilde{X}_T \mathbb{I}_{\{T < T^\epsilon\}} + (1+\epsilon)\widetilde{X}_T \mathbb{I}_{\{T^\epsilon \leq T\}},$$

which implies that $\widetilde{X}^\epsilon_T / \widetilde{X}_T = 1 + \epsilon \mathbb{I}_{\{T^\epsilon \leq T\}}$ and, as a consequence, $\mathbb{E}\mathsf{rr}_T(\widetilde{X}^\epsilon | \widetilde{X}) = \epsilon \mathbb{P}[T^\epsilon \leq T]$. In case $\mathbb{P}[T^\epsilon \leq T] > 0$, it would follow that $\widetilde{X}$ fails to have the numéraire property in ${}^\alpha\mathcal{X}$ for investment in $[0, T]$. Therefore, $\mathbb{P}[T^\epsilon \leq T] = 0$, which implies that $\widetilde{Z} \leq \widetilde{X}$ holds on $[\![0, T]\!]$, as already mentioned. The proof of Theorem 2.4 is complete.

A.3. **Proof of Theorem 2.8.** The main tool towards proving assertion (1) of Theorem 2.8 is the following auxiliary result.

Lemma A.7. *For any $X \in \mathcal{X}$, $\lim_{t\to\infty}(X^*_t / \widehat{X}^*_t)$ a.s. exists. Moreover, it a.s. holds that*

$$\lim_{t\to\infty} \left(\frac{X^*_t}{\widehat{X}^*_t} \right) = \lim_{t\to\infty} \left(\frac{X_t}{\widehat{X}_t} \right).$$

Proof. For $t \in \mathbb{R}_+$, define the $[0, t]$-valued random time $\widehat{\rho}_t := \sup\{s \in [0, t] \mid \widehat{X}_s = \widehat{X}^*_s\}$; then, $\widehat{X}^*_t = \widehat{X}_{\widehat{\rho}_t}$. Note that $\mathbb{P}[\uparrow \lim_{t\to\infty} \widehat{\rho}_t = \infty] = 1$ holds in view of Assumption 1.2. It follows that, for any $X \in \mathcal{X}$, it a.s. holds that

$$\liminf_{t\to\infty} \left(\frac{X^*_t}{\widehat{X}^*_t} \right) = \liminf_{t\to\infty} \left(\frac{X^*_t}{\widehat{X}^*_{\widehat{\rho}_t}} \right) \geq \liminf_{t\to\infty} \left(\frac{X_{\widehat{\rho}_t}}{\widehat{X}_{\widehat{\rho}_t}} \right) = \lim_{t\to\infty} \left(\frac{X_t}{\widehat{X}_t} \right). \tag{A.1}$$

In what follows, fix $X \in \mathcal{X}$. For $t \in \mathbb{R}_+$ define $\rho_t := \sup\{s \in [0, t] \mid X_s = X^*_s\}$, which is a $[0, t]$-valued random time. For each $t \in \mathbb{R}_+$, $X^*_t = X_{\rho_t}$. Note that the set-inclusions $\{\uparrow \lim_{t\to\infty} \rho_t < \infty\} \subseteq \left\{\sup_{t\in\mathbb{R}_+} X_t < \infty\right\} \subseteq \{\mathsf{rr}_\infty(X|\widehat{X}) = -1\}$ are valid a.s., the last in view of Assumption 1.2. Therefore,

$$\lim_{t\to\infty} \left(\frac{X_t}{\widehat{X}_t} \right) = \lim_{t\to\infty} \left(\frac{X^*_t}{\widehat{X}^*_t} \right) = 0 \text{ holds on } \left\{ \lim_{t\to\infty} \rho_t < \infty \right\}. \tag{A.2}$$

Furthermore,

$$\limsup_{t\to\infty} \left(\frac{X^*_t}{\widehat{X}^*_t} \right) = \limsup_{t\to\infty} \left(\frac{X^*_{\rho_t}}{\widehat{X}^*_t} \right) \leq \limsup_{t\to\infty} \left(\frac{X_{\rho_t}}{\widehat{X}_{\rho_t}} \right) = \lim_{t\to\infty} \left(\frac{X_t}{\widehat{X}_t} \right) \text{ holds on } \left\{ \lim_{t\to\infty} \rho_t = \infty \right\}. \tag{A.3}$$

The claim now readily follows from (A.1), (A.2), and (A.3). □

Proof of Theorem 2.8, statement (1). In the sequel, fix $X \in \mathcal{X}$ and assume that $\alpha \in (0, 1)$. Results for the case $\alpha = 0$ are well-understood and not discussed.

To ease notation, let $D := X/X^*$ and $\widehat{D} := \widehat{X}/\widehat{X}^*$. The process D is $[0, 1]$-valued and $\widehat{D}$ is $(0, 1]$-valued. Observe that

$$\frac{{}^\alpha X}{{}^\alpha\widehat{X}} = \frac{\alpha(X^*)^{1-\alpha} + \alpha(X^*)^{-\alpha}X}{\alpha(\widehat{X}^*)^{1-\alpha} + \alpha(\widehat{X}^*)^{-\alpha}\widehat{X}} = \left(\frac{X^*}{\widehat{X}^*} \right)^{1-\alpha} \left(\frac{\alpha + (1-\alpha)D}{\alpha + (1-\alpha)\widehat{D}} \right).$$

In view of Lemma A.7, $\lim_{t\to\infty}(X_t^*/\widehat{X}_t^*)^{1-\alpha} = (1+\mathsf{rr}_\infty(X|\widehat{X}))^{1-\alpha}$ holds. Firstly, the fact that

$$\frac{\alpha+(1-\alpha)D}{\alpha+(1-\alpha)\widehat{D}} \leq \frac{1}{\alpha}$$

implies that ${}^\alpha X/{}^\alpha\widehat{X} \leq (1/\alpha)(X^*/\widehat{X}^*)^{1-\alpha}$, which readily gives (2.3) on $\{\mathsf{rr}_\infty(X|\widehat{X}) = -1\}$. Furthermore, the facts that $0 \leq D \leq 1$, $0 < \widehat{D} \leq 1$ and $\lim_{t\to\infty}(D_t/\widehat{D}_t) = 1$, the latter holding a.s. on $\{\mathsf{rr}_\infty(X|\widehat{X}) > -1\}$ in view of Lemma A.7, imply that

$$\limsup_{t\to\infty}\left|\frac{\alpha+(1-\alpha)D_t}{\alpha+(1-\alpha)\widehat{D}_t} - 1\right| \leq \frac{1-\alpha}{\alpha}\limsup_{t\to\infty}|D_t-\widehat{D}_t| = 0 \text{ holds on } \left\{\mathsf{rr}_\infty(X|\widehat{X}) > -1\right\}.$$

Therefore, $\lim_{t\to\infty}({}^\alpha X_t/{}^\alpha\widehat{X}_t) = (1+\mathsf{rr}_\infty(X|\widehat{X}))^{1-\alpha}$ also holds on the event $\{\mathsf{rr}_\infty(X|\widehat{X}) > -1\}$, which completes the proof of statement (1) of Theorem 2.8. □

Proof of Theorem 2.8, statement (2). Let τ be a time of maximum of $\widehat{X}$. Recall the definition of the stopping times $(\tau_\ell)_{\ell\in\mathbb{R}_+}$ from (2.5). In view of statement (1) of Theorem 2.8,

$$\mathsf{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X}) = \lim_{\ell\to\infty}\left(\frac{{}^\alpha X_{\tau\wedge\tau_\ell}}{{}^\alpha\widehat{X}_{\tau\wedge\tau_\ell}}\right) - 1 \tag{A.4}$$

a.s. holds. Now, observe that $\tau\wedge\tau_\ell$ is a time of maximum of $\widehat{X}$ for each $\ell\in\mathbb{R}_+$; therefore, ${}^\alpha\widehat{X}_{\tau\wedge\tau_\ell} = (\widehat{X}_{\tau\wedge\tau_\ell})^{1-\alpha} = (\widehat{X}^*_{\tau\wedge\tau_\ell})^{1-\alpha}$. It then follows that

$$\frac{{}^\alpha X_{\tau\wedge\tau_\ell}}{{}^\alpha\widehat{X}_{\tau\wedge\tau_\ell}} = \alpha\left(\frac{X^*_{\tau\wedge\tau_\ell}}{\widehat{X}^*_{\tau\wedge\tau_\ell}}\right)^{1-\alpha} + (1-\alpha)\left(\frac{X_{\tau\wedge\tau_\ell}}{\widehat{X}_{\tau\wedge\tau_\ell}}\right)\left(\frac{X^*_{\tau\wedge\tau_\ell}}{\widehat{X}^*_{\tau\wedge\tau_\ell}}\right)^{-\alpha}. \tag{A.5}$$

Define $\chi := X/\widehat{X}$ and, in the obvious way, $\chi^* := \sup_{t\in[0,\cdot]}(X_t/\widehat{X}_t)$. For $y\in\mathbb{R}_+$, the function $[y,\infty)\ni z\mapsto \alpha z^{1-\alpha}+(1-\alpha)yz^{-\alpha}$ is nondecreasing, which can be shown upon simple differentiation. With $y = \chi_{\tau\wedge\tau_\ell}$, $z_1 = X^*_{\tau\wedge\tau_\ell}/\widehat{X}^*_{\tau\wedge\tau_\ell} = X^*_{\tau\wedge\tau_\ell}/\widehat{X}_{\tau\wedge\tau_\ell} \geq y$ and $z_2 = \chi^*_{\tau\wedge\tau_\ell} \geq X^*_{\tau\wedge\tau_\ell}/\widehat{X}^*_{\tau\wedge\tau_\ell} = z_1$, (A.5) then implies that

$$\frac{{}^\alpha X_{\tau\wedge\tau_\ell}}{{}^\alpha\widehat{X}_{\tau\wedge\tau_\ell}} \leq \alpha\left(\chi^*_{\tau\wedge\tau_\ell}\right)^{1-\alpha} + (1-\alpha)\chi_{\tau\wedge\tau_\ell}\left(\chi^*_{\tau\wedge\tau_\ell}\right)^{-\alpha}.$$

Define the process $\phi := \alpha\left(\chi^*\right)^{1-\alpha} + (1-\alpha)\chi\left(\chi^*\right)^{-\alpha}$; then, by the last estimate and (A.4),

$$\mathsf{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X}) \leq \liminf_{\ell\to\infty}\left(\phi_{\tau\wedge\tau_\ell}\right) - 1.$$

Since $\int_0^\infty \mathbb{I}_{\{\chi_t<\chi_t^*\}}\mathrm{d}\chi_t^* = 0$ a.s. holds, a straightforward use of Itô's formula gives

$$\phi = 1 + \int_0^\cdot (1-\alpha)\left(\chi_t^*\right)^{-\alpha}\mathrm{d}\chi_t;$$

since χ is a local martingale, ϕ is a local martingale as well. Since ϕ is nonnegative, it is a supermartingale with $\phi_0 = 1$, which implies that $\mathbb{E}\left[\phi_{\tau\wedge\tau_\ell}\right] \leq 1$ holds for all $\ell\in\mathbb{R}_+$. It follows that

$$\mathbb{E}\mathsf{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X}) = \mathbb{E}\left[\mathsf{rr}_\tau({}^\alpha X|{}^\alpha\widehat{X})\right] \leq \mathbb{E}\left[\liminf_{\ell\to\infty}\left(\phi_{\tau\wedge\tau_\ell}\right)\right] - 1 \leq \liminf_{\ell\to\infty}\left(\mathbb{E}\left[\phi_{\tau\wedge\tau_\ell}\right]\right) - 1 \leq 0,$$

Now, let σ be a time of maximum of $\widehat{X}$ with $\sigma \le \tau$. Fix $X \in \mathcal{X}$ and $A \in \mathcal{F}_\sigma$; by Lemma A.4, the process $\xi := {}^\alpha\widehat{X}\mathbb{I}_{[\![0,\sigma[\![} + \left({}^\alpha\widehat{X}\mathbb{I}_{\Omega \setminus A} + {}^\alpha\widehat{X}_\sigma \left({}^\alpha X / {}^\alpha X_\sigma\right)\mathbb{I}_A\right)\mathbb{I}_{[\![\sigma,\infty[\![}$ is an element of ${}^\alpha\mathcal{X}$. Furthermore, it is straightforward to check that

$$\mathsf{rr}_\tau(\xi | {}^\alpha\widehat{X}) = \left(\frac{1 + \mathsf{rr}_\tau({}^\alpha X | {}^\alpha\widehat{X})}{1 + \mathsf{rr}_\sigma({}^\alpha X | {}^\alpha\widehat{X})} - 1 \right) \mathbb{I}_{A \cap \{\sigma < \infty\}}.$$

Since $\mathbb{E}\mathsf{rr}_\tau(\xi | {}^\alpha\widehat{X}) \le 0$ has to hold by the result previously established, we obtain

$$\mathbb{E}\left[\frac{1 + \mathsf{rr}_\tau({}^\alpha X | {}^\alpha\widehat{X})}{1 + \mathsf{rr}_\sigma({}^\alpha X | {}^\alpha\widehat{X})} \mathbb{I}_{A \cap \{\sigma < \infty\}} \right] \le \mathbb{P}[A \cap \{\sigma < \infty\}].$$

Since the previous holds for all $A \in \mathcal{F}_\sigma$, we obtain that $\mathbb{E}\left[\mathsf{rr}_\tau({}^\alpha X | {}^\alpha\widehat{X}) \mid \mathcal{F}_\sigma\right] \le \mathsf{rr}_\sigma({}^\alpha X | {}^\alpha\widehat{X})$ holds on $\{\sigma < \infty\}$. On $\{\sigma = \infty\}$, we have $\sigma = \tau$ and $\mathbb{E}\left[\mathsf{rr}_\tau({}^\alpha X | {}^\alpha\widehat{X}) \mid \mathcal{F}_\sigma\right] = \mathsf{rr}_\infty({}^\alpha X | {}^\alpha\widehat{X}) = \mathsf{rr}_\sigma({}^\alpha X | {}^\alpha\widehat{X})$. Therefore, $\mathbb{E}\left[\mathsf{rr}_\tau({}^\alpha X | {}^\alpha\widehat{X}) \mid \mathcal{F}_\sigma\right] \le \mathsf{rr}_\sigma({}^\alpha X | {}^\alpha\widehat{X})$ holds. □

A.4. **Proof of Theorem 3.7.** In the setting of Definition 3.5, consider a sequence $(\xi^n)_{n \in \mathbb{N}}$ of semimartingales and another semimartingale ξ. It is straightforward to check that $\mathcal{S}_{\mathsf{loc}}\text{-}\lim_{n \to \infty} \xi^n = \xi$ holds if and only if there exists a nondecreasing sequence $(\tau_k)_{k \in \mathbb{N}}$ of finitely-valued stopping times with $\mathbb{P}\left[\lim_{k \to \infty} \tau_k = \infty\right] = 1$ such that $\mathcal{S}_{\tau_k}\text{-}\lim_{n \to \infty} \xi^n = \xi$ holds for all $k \in \mathbb{N}$. For the proof of Theorem 3.7, we shall use the previous observation along the sequence $(\tau_\ell)_{\ell \in \mathbb{N}}$ of finitely-valued stopping times defined in (2.5). Therefore, in the course of the proof, we keep $\ell \in \mathbb{R}_+$ fixed and will show that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n \to \infty} {}^\alpha\widetilde{X}^n = {}^\alpha\widehat{X}$.

As a first step, we shall show that $\mathbb{P}\text{-}\lim_{n \to \infty} {}^\alpha\widetilde{X}^n_{\tau_\ell} = {}^\alpha\widehat{X}_{\tau_\ell}$, where "$\mathbb{P}\text{-}\lim$" denotes limit in probability. For each $n \in \mathbb{N}$, consider the process $\xi^n := {}^\alpha\widehat{X}\mathbb{I}_{[\![0,\tau_\ell[\![} + {}^\alpha\widehat{X}_{\tau_\ell}\left({}^\alpha\widetilde{X}^n / {}^\alpha\widetilde{X}^n_{\tau_\ell}\right)\mathbb{I}_{[\![\tau_\ell,\infty[\![}$. By Lemma A.4, $\xi^n \in {}^\alpha\mathcal{X}$ for all $n \in \mathbb{N}$. Furthermore, note that

$$\mathsf{rr}_{T_n}(\xi^n | {}^\alpha\widetilde{X}) = \mathsf{rr}_{\tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n)\mathbb{I}_{\{\tau_\ell < T_n\}} + \mathsf{rr}_{T_n}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n)\mathbb{I}_{\{T_n \le \tau_\ell\}} = \mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n).$$

Using the previous relationship, the assumptions of Theorem 3.7 give $\mathbb{E}\mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n) \le 0$ for all $n \in \mathbb{N}$. Furthermore, by Theorem 2.8, $\mathbb{E}\mathsf{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n | {}^\alpha\widehat{X}) \le 0$ holds for all $n \in \mathbb{N}$. Therefore, $\mathbb{E}\left[\mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n) + \mathsf{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n | {}^\alpha\widehat{X})\right] \le 0$ holds for all $n \in \mathbb{N}$. Observe that the equality $\mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n) + \mathsf{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n | {}^\alpha\widehat{X}) = ({}^\alpha\widetilde{X}^n_{\tau_\ell} - {}^\alpha\widehat{X}_{\tau_\ell})^2 / ({}^\alpha\widehat{X}_{\tau_\ell}{}^\alpha\widetilde{X}^n_{\tau_\ell})$ holds on $\{\tau_\ell < T_n\}$, and that the inequality $\mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n) + \mathsf{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n | {}^\alpha\widehat{X}) \ge -2$ is always true; therefore,

$$\mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n) + \mathsf{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n | {}^\alpha\widehat{X}) \ge \frac{({}^\alpha\widetilde{X}^n_{\tau_\ell} - {}^\alpha\widehat{X}_{\tau_\ell})^2}{{}^\alpha\widehat{X}_{\tau_\ell}{}^\alpha\widetilde{X}^n_{\tau_\ell}}\mathbb{I}_{\{\tau_\ell < T_n\}} - 2\mathbb{I}_{\{T_n \le \tau_\ell\}}.$$

Since $\mathbb{E}\left[\mathsf{rr}_{T_n \wedge \tau_\ell}({}^\alpha\widehat{X} | {}^\alpha\widetilde{X}^n) + \mathsf{rr}_{\tau_\ell}({}^\alpha\widetilde{X}^n | {}^\alpha\widehat{X})\right] \le 0$ holds for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \mathbb{P}\left[T_n \le \tau_\ell\right] = 0$ holds in view of Theorem 1.3, we obtain that

$$\lim_{n \to \infty} \mathbb{E}\left[\frac{({}^\alpha\widetilde{X}^n_{\tau_\ell} - {}^\alpha\widehat{X}_{\tau_\ell})^2}{{}^\alpha\widehat{X}_{\tau_\ell}{}^\alpha\widetilde{X}^n_{\tau_\ell}}\mathbb{I}_{\{\tau_\ell < T_n\}} \right] = 0.$$

Using again the fact that $\lim_{n \to \infty} \mathbb{P}\left[\tau_\ell < T_n\right] = 1$, we obtain that $\mathbb{P}\text{-}\lim_{n \to \infty} {}^\alpha\widetilde{X}^n_{\tau_\ell} = {}^\alpha\widehat{X}_{\tau_\ell}$.

Given $\mathbb{P}\text{-}\lim_{n\to\infty} {}^{\alpha}\widetilde{X}^n_{\tau_\ell} = {}^{\alpha}\widehat{X}_{\tau_\ell}$, we now proceed in showing that $\mathbb{P}\text{-}\lim_{n\to\infty}(\widetilde{X}^n_{\tau_\ell}/\widehat{X}_{\tau_\ell}) = 1$. We use some arguments similar to the first part of the proof of statement (2) of Theorem 2.8, where the reader is referred to for certain details that are omitted here. Define $\chi^n := \widetilde{X}^n/\widehat{X}$ and $(\chi^n)^* := \sup_{t\in[0,\cdot]}(\widetilde{X}^n_t/\widehat{X}_t)$. It then follows that

$$\frac{{}^{\alpha}\widetilde{X}^n_{\tau_\ell}}{{}^{\alpha}\widehat{X}_{\tau_\ell}} \leq \alpha\left((\chi^n)^*_{\tau_\ell}\right)^{1-\alpha} + (1-\alpha)\chi^n_{\tau_\ell}\left((\chi^n)^*_{\tau_\ell}\right)^{-\alpha} =: \phi^n_{\tau_\ell}, \tag{A.6}$$

where the process $\phi^n := \alpha\left((\chi^n)^*\right)^{1-\alpha} + (1-\alpha)\chi^n\left((\chi^n)^*\right)^{-\alpha}$ is a nonnegative local martingale for each $n \in \mathbb{N}$. We claim that $\mathbb{P}\text{-}\lim_{n\to\infty}\phi^n_{\tau_\ell} = 1$. To see this, first observe that $\mathbb{P}\text{-}\liminf_{n\to\infty}\phi^n_{\tau_\ell} \geq 1$ holds, in the sense that $\liminf_{n\to\infty}\mathbb{P}\left[\phi^n_{\tau_\ell} > 1-\epsilon\right] \geq \liminf_{n\to\infty}\mathbb{P}\left[{}^{\alpha}\widetilde{X}^n_{\tau_\ell}/{}^{\alpha}\widehat{X}^n_{\tau_\ell} > 1-\epsilon\right] = 1$ holds for all $\epsilon \in (0,1)$. Then, given that $\mathbb{P}\text{-}\liminf_{n\to\infty}\phi^n_{\tau_\ell} \geq 1$, if $\limsup_{n\to\infty}\mathbb{P}\left[\phi^n_{\tau_\ell} > 1+\epsilon\right] > 0$ was true, one would conclude that $\limsup_{n\to\infty}\mathbb{E}\left[\phi^n_{\tau_\ell}\right] > 1$, which contradicts the fact that $\mathbb{E}\left[\phi^n_{\tau_\ell}\right] \leq \phi^n_0 = 1$ holds for all $n \in \mathbb{N}$. Therefore, $\limsup_{n\to\infty}\mathbb{P}\left[\phi^n_{\tau_\ell} > 1+\epsilon\right] = 0$ holds for all $\epsilon \in (0,1)$, which combined with $\mathbb{P}\text{-}\liminf_{n\to\infty}\phi^n_{\tau_\ell} \geq 1$ gives $\mathbb{P}\text{-}\lim_{n\to\infty}\phi^n_{\tau_\ell} = 1$. To recapitulate, the setting is the following: $(\phi^n)_{n\in\mathbb{N}}$ is a sequence of nonnegative local martingales with $\phi^n_0 = 1$, and $\mathbb{P}\text{-}\lim_{n\to\infty}\phi^n_{\tau_\ell} = 1$ holds. In that case, (Kardaras, 2012, Lemma 2.11) implies that $\mathbb{P}\text{-}\lim_{n\to\infty}(\phi^n)^*_{\tau_\ell} = 1$ holds as well. Note that $(\phi^n)^* = ((\chi^n)^*)^{1-\alpha}$, so that $\mathbb{P}\text{-}\lim_{n\to\infty}(\chi^n)^*_{\tau_\ell} = 1$ holds as well. Then, the bounds in (A.6) imply that $\mathbb{P}\text{-}\lim_{n\to\infty}\chi^n_{\tau_\ell} = 1$.

Once again, we are in the following setting: $(\chi^n)_{n\in\mathbb{N}}$ is a sequence of nonnegative local martingales with $\chi^n_0 = 1$, and $\mathbb{P}\text{-}\lim_{n\to\infty}\chi^n_{\tau_\ell} = 1$ holds. An application of (Kardaras, 2012, Proposition 2.7 and Lemma 2.12) gives that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty}\chi^n = 1$, which also implies that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty}\widetilde{X}^n = \widehat{X}$ by (Kardaras, 2012, Proposition 2.10). This also implies that $\lim_{n\to\infty}\mathbb{P}\big[\sup_{t\in[0,\tau_\ell]}|(\widetilde{X}^n)^*_t - \widehat{X}^*_t| > \epsilon\big] = 0$ also holds for all $\epsilon > 0$ by Remark 3.6. Therefore, by (1.5) and (Kardaras, 2012, Lemma 2.9), we obtain that $\mathcal{S}_{\tau_\ell}\text{-}\lim_{n\to\infty}{}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$, which completes the proof of Theorem 3.7.

Appendix B. A Cautionary Note Regarding Theorem 3.7

In this Section, we elaborate on the point that is made in Remark 3.9 via use of an example. In the discussion that follows, fix $\alpha \in (0,1)$. The model is the general one described in Subsection 1.1, and Assumption 1.2 is always in force.

Let $T_{1/2} = 0$ and, using induction, for $n \in \mathbb{N}$ define

$$T_n := \inf\{t \in (T_{n-1/2}, \infty) \mid \widehat{X}_t = \alpha\widehat{X}^*_t\}, \quad T_{n+1/2} := \inf\{t \in (T_n, \infty) \mid \widehat{X}_t = \widehat{X}^*_{T_n}\}.$$

(In the setting of Example 2.6, T there is exactly T_1 defined above.) Note the following: $T_{n-1/2}$ is a time of maximum of $\widehat{X}$ for all $n \in \mathbb{N}$, $(T_{k/2})_{k\in\mathbb{N}}$ is an increasing sequence, and $\mathbb{P}\left[\lim_{n\to\infty}T_n = \infty\right] = 1$ holds. Under Assumption 1.2, Lemma A.1 implies that $\mathbb{P}\left[T_n < \infty\right] = 1$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, one can explicitly describe the wealth process ${}^{\alpha}\widetilde{X}^n$ that has the numéraire property in the class ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T_n]$. In words, ${}^{\alpha}\widetilde{X}^n$ will follow ${}^{\alpha}\widehat{X}$ until time $T_{n-1/2}$, then switch to investing like the numéraire portfolio $\widehat{X}$ up to time T_n and, since at time T_n one hits the hard drawdown constraint, ${}^{\alpha}\widetilde{X}^n$ will remain constant from T_n onwards. In mathematical terms, define

$$\begin{aligned} {}^{\alpha}\widetilde{X}^n := {}& {}^{\alpha}\widehat{X}\mathbb{I}_{[\![0,T_{n-1/2}[\![} + \left(\frac{{}^{\alpha}\widehat{X}_{T_{n-1/2}}}{\widehat{X}_{T_{n-1/2}}}\right)\widehat{X}\mathbb{I}_{[\![T_{n-1/2},T_n[\![} + \left(\frac{{}^{\alpha}\widehat{X}_{T_{n-1/2}}}{\widehat{X}_{T_{n-1/2}}}\right)\widehat{X}_{T_n}\mathbb{I}_{[\![T_n,\infty[\![} \\ = {}& {}^{\alpha}\widehat{X}\mathbb{I}_{[\![0,T_{n-1/2}[\![} + \left(\widehat{X}_{T_{n-1/2}}\right)^{-\alpha}\widehat{X}\mathbb{I}_{[\![T_{n-1/2},T_n[\![} + \left(\widehat{X}_{T_{n-1/2}}\right)^{-\alpha}\alpha\widehat{X}^*_{T_n}\mathbb{I}_{[\![T_n,\infty[\![}, \end{aligned}$$

where for the equality in the second line the facts that ${}^{\alpha}\widehat{X}_{T_{n-1/2}} = (\widehat{X}_{T_{n-1/2}})^{1-\alpha}$ and $\widehat{X}_{T_n} = \alpha\widehat{X}^*_{T_n}$ were used. It is straightforward to check that ${}^{\alpha}\widetilde{X} \in {}^{\alpha}\mathcal{X}$, in view of the definition of the stopping times $(T_{k/2})_{k\in\mathbb{N}}$. Pick any $X \in \mathcal{X}$. The global (in time) numéraire property of $\widehat{X}$ in $\mathcal{X}$ will give

$$\mathbb{E}\left[\frac{{}^{\alpha}X_{T_n}}{{}^{\alpha}\widetilde{X}^n_{T_n}} - 1 \,\middle|\, \mathcal{F}_{T_{n-1/2}}\right] \leq \frac{{}^{\alpha}X_{T_{n-1/2}}}{{}^{\alpha}\widetilde{X}^n_{T_{n-1/2}}} - 1 = \frac{{}^{\alpha}X_{T_{n-1/2}}}{{}^{\alpha}\widehat{X}^n_{T_{n-1/2}}} - 1.$$

Upon taking expectation on both sides of the previous inequality, we obtain $\mathbb{E}\mathsf{rr}_{T_n}({}^{\alpha}X|{}^{\alpha}\widetilde{X}^n) \leq \mathbb{E}\mathsf{rr}_{T_{n-1/2}}({}^{\alpha}X|{}^{\alpha}\widehat{X}^n) \leq 0$, the last inequality holding in view of statement (2) of Theorem 2.8, given that $T_{n-1/2}$ is a time of maximum of $\widehat{X}$. We have shown that ${}^{\alpha}\widetilde{X}^n$ indeed has the numéraire property in the class ${}^{\alpha}\mathcal{X}$ for investment over the interval $[0, T_n]$.

Note that ${}^{\alpha}\widetilde{X}^n = {}^{\alpha}\widehat{X}$ identically holds in the stochastic interval $[\![0, T_{n-1/2}]\!]$ for each $n \in \mathbb{N}$; therefore, the conclusion of Theorem 3.7 in this case is valid in a quite strong sense. However, the behavior of ${}^{\alpha}\widetilde{X}^n$ and ${}^{\alpha}\widehat{X}$ in the stochastic interval $[\![T_{n-1/2}, T_n]\!]$ is different and results in quite diverse outcomes at time T_n, as we shall now show. At time T_n one has

$${}^{\alpha}\widehat{X}_{T_n} = \alpha(\widehat{X}^*_{T_n})^{1-\alpha} + (1-\alpha)(\widehat{X}^*_{T_n})^{-\alpha}\widehat{X}_{T_n} = \alpha(2-\alpha)(\widehat{X}^*_{T_n})^{1-\alpha},$$

where the fact that $\widehat{X}_{T_n} = \alpha\widehat{X}^*_{T_n}$ was again used. Furthermore, ${}^{\alpha}\widetilde{X}^n_{T_n} = (\widehat{X}_{T_{n-1/2}})^{-\alpha}\alpha\widehat{X}^*_{T_n}$. It then follows that

$$\frac{{}^{\alpha}\widetilde{X}^n_{T_n}}{{}^{\alpha}\widehat{X}_{T_n}} = \frac{(\widehat{X}_{T_{n-1/2}})^{-\alpha}\alpha\widehat{X}^*_{T_n}}{\alpha(2-\alpha)(\widehat{X}^*_{T_n})^{1-\alpha}} = \frac{1}{2-\alpha}\left(\frac{\widehat{X}^*_{T_n}}{\widehat{X}_{T_{n-1/2}}}\right)^{\alpha} =: \zeta_n.$$

In view of Assumption 1.2 and the result of Dambis, Dubins and Schwarz (Karatzas and Shreve, 1991, Theorem 3.4.6), the law of the random variable ζ_n is the same for all $n \in \mathbb{N}$. In fact, universal distributional properties of the maximum of a non-negative local martingale stopped at first hitting time, see (Carraro et al., 2012, Proposition 4.3), imply that $\zeta_n = (2-\alpha)^{-1}(\alpha + (1-\alpha)\,(1/\eta_n))^{\alpha}$, where η_n has the uniform law on $(0, 1)$. In particular, $\mathbb{P}[\zeta_n < (2-\alpha)^{-1} + \epsilon] > 0$ and $\mathbb{P}[\zeta_n > (2-\alpha)^{-1} + \epsilon^{-1}] > 0$ holds for all $\epsilon \in (0, 1)$. Furthermore, ζ_n is $\mathcal{F}_{T_n}$-measurable and independent of $\mathcal{F}_{T_{n-1/2}} \supseteq \mathcal{F}_{T_{n-1}}$ for each $n \in \mathbb{N}$, which implies that $(\zeta_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables. By an application of the second Borel-Cantelli lemma, it follows that

$$\frac{1}{2-\alpha} = \liminf_{n\to\infty}\left(\frac{{}^{\alpha}\widetilde{X}^n_{T_n}}{{}^{\alpha}\widehat{X}_{T_n}}\right) < \limsup_{n\to\infty}\left(\frac{{}^{\alpha}\widetilde{X}^n_{T_n}}{{}^{\alpha}\widehat{X}_{T_n}}\right) = \infty,$$

demonstrating the claim made at Remark 3.9.

References

Bansal, R. and Lehmann, B. N. (1997), 'Growth-optimal portfolio restrictions on asset pricing models', *Macroeconomic Dynamics* **1**(02), 333–354.

Carraro, L., El Karoui, N. and Obłój, J. (2012), 'On Azéma-Yor processes, their optimal properties and the Bachelier-Drawdown equation', *The Annals of Probability* **40**(1), 372–400.

Chekhlov, A., Uryasev, S. and Zabarankin, M. (2005), 'Drawdown measure in portfolio optimization', *International J. Theoretical and Applied Finance* **8**(1), 13–58.

Cherny, V. and Obłój, J. (2011), Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model. arXiv:1110.6289.

Cvitanić, J. and Karatzas, I. (1994), 'On portfolio optimization under "draw-down" constraints', *IMA Volumes in Mathematics and Its Applications* **65**(3), 35–46.

Delbaen, F. and Schachermayer, W. (1994), 'A general version of the fundamental theorem of asset pricing', *Math. Ann.* **300**(3), 463–520.

Elie, R. (2008), 'Finite time Merton strategy under drawdown constraint: A viscosity solution approach', *Applied Mathematics & Optimization* **58**(3), 411–431.

Elie, R. and Touzi, N. (2008), 'Optimal lifetime consumption and investment under a drawdown constraint', *Finance and Stochastics* **12**(3), 299–330.

Emery, M. (1979), 'Une topologie sur l'espace des semimartingales', **721**, 260–280.

Föllmer, H. and Kramkov, D. (1997), 'Optional decompositions under constraints', *Probab. Theory Related Fields* **109**(1), 1–25.

Grossman, S. J. and Zhou, Z. (1993), 'Optimal investment strategies for controlling drawdowns', *Mathematical Finance* **3**(3), 241–276.

Guasoni, P. and Obłój, J. (2011), The incentives of hedge funds fees and high-water marks. submitted, available at `http://ssrn.com/abstract=1784429`.

Karatzas, I. and Kardaras, C. (2007), 'The numéraire portfolio in semimartingale financial models', *Finance Stoch.* **11**(4), 447–493.

Karatzas, I. and Shreve, S. E. (1991), *Brownian motion and stochastic calculus*, Vol. 113 of *Graduate Texts in Mathematics*, second edn, Springer-Verlag, New York.

Kardaras, C. (2010*a*), 'Finitely additive probabilities and the fundamental theorem of asset pricing', *Contemporary Quantitative Finance (Platen Festschrift)* pp. 19–34.

Kardaras, C. (2010*b*), 'Numéraire-invariant preferences in financial modeling', *Ann. Appl. Probab.* **20**(5), 1697–1728.

Kardaras, C. (2012), On the closure in the Emery topology of semimartingale wealth-process sets. Forthcoming in Ann. Appl. Probab., available at `http://arxiv.org/abs/1108.0945`.

Kelly, Jr, J. L. (1956), 'A new interpretation of information rate', *Bell System Technical Journal* **35**, 917–926.

Kramkov, D. and Schachermayer, W. (1999), 'The asymptotic elasticity of utility functions and optimal investment in incomplete markets', *Ann. Appl. Probab.* **9**(3), 904–950.

Leland, H. (1972), On turnpike portfolios, *in* K. S. G. Szego, ed., 'Mathematical methods in investment and finance', North-Holland, Amsterdam.

MacLean, L. C., Thorp, E. O. and Ziemba, W. T., eds (2011), *The Kelly Capital Growth Investment Criterion: Theory and Practice*, World Scientific: Singapore.

Magdon-Ismail, M. and Atiya, A. (2004), 'Maximum drawdown', *Risk* **17**(10), 99–102.

Markowitz, H. M. (2006), Samuelson and investment for the long run, *in* M. Szenberg, L. Ramrattan and A. A. Gottesman, eds, 'Samuelsonian Economics and the Twenty-First Century', Oxford University Press, Oxford, pp. 252–261.

Merton, R. C. (1971), 'Optimum consumption and portfolio rules in a continuous-time model', *Journal of Economic Theory* **3**, 373–413.

Musiela, M. and Zariphopoulou, T. (2009), 'Portfolio choice under dynamic investment performance criteria', *Quantitative Finance* **9**(2), 161–170.

Pirvu, T. A. and Žitković, G. (2009), 'Maximizing the growth rate under risk constraints', *Math. Finance* **19**(3), 423–455.

Platen, E. and Heath, D. (2006), *A benchmark approach to quantitative finance*, Springer Finance, Springer-Verlag, Berlin.

Samuelson, P. A. (1979), 'Why we should not make mean log of wealth big though years to act are long', *Journal of Banking and Finance* **3**(4), 305–307.

Stricker, C. and Yan, J. A. (1998), Some remarks on the optional decomposition theorem, *in* 'Séminaire de Probabilités, XXXII', Vol. 1686 of *Lecture Notes in Math.*, Springer, Berlin, pp. 56–66.

Vecer, J. (2006), 'Maximum drawdown and directional trading', *Risk* **19**(12), 88–92.

Žitković, G. (2002), 'A filtered version of the bipolar theorem of Brannath and Schachermayer', *J. Theoret. Probab.* **15**(1), 41–61.

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