Second Order BSDEs with Jumps, Part II: Existence and Applications

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Abstract

In this paper, we follow the study of second order BSDEs with jumps started in our accompanying paper [17]. We prove existence of these equations by a direct method, thus providing complete wellposedness for second order BSDEs. These equations are the natural candidates for the probabilistic interpretation of fully non-linear partial integro-differential equations, which is the point of our paper [18]. Finally, we give an application of second order BSDEs to the study of a robust exponential utility maximization problem under model uncertainty. The uncertainty affects both the volatility process and the jump measure compensator. We prove existence of an optimal strategy, and that the value function of the problem is the unique solution of a particular second order BSDE with jumps.

Key words: Second order backward stochastic differential equation, backward stochastic differential equation with jumps

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1 Introduction

Motivated by duality methods and maximum principles for optimal stochastic control, Bismut studied in [4] a linear backward stochastic differential equation (BSDE). In their seminal paper [23], Pardoux and Peng generalized such equations to the non-linear Lipschitz case and proved existence and uniqueness results in a Brownian framework. Since then, a lot of attention has been given to BSDEs and their applications, not only in stochastic control, but also in theoretical economics, stochastic differential games and financial mathematics. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion B, solving a BSDE with generator g, and terminal condition ξ consists in finding a pair of progressively measurable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ \mathbb{P}-a.s,\ t \in [0, T]. \tag{1.1}$$

The process Y we define this way is a possible generalization of the conditional expectation of ξ, since when g is the null function, we have $Y_t = \mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t]$, and in that case, Z is the process appearing in the $(\mathcal{F}_t)$-martingale representation property of $\{\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t], t \geq 0\}$. In the case of a filtered probability space generated by both a Brownian motion B and a Poisson random measure μ with compensator ν, the martingale representation for $\{\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t], t \geq 0\}$ becomes

$$\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t] = \int_0^t Z_s dB_s + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \psi_s(x)(\mu - \nu)(ds, dx),$$

where ψ is a predictable function. This leads to the following natural generalization of equation (1.1) to the case with jumps. We will say that (Y, Z, U) is a solution of the BSDE with generator g and terminal condition ξ if for all $t \in [0, T]$, we have $\mathbb{P} - a.s.$

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_t^T \int_{\mathbb{R}^d\setminus\{0\}} U_s(x)(\mu - \nu)(ds, dx). \tag{1.2}$$

Li and Tang [30] were the first to prove existence and uniqueness of a solution for (1.2) in the case where g is Lipschitz in (y, z, u). In the continuous framework, Soner, Touzi and Zhang [26] generalized the BSDE (1.1) to the second order case. Their key idea in the definition of the second order BSDE is that the equation has to hold $\mathbb{P}$-almost surely, for every $\mathbb{P}$ in a class of non dominated probability measures. Furthermore, they proved a uniqueness result using a representation result of the 2BSDEs as essential supremum of standard BSDEs.

In [17], we extended these definitions and properties to the case with jumps. For this purpose, we first proved that if $\{X^{a,\nu},\ (a, \nu) \in \mathcal{A}\}$ is a family of processes on the Skorohod space $\mathbb{D}$ indexed by volatility processes and jump measures compensators belonging to some admissible space $\mathcal{A}$, then there exists a unique process X such that $X = X^{a,\nu}$, $\mathbb{P} - a.s.$, for any measure $\mathbb{P}$ in a given space of mutually singular probability measures. In other words, we extended to the space $\mathbb{D}$ the aggregation result proved in [28] (we refer to [17] for more details).

This allowed us to find an aggregated version $\widehat{\nu}$ of the compensator of the jump measure of the canonical process. On the other hand, it is always possible to define an aggregated version $\widehat{a}$ of the density with respect to the Lebesgue measure on $\mathbb{R}^+$ of the quadratic variation of the continuous part of the canonical process. Equipped with the pair $(\widehat{a}, \widehat{\nu})$, we were able to define a generator $\widehat{F}$ depending on both $\widehat{a}$ and $\widehat{\nu}$, and to give a good formulation of second order BSDEs with jumps (see equation (3.3)). With this definition, we proved some a priori estimates and a uniqueness result.

Our aim in this paper is to pursue the study undertaken in [17]. More precisely, we prove existence of a solution to equation (3.3) by a direct approach. We construct a solution inspired by the representation obtained in Theorem 4.1 of [17], and using the tool of regular conditional probability distributions. This gives complete wellposedness for second order BSDEs with jumps. As an application of these results, in the spirit of [12] and [20], we treat a robust exponential utility maximization problem in a market with jumps, under model uncertainty. The uncertainty affects both the volatility process and the jump measure compensator. We prove existence of an optimal strategy, and that the value function of the problem is the unique solution of a particular second order BSDE with jumps.

The last part of our study is to establish a connection with partial integro-differential equations (PIDEs for short). Indeed, Soner, Touzi and Zhang proved in [26] that Markovian second order BSDEs, are connected in the continuous case to a class of parabolic fully non-linear PDEs. On the other hand, we know that solutions to standard Markovian BSDEJs provide viscosity solutions to some parabolic partial integro-differential equations whose non local operator is given by a quantity similar to $< \tilde{v}, \nu >$ defined in (3.1) (see [1] for more details). Then in the Markovian case, second order BSDEs are the natural candidate for the probabilistic interpretation of fully non-linear PIDEs. This is the purpose of our paper [18].

The rest of the paper is organized as follows. In Section 2, we give some preliminaries on the set of probability measures on the Skorohod space $\mathbb{D}$ that we will work with. In Section 3, we introduce the generator of our BSDEs and the assumptions we make, we recall from [17] the spaces in which we look for a solution, and give the formulation of the second order BSDEs with jumps. Section 4 is devoted to the proof of our existence result. Finally, in Section 5, we give an application to the resolution of a robust utility maximization problem. The Appendix is dedicated to the proof of some important technical results needed throughout the paper.

2 Preliminaries on probability measures

2.1 The stochastic basis

Let $\Omega := \mathbb{D}([0,T], \mathbb{R}^d)$ be the space of càdlàg paths defined on $[0,T]$ with values in R^d and such that $w(0) = 0$, equipped with the Skorokhod topology, so that it is a complete, separable metric space (see [3] for instance). The uniform norm on Ω is defined by $\|\omega\|_\infty := \sup_{0 \leq t \leq T} |\omega_t|$.

We denote B the canonical process, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B, $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0\leq t\leq T}$ the right limit of $\mathbb{F}$. We then define as in [26] a local martingale measure $\mathbb{P}$ as a probability measure such that B is a $\mathbb{P}$-local martingale. Since we are working in the Skorohod space, we can then define the continuous martingale part of B, noted B^c, and its purely discontinuous part, noted B^d, both being local martingales under each local martingale measures (see [13]). We then associate to the jumps of B a counting measure μ_{B^d}, which is a random measure on $\mathcal{B}(\mathbb{R}^+) \times E$ (where $E := \mathbb{R}^r\backslash\{0\}$ for some $r \in \mathbb{N}^*$), defined pathwise by

$$\mu_{B^d}([0,t], A) := \sum_{0<s\leq t} \mathbf{1}_{\{\Delta B^d_s \in A\}},\ \forall t\geq 0,\ \forall A\subset E. \tag{2.1}$$

We also denote by $\nu_t^{\mathbb{P}}(dt,dx)$ the compensator of $\mu_{B^d}(dt,dx)$, which is a predictable random measure, under $\mathbb{P}$ and by $\widetilde{\mu}^{\mathbb{P}}_{B^d}(dt,dx)$ the corresponding compensated measure.

We then denote $\overline{\mathcal{P}}_W$ the set of all local martingale measures $\mathbb{P}$ such that $\mathbb{P}$-a.s.

(i) The quadratic variation of B^c is absolutely continuous with respect to the Lebesgue measure dt and its density takes values in $\mathbb{S}_d^{>0}$.

(ii) The compensator $\nu_t^{\mathbb{P}}(dt,dx)$ under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure dt.

2.2 Martingale problems and probability measures

In this section, we recall the families of probability measures introduced in [17]. Let $\mathcal{N}$ be the set of $\mathbb{F}$-predictable random measures ν on $\mathcal{B}(E)$ satisfying

$$\int_0^t\int_E (1\wedge|x|^2)\nu_s(dx)ds < +\infty \text{ and } \int_0^t\int_{|x|>1} x\nu_s(dx)ds < +\infty,\ \forall\omega\in\Omega, \tag{2.2}$$

and let $\mathcal{D}$ be the set of $\mathbb{F}$-predictable processes α in $\mathbb{S}_d^{>0}$ with $\int_0^T|\alpha_t|dt < +\infty,\ \forall\ \omega\in\Omega$.

We define a martingale problem as follows

Definition 2.1. *For $\mathbb{F}$-stopping times τ_1 and τ_2, for $(\alpha,\nu)\in\mathcal{D}\times\mathcal{N}$ and for a probability measure $\mathbb{P}_1$ on $\mathcal{F}_{\tau_1}$, we say that $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}_1,\tau_1,\tau_2,\alpha,\nu)$ if*

(i) $\mathbb{P} = \mathbb{P}_1$ *on* $\mathcal{F}_{\tau_1}$.

(ii) *The canonical process B on $[\tau_1,\tau_2]$ is a semimartingale under $\mathbb{P}$ with characteristics*

$$\left(-\int_{\tau_1}^{\cdot}\int_E x\mathbf{1}_{|x|>1}\nu_s(dx)ds,\int_{\tau_1}^{\cdot}\alpha_s ds,\ \nu_s(dx)ds\right).$$

Remark 2.1. *We refer to Theorem $II.2.21$ in [13] for the fact that $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}_1,\tau_1,\tau_2,\alpha,\nu)$ if and only if the following properties hold:*

(i) $\mathbb{P} = \mathbb{P}_1$ *on* $\mathcal{F}_{\tau_1}$.

(ii) *The processes M, J and L defined below are $\mathbb{P}$-local martingales on $[\tau_1, \tau_2]$*

$$M_t := B_t - \sum_{\tau_1 \le s \le t} \mathbf{1}_{|\Delta B_s|>1} \Delta B_s + \int_{\tau_1}^t \int_E x \mathbf{1}_{|x|>1} \nu_s(dx) ds, \; \tau_1 \le t \le \tau_2$$

$$J_t := M_t^2 - \int_{\tau_1}^t \alpha_s ds - \int_{\tau_1}^t \int_E x^2 \nu_s(dx) ds, \; \tau_1 \le t \le \tau_2$$

$$Q_t := \int_{\tau_1}^t \int_E g(x) \mu_B(ds, dx) - \int_{\tau_1}^t \int_E g(x) \nu_s(dx) ds, \; \tau_1 \le t \le \tau_2, \; \forall g \in \mathcal{C}^+(\mathbb{R}^d),$$

where $\mathcal{C}^+(\mathbb{R}^d)$ is discriminating family of bounded Borel functions (see Remark $II.2.20$ in [13] for more details).

We say that the martingale problem associated to (α, ν) has a unique solution if, for every stopping times τ_1, τ_2 and for every probability measure $\mathbb{P}_1$, the martingale problem $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ has a unique solution.

Let now $\overline{\mathcal{A}}_W$ be the set of $(\alpha, \nu) \in \mathcal{D} \times \mathcal{N}$, such that there exists a solution to the martingale problem $(\mathbb{P}^0, 0, +\infty, \alpha, \nu)$, where $\mathbb{P}^0$ is such that $\mathbb{P}^0(B_0 = 0) = 1$. We also denote by $\mathcal{A}_W$ the set of $(\alpha, \nu) \in \overline{\mathcal{A}}_W$ such that there exists a unique solution to the martingale problem $(\mathbb{P}_1, 0, +\infty, \alpha, \nu)$, where $\mathbb{P}_1$ is such that $\mathbb{P}_1(B_0 = 0) = 1$.

We now recall the so-called strong formulation in our context. For this purpose, we define

$$\mathcal{V} := \{\nu \in \mathcal{N}, \; (I_d, \nu) \in \mathcal{A}_W\}.$$

For each $\nu \in \mathcal{V}$, we denote $\mathbb{P}^\nu := \mathbb{P}_\nu^{I_d}$ and for each $\alpha \in \mathcal{D}$, we define

$$\mathbb{P}^{\alpha,\nu} := \mathbb{P}^\nu \circ (X_\cdot^\alpha)^{-1}, \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s^c + B_t^d, \; \mathbb{P}^\nu - a.s. \tag{2.3}$$

Let us now define $\mathcal{P}_S := \{\mathbb{P}^{\alpha,\nu}, \; (\alpha, \nu) \in \mathcal{A}_W\}$. We also define for each $\mathbb{P} \in \mathcal{P}_W$ the following process

$$L_t^{\mathbb{P}} := W_t^{\mathbb{P}} + B_t^d, \; \mathbb{P} - a.s., \tag{2.4}$$

where $W_t^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion defined by

$$W_t^{\mathbb{P}} := \int_0^t \widehat{a}_s^{-1/2} dB_s^c.$$

Then, we have by definition that the $\mathbb{P}^{\alpha,\nu}$-distribution of $(B, \widehat{a}, \widehat{\nu}, L^{\mathbb{P}^{\alpha,\nu}})$ is equal to the $\mathbb{P}^\nu$-distribution of $(W^\alpha, \alpha, \nu, B)$. Let us now consider the sets

$$\tilde{\mathcal{A}}_0 := \{(\alpha, \nu) \in \mathcal{D} \times \mathcal{N} \text{ which are deterministic}\}, \; \mathcal{P}_{\tilde{\mathcal{A}}_0} := \{\mathbb{P}_\nu^\alpha, \; (\alpha, \nu) \in \mathcal{A}_0\}.$$

$\tilde{\mathcal{A}}_0$ is a generating class of coefficients in the sense of [17]. Consider $\tilde{\mathcal{A}}$ the separable class of coefficients generated by $\tilde{\mathcal{A}}_0$ and $\mathcal{P}_{\tilde{\mathcal{A}}}$ the corresponding set of probability measures (we refer once more the reader to [17] for the precise definition). One of the main results of [17] is then

Proposition 2.1. *$\mathcal{P}_{\tilde{\mathcal{A}}} \subset \mathcal{P}_S$ and every probability measure in $\mathcal{P}_{\tilde{\mathcal{A}}}$ satisfies the martingale representation property and the $0-1$ Blumenthal law.*

Remark 2.2. *In our jump framework, we need to impose this separability structure on both α and ν, in order to be able to retrieve not only the aggregation result but also the property that all our probability measures satisfy the Blumenthal $0-1$ law and the martingale representation property. However, if one is only interested in being able to consider a standard BSDE with jumps under each $\mathbb{P}$ in a set of probability measures, then we do not need the aggregation result and we can work with a set larger than $\mathcal{P}_{\tilde{\mathcal{A}}}$. Namely, let us define*

$$\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}} := \left\{ \mathbb{P}^{\alpha,\nu}, \ \alpha \in \mathcal{D}, \ (I_d, \nu) \in \widetilde{\mathcal{A}} \right\}.$$

Then we can show as above that $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}} \subset \mathcal{P}_S$ and that all the probability measures in $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}}$ satisfy the Blumenthal $0-1$ law and the martingale representation property. This is going to be useful for us in Section 4.4.

3 Preliminaries on 2BSDEs

3.1 The non-linear Generator

In this subsection we will introduce the function which will serve as the generator of our 2BSDE with jumps. Let us define the spaces

$$\hat{L}^2 := \bigcap_{\nu \in \mathcal{N}} L^2(\nu) \text{ and } \hat{L}^1 := \bigcap_{\nu \in \mathcal{N}} L^1(\nu).$$

For any $\mathcal{C}^1$ function v with bounded gradient, any $\omega \in \Omega$ and any $0 \leq t \leq T$, we denote by $\tilde{v}$

$$\tilde{v}(e) := v(e + \omega(t)) - v(\omega(t)) - \mathbf{1}_{\{|e| \leq 1\}} \, e.(\nabla v)(\omega(t)), \text{ for } e \in E.$$

The hypothesis on v ensure that $\tilde{v}$ is an element of $\hat{L}^1$. We then consider a map

$$H_t(\omega, y, z, u, \gamma, \tilde{v}) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \hat{L}^2 \times D_1 \times D_2 \to \mathbb{R},$$

where $D_1 \subset \mathbb{R}^{d \times d}$ is a given subset containing 0 and $D_2 \subset \hat{L}^1 \cap \mathcal{N}^*$, where $\mathcal{N}^*$ denotes the topological dual of $\mathcal{N}$.

Define the following conjugate of H with respect to γ and v by

$$F_t(\omega, y, z, u, a, \nu) := \sup_{\{\gamma, \tilde{v}\} \in D_1 \times D_2} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) + < \tilde{v}, \nu > - H_t(\omega, y, z, u, \gamma, \tilde{v}) \right\},$$

for $a \in \mathbb{S}_d^{>0}$ and $\nu \in \mathcal{N}$, and where $< \tilde{v}, \nu >$ is defined by

$$< \tilde{v}, \nu > := \int_E \tilde{v}(e) \nu(de). \tag{3.1}$$

The quantity $< \tilde{v}, \nu >$ will not appear again in the paper, since we formulate the needed hypothesis for the backward equation generator directly on the function F. But the particular form of $< \tilde{v}, \nu >$ comes from the intuition that the second order BSDE with jumps is an essential supremum of classical BSDEs with jumps (BSDEJs). Indeed, solutions to markovian BSDEJs provide viscosity solutions to some parabolic partial integro-differential equations whose non local operator is given by a quantity similar to $< \tilde{v}, \nu >$ (see [1] for more details). We define

$$\widehat{F}_t(y,z,u) := F_t(y,z,u,\widehat{a}_t,\widehat{\nu}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0,0,0),\ \mathcal{P}_{\tilde{\mathcal{A}}}\text{-q.s.} \tag{3.2}$$

We denote by $D^1_{F_t(y,z,u)}$ the domain of F in a and by $D^2_{F_t(y,z,u)}$ the domain of F in ν, for a fixed (t,ω,y,z,u). As in [26] we fix a constant $\kappa \in (1,2]$ and restrict the probability measures in $\mathcal{P}^\kappa_H \subset \mathcal{P}_{\tilde{\mathcal{A}}}$

Definition 3.1. *$\mathcal{P}^\kappa_H$ consists of all $\mathbb{P} \in \mathcal{P}_{\tilde{\mathcal{A}}}$ such that*

(i) $\mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_E x^2 \widehat{\nu}_t(dx)dt\right] < +\infty.$

(ii) $\underline{a}^{\mathbb{P}} \leq \widehat{a} \leq \bar{a}^{\mathbb{P}},\ dt \times d\mathbb{P} - as\ for\ some\ \underline{a}^{\mathbb{P}}, \bar{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0},\ and\ \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^\kappa dt\right)^{\frac{2}{\kappa}}\right] < +\infty.$

Remark 3.1. *The above conditions assumed on the probability measures in $\mathcal{P}^\kappa_H$ ensure that under any $\mathbb{P} \in \mathcal{P}^\kappa_H$, the canonical process B is actually a true càdlàg martingale. This will be important when we will define standard BSDEs under each of those probability measures.*

We now state our main assumptions on the function F

Assumption 3.1. (i) The domains $D^1_{F_t(y,z,u)} = D^1_{F_t}$ and $D^2_{F_t(y,z,u)} = D^2_{F_t}$ are independent of (ω, y, z, u).

(ii) For fixed (y,z,a,ν), F is $\mathbb{F}$-progressively measurable in $D^1_{F_t} \times D^2_{F_t}$.

(iii) The following uniform Lipschitz-type property holds. For all $(y,y',z,z',u,t,a,\nu,\omega)$

$$\left|F_t(\omega,y,z,u,a,\nu) - F_t(\omega,y',z',u,a,\nu)\right| \leq C\left(\left|y-y'\right| + \left|a^{1/2}\left(z-z'\right)\right|\right).$$

(iv) For all $(t,\omega,y,z,u^1,u^2,a,\nu)$, there exist two processes γ and γ' such that

$$\int_E \delta^{1,2}u(x)\gamma'_t(x)\nu(dx) \leq F_t(\omega,y,z,u^1,a,\nu) - F_t(\omega,y,z,u^2,a,\nu) \leq \int_E \delta^{1,2}u(x)\gamma_t(x)\nu(dx),$$

where $\delta^{1,2}u := u^1 - u^2$ and $c_1(1 \wedge |x|) \leq \gamma_t(x) \leq c_2(1 \wedge |x|)$ with $-1+\delta \leq c_1 \leq 0,\ c_2 \geq 0$, and $c'_1(1 \wedge |x|) \leq \gamma'_t(x) \leq c'_2(1 \wedge |x|)$ with $-1+\delta \leq c'_1 \leq 0,\ c'_2 \geq 0$, for some $\delta > 0$.

(v) F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

3.2 The Spaces and Norms

We now define as in [26], the spaces and norms which will be needed for the formulation of the second order BSDEs.

For $p \geq 1$, $L^{p,\kappa}_H$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|^p_{L^{p,\kappa}_H} := \sup_{\mathbb{P} \in \mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-predictable $\mathbb{R}^d$-valued predictable processes Z with

$$\|Z\|^p_{\mathbb{H}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \|Y\|^p_{\mathbb{D}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t|^p\right] < +\infty.$$

$\mathbb{J}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-predictable functions U with

$$\|U\|^p_{\mathbb{J}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \int_E |U_s(x)|^2\, \widehat{\nu}_t(dx)ds\right)^{\frac{p}{2}}\right] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $t\in[0,T]$ denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \left\{\mathbb{P}'\in\mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

Then we define for each $p\geq\kappa$,

$$\mathbb{L}_H^{p,\kappa} := \left\{\xi\in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\right\} \text{ where } \|\xi\|^p_{\mathbb{L}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup^{\mathbb{P}}}_{0\leq t\leq T}\left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi:\Omega\to\mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1<\kappa\leq p.$$

For a given probability measure $\mathbb{P}\in\mathcal{P}_H^\kappa$, the spaces $L^p(\mathbb{P})$, $\mathbb{D}^p(\mathbb{P})$, $\mathbb{H}^p(\mathbb{P})$ and $\mathbb{J}^p(\mathbb{P})$ correspond to the above spaces when the set of probability measures is only the singleton $\{\mathbb{P}\}$. Finally, we have $\mathbb{H}^p_{loc}(\mathbb{P})$ denotes the space of all $\mathbb{F}^+$-predictable $\mathbb{R}^d$-valued processes Z with

$$\left(\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right)^{\frac{p}{2}} < +\infty,\ \mathbb{P}-a.s.$$

$\mathbb{J}^p_{loc}(\mathbb{P})$ denotes the space of all $\mathbb{F}^+$-predictable functions U with

$$\left(\int_0^T \int_E |U_s(x)|^2\, \widehat{\nu}_t(dx)ds\right)^{\frac{p}{2}} < +\infty,\ \mathbb{P}-a.s.$$

3.3 Formulation

We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0\leq t\leq T$ and $\mathcal{P}_H^\kappa$-q.s.

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s,Z_s,U_s)ds - \int_t^T Z_s dB_s^c - \int_t^T\int_E U_s(x)\tilde{\mu}_{B^d}(ds,dx) + K_T - K_t. \quad (3.3)$$

Definition 3.2. *We say* $(Y, Z, U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ *is a solution to 2BSDEJ* (3.3) *if*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa$*-q.s.*
- *For all* $\mathbb{P} \in P_H^\kappa$ *and* $0 \le t \le T$, *the process* $K^{\mathbb{P}}$ *defined below is predictable and has non-decreasing paths* $\mathbb{P} - a.s.$

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s, U_s)ds + \int_0^t Z_s dB_s^c + \int_0^t \int_E U_s(x)\tilde{\mu}_{B^d}(ds, dx). \quad (3.4)$$

- *The family* $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ *satisfies the minimum condition*

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right], \ 0 \le t \le T, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \quad (3.5)$$

Moreover if the family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ *can be aggregated into a universal process* K, *we call* (Y, Z, U, K) *a solution of the 2BSDEJ* (3.3).

Remark 3.2. *Since with our set* $\mathcal{P}_H^\kappa$ *we have the aggregation property of Theorem* 2.1 *in [17], and since the minimum condition* (3.5) *implies easily that the family* $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ *satisfies the consistency condition, we can apply Theorem* 2.1 *of [17] and find an aggregator for the family. This is different from [26], because we are working with a smaller set of probability measures. Therefore, from now on, we will supress the dependence in* $\mathbb{P}$ *of* K, *when it will be clear that we are dealing with a solution to a 2BSDE.*

Following [26], in addition to Assumption 3.1, we will always assume

Assumption 3.2. (i) $\mathcal{P}_H^\kappa$ is not empty.

(ii) The process $\widehat{F}^0$ satisfy the following integrability condition

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0 \le t \le T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right] < +\infty \quad (3.6)$$

We recall the uniqueness result proved in [17].

Theorem 3.1. *Let Assumptions 3.1 and 3.2 hold. Assume* $\xi \in \mathbb{L}_H^{2,\kappa}$ *and that* (Y, Z, U) *is a solution to 2BSDE with jumps* (3.3). *Then, for any* $\mathbb{P} \in \mathcal{P}_H^\kappa$ *and* $0 \le t_1 < t_2 \le T$,

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \ \mathbb{P} - a.s., \quad (3.7)$$

where, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}^+$*-stopping time* τ, *and* $\mathcal{F}_\tau^+$*-measurable random variable* $\xi \in \mathbb{L}^2(\mathbb{P})$, $(y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi))$ *denotes the solution to the following standard BSDE on* $0 \le t \le \tau$

$$y_t^{\mathbb{P}} = \xi - \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, u_s^{\mathbb{P}})ds + \int_t^\tau z_s^{\mathbb{P}} dB_s^c + \int_t^\tau \int_E u_s^{\mathbb{P}}(x)\tilde{\mu}_{B^d}(ds, \ dx), \ \mathbb{P} - a.s. \quad (3.8)$$

Remark 3.3. *We first emphasize that existence and uniqueness results for the standard BSDEs* (3.8) *are not given directly by the existing litterature, since the compensator of the counting measure associated to the jumps of B is not deterministic. However, since all the probability measure we consider satisfy the martingale representation property and the Blumenthal $0-1$ law, it is clear that we can straightforwardly generalize the proof of existence and uniqueness of Li and Tang [30] (see also [2] and [5] for related results). Furthermore, the usual a priori estimates and comparison Theorems will also hold.*

4 A direct existence argument

In the article [26], the main tool to prove existence of a solution is the so called regular conditional probability distributions of Stroock and Varadhan [29]. Indeed, these tools allow to give a pathwise construction for conditional expectations. Since, at least when the generator is null, the y component of the solution of a BSDE can be written as a conditional expectation, the r.c.p.d. allows us to construct solutions of BSDEs pathwise. Earlier in the paper, we have identified a candidate solution to the 2BSDEJ as an essential supremum of solutions of classical BSDEs with jumps (see (3.7)). However those BSDEs with jumps are written under mutually singular probability measures. Hence, being able to construct them pathwise allows us to avoid the problems related to negligible-sets. In this section we will generalize the approach of [26] to the jump case.

4.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [26]. Remember that we are working in the Skorohod space $\Omega = \mathbb{D}\left([0,T],\mathbb{R}^d\right)$ endowed with the Skorohod metric which makes it a separable space.

• For $0 \leq t \leq T$, we denote by $\Omega^t := \left\{\omega \in \mathbb{D}\left([t,T],\mathbb{R}^d\right)\right\}$ the shifted canonical space of càdlàg paths on $[t,T]$ which are null at t, B^t the shifted canonical process. Let $\mathcal{N}^t$ be the set of measures ν on $\mathcal{B}(E)$ satisfying

$$\int_t^T \int_E (1\wedge |x|^2)\nu_s(dx)ds < +\infty \text{ and } \int_t^T \int_{|x|>1} x\nu_s(dx)ds < +\infty,\ \forall \tilde{\omega} \in \Omega^t, \tag{4.1}$$

and let $\mathcal{D}^t$ be the set of $\mathbb{F}^t$-progressively measurable processes α taking values in $\mathbb{S}_d^{>0}$ with $\int_t^T |\alpha_s| ds < +\infty$, for every $\tilde{\omega} \in \Omega^t$.

$\mathbb{F}^t$ is the filtration generated by B^t. We define similarly the continuous part of B^t, denoted $B^{t,c}$, its discontinuous part denoted $B^{t,d}$, the density of the quadratic variation of $B^{t,c}$, denoted $\widehat{a}^t$, and $\mu_{B^{t,d}}$ the counting measure associated to the jumps of B^t.

Exactly as in Section 2, we can define semimartingale problems and the corresponding probability measures. We then restrict ourselves to deterministic (α,ν) and we let $\tilde{\mathcal{A}}^t$ be the corresponding separable class of coefficients and $\mathcal{P}_{\tilde{\mathcal{A}}^t}$ the corresponding family of probability measures, which will be noted $\mathbb{P}^{t,\alpha,\nu}$. Then, this family also satisfies the aggregation property of Theorem 2.1 in [17], and we can define $\widehat{\upsilon}^t$, the aggregator of the predictable compensators of B^t.

• For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, we define the shifted path $\omega^t \in \Omega^t$ by

$$\omega^t_r := \omega_r - \omega_t, \ \forall r \in [t, T].$$

• For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, $\widetilde{\omega} \in \Omega^t$ we define the concatenation path $\omega \otimes_t \widetilde{\omega} \in \Omega^s$ by

$$(\omega \otimes_t \widetilde{\omega})(r) := \omega_r 1_{[s,t)}(r) + (\omega_t + \widetilde{\omega}_r) 1_{[t,T]}(r), \ \forall r \in [s, T].$$

• For $0 \leq s \leq t \leq T$ and a $\mathcal{F}^s_T$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, we define the shifted $\mathcal{F}^t_T$-measurable random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_t \widetilde{\omega}), \ \forall \ \widetilde{\omega} \in \Omega^t.$$

Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s, T]$ and $(t, \omega) \in [s, T] \times \Omega^s$, we can define the shifted process $\left\{ X^{t,\omega}_r, r \in [t, T] \right\}$, which is $\mathbb{F}^t$-progressively measurable.

• For a $\mathbb{F}$-stopping time τ, we use the same simplification as [26]

$$\omega \otimes_\tau \widetilde{\omega} := \omega \otimes_{\tau(\omega)} \widetilde{\omega}, \ \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, \ X^{\tau,\omega} := X^{\tau(\omega),\omega}.$$

• We define our "shifted" generator

$$\widehat{F}^{t,\omega}_s(\widetilde{\omega}, y, z, u) := F_s(\omega \otimes_t \widetilde{\omega}, y, z, u, \widehat{a}^t_s(\widetilde{\omega}), \widehat{\nu}^t_s(\widetilde{\omega})), \ \forall (s, \widetilde{\omega}) \in [t, T] \times \Omega^t.$$

Then note that since F is assumed to be uniformly continuous in ω under the $\mathbb{L}^\infty$ norm, then so is $\widehat{F}^{t,\omega}$. Notice that this implies that for any $\mathbb{P} \in \mathcal{P}_{\tilde{\mathcal{A}}^t}$

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left|\widehat{F}^{t,\omega}_s(0,0,0)\right|^\kappa ds\right)^{\frac{2}{\kappa}}\right] < +\infty,$$

for some ω if and only if it holds for all $\omega \in \Omega$.

• Finally, we extend Definition 3.1 in the shifted spaces

Definition 4.1. *$\mathcal{P}^{t,\kappa}_H$ consists of all $\mathbb{P} := \mathbb{P}^{t,\alpha,\nu} \in \mathcal{P}_{\tilde{\mathcal{A}}^t}$ such that*

$$\underline{a}^{\mathbb{P}} \leq \widehat{a}^t \leq \overline{a}^{\mathbb{P}}, \ dt \times d\mathbb{P} - a.s. \ \text{for some} \ \underline{a}^{\mathbb{P}}, \overline{a}^{\mathbb{P}} \in \mathbb{S}^{>0}_d$$
$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left|\widehat{F}^{t,\omega}_s(0,0,0)\right|^\kappa ds\right)^{\frac{2}{\kappa}}\right] < +\infty, \ \text{for all} \ \omega \in \Omega.$$

For given $\omega \in \Omega$, $\mathbb{F}$-stopping time τ and $\mathbb{P} \in \mathcal{P}^\kappa_H$, the r.c.p.d. of $\mathbb{P}$ is a probability measure $\mathbb{P}^\omega_\tau$ on $\mathcal{F}_T$ such that for every bounded $\mathcal{F}_T$-measurable random variable ξ

$$\mathbb{E}^{\mathbb{P}}_\tau [\xi](\omega) = \mathbb{E}^{\mathbb{P}^\omega_\tau}[\xi], \ \text{for } \mathbb{P}\text{-a.e. } \omega.$$

Furthermore, $\mathbb{P}^\omega_\tau$ naturally induces a probability measure $\mathbb{P}^{\tau,\omega}$ on $\mathcal{F}^{\tau(\omega)}_T$ such that the $\mathbb{P}^{\tau,\omega}$-distribution of $B^{\tau(\omega)}$ is equal to the $\mathbb{P}^\omega_\tau$-distribution of $\left\{ B_t - B_{\tau(\omega)}, \ t \in [\tau(\omega), t] \right\}$. Besides, we have

$$\mathbb{E}^{\mathbb{P}^\omega_\tau}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}].$$

Remark 4.1. *We emphasize that the above notations correspond to the ones used in [26] when we consider the subset of* Ω *consisting of all continuous paths from* $[0,T]$ *to* $\mathbb{R}^d$ *whose value at time* 0 *is* 0.

We now prove the following Proposition which gives a relation between $(\widehat{a}^{t,\omega}, \widehat{\nu}^{t,\omega})$ and $(\widehat{a}^{t}, \widehat{\nu}^{t})$.

Proposition 4.1. *Let* $\mathbb{P} \in \mathcal{P}_H^\kappa$ *and* τ *be an* $\mathbb{F}$*-stopping time. Then, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, we have for* $ds \times d\mathbb{P}^{\tau,\omega}$*-a.e.* $(s,\widetilde{\omega}) \in [\tau(\omega), T] \times \Omega^{\tau(\omega)}$

$$\widehat{a}_s^{\tau,\omega}(\widetilde{\omega}) = \widehat{a}_s^{\tau(\omega)}(\widetilde{\omega}), \text{ and } \widehat{\nu}_s^{\tau,\omega}(\widetilde{\omega}, A) = \widehat{\nu}_s^{\tau(\omega)}(\widetilde{\omega}, A) \text{ for every } A \in \mathcal{B}(E).$$

This result is important for us, because it implies that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $ds \times d\mathbb{P}^{t,\omega}-$ *a.e.* $(s, \widetilde{\omega}) \in [t,T] \times \Omega^t$

$$F_s\left(\omega \otimes_t \widetilde{\omega}, y, z, u, \widehat{a}_s(\omega \otimes_t \widetilde{\omega}), \widehat{\nu}_s(\omega \otimes_t \widetilde{\omega})\right) = F_s\left(\omega \otimes_t \widetilde{\omega}, y, z, u, \widehat{a}_s^t(\widetilde{\omega}), \widehat{\nu}_s^t(\widetilde{\omega})\right).$$

Whereas the left-hand side has in general no regularity in ω, the right-hand side, that we choose as our shifted generator, is uniformly continuous in ω.

Proof. The proof of the equality for $\widehat{a}$ is the same as in Lemma 4.1 of [27], so we omit it.

Now, for $s \geq \tau$ and for any $A \in \mathcal{B}(E)$, we know by the Doob-Meyer decomposition that there exist a $\mathbb{P}$-local martingale M and a $\mathbb{P}^{\tau,\omega}$-martingale N such that

$$\mu_{B^d}([0,s], A) = M_s + \int_0^s \widehat{\nu}_r(A)dr, \ \mathbb{P} - a.s.$$
$$\mu_{B^{\tau(\omega),d}}([\tau(\omega), s], A) = N_s + \int_\tau^s \widehat{\nu}_r^{\tau(\omega)}(A)dr, \ \mathbb{P}^{\tau,\omega} - a.s.$$

Then, we can rewrite the first equation above for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $\mathbb{P}^{\tau,\omega}$-a.e. $\widetilde{\omega} \in \Omega^{\tau(\omega)}$

$$\mu_{B^d}(\omega \otimes_\tau \widetilde{\omega}, [0,s], A) = M_s^{\tau,\omega}(\widetilde{\omega}) + \int_0^s \widehat{\nu}_r^{\tau,\omega}(\widetilde{\omega}, A)dr. \tag{4.2}$$

Now, by definition of the measures μ_{B^d} and $\mu_{B^{\tau(\omega),d}}$, we have

$$\mu_{B^d}(\omega \otimes_\tau \widetilde{\omega}, [0,s], A) = \mu_{B^d}(\omega, [0,\tau], A) + \mu_{B^{\tau(\omega),d}}(\widetilde{\omega}, [\tau, s], A).$$

Hence, we obtain from (4.2) that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $\mathbb{P}^{\tau,\omega}$-a.e. $\widetilde{\omega} \in \Omega^{\tau(\omega)}$

$$\mu_{B^d}(\omega, [0,\tau], A) - \int_0^\tau \widehat{\nu}_r(\omega, A)dr + N_s(\widetilde{\omega}) - M_s^{\tau,\omega}(\widetilde{\omega}) = \int_\tau^s \left(\widehat{\nu}_r^{\tau,\omega}(\widetilde{\omega}, A) - \widehat{\nu}_r^{\tau(\omega)}(\widetilde{\omega}, A)\right) dr$$

In the left-hand side above, the terms which are $\mathcal{F}_\tau$-measurable are constants in $\Omega^{\tau(\omega)}$ and using the same arguments as in Step 1 of the proof of Lemma A.1, we can show that $M^{\tau,\omega}$ is a $\mathbb{P}^{\tau,\omega}$-local martingale for $\mathbb{P}$-a.e. $\omega \in \Omega$. This means that the left-hand side is a $\mathbb{P}^{\tau,\omega}$-local martingale while the right-hand side is a predictable finite variation process. By the martingale representation property which still holds in the shifted canonical spaces, we deduce that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $ds \times d\mathbb{P}^{\tau,\omega}$-a.e. $(s, \widetilde{\omega}) \in [\tau(\omega), T] \times \Omega^{\tau(\omega)}$

$$\int_\tau^s \left(\widehat{\nu}_r^{\tau,\omega}(\widetilde{\omega}, A) - \widehat{\nu}_r^{\tau(\omega)}(\widetilde{\omega}, A)\right) dr = 0,$$

which is the desired result. □

4.2 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

When ξ is in $\mathrm{UC_b}(\Omega)$, we know that there exists a modulus of continuity function ρ for ξ and F in ω. Then, for any $0 \leq t \leq s \leq T$, $(y, z, \nu) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{V}$ and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})\right| \leq \rho\left(\|\omega - \omega'\|_t\right), \ \left|\widehat{F}^{t,\omega}_s(\tilde{\omega}, y, z, u) - \widehat{F}^{t,\omega'}_s(\tilde{\omega}, y, z, u)\right| \leq \rho\left(\|\omega - \omega'\|_t\right)$$

We then define for all $\omega \in \Omega$

$$\Lambda(\omega) := \sup_{0 \leq s \leq t} \Lambda_t(\omega) := \sup_{0 \leq s \leq t} \sup_{\mathbb{P} \in \mathcal{P}^t_H} \left(\mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T |\widehat{F}^{t,\omega}_s(0,0,0)|^2 ds\right]\right)^{1/2}. \tag{4.3}$$

Now since $\widehat{F}^{t,\omega}$ is also uniformly continuous in ω, it is easily verified that

$$\Lambda(\omega) < \infty \text{ for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega. \tag{4.4}$$

Moreover, when Λ is finite, it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm and is therefore $\mathcal{F}_T$-measurable. Now, by Assumption 3.2, we have

$$\Lambda_t(\omega) < \infty \text{ for all } (t, \omega) \in [0, T] \times \Omega. \tag{4.5}$$

To prove existence, we define the following value process V_t pathwise

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}^{t,\kappa}_H} \mathcal{Y}^{\mathbb{P},t,\omega}_t(T, \xi), \text{ for all } (t, \omega) \in [0, T] \times \Omega, \tag{4.6}$$

where, for any $(t_1, \omega) \in [0, T] \times \Omega$, $\mathbb{P} \in \mathcal{P}^{t_1,\kappa}_H, t_2 \in [t_1, T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta \in \mathbb{L}^2(\mathbb{P})$, we denote $\mathcal{Y}^{\mathbb{P},t_1,\omega}_{t_1}(t_2, \eta) := y^{\mathbb{P},t_1,\omega}_{t_1}$, where $\left(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}, u^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following BSDE with jumps on the shifted space Ω^{t_1} under $\mathbb{P}$

$$\begin{aligned} y^{\mathbb{P},t_1,\omega}_s &= \eta^{t_1,\omega} + \int_s^{t_2} \widehat{F}^{t_1,\omega}_r\left(y^{\mathbb{P},t_1,\omega}_r, z^{\mathbb{P},t_1,\omega}_r, \nu\right) dr - \int_s^{t_2} z^{\mathbb{P},t_1,\omega}_r dB^{t_1,c}_r \\ &\quad - \int_s^{t_2}\int_{\mathbb{R}^d} u^{\mathbb{P},t_1,\omega}_s(x)\widetilde{\mu}_{B^{t_1,d}}(ds, dx), \ \mathbb{P}-a.s., \ s \in [t, T], \end{aligned} \tag{4.7}$$

where as usual $\widetilde{\mu}_{B^{t_1,d}}(ds, dx) := \mu_{B^{t_1,d}}(ds, dx) - \widehat{\nu}^t_s(dx)ds$. In view of the Blumenthal zero-one law, $y^{\mathbb{P},t,\omega}_t$ is constant for any given (t, ω) and $\mathbb{P} \in \mathcal{P}^{t,\kappa}_H$, and therefore the value process V is well defined. Let us now show that V inherits some properties from ξ and F.

Lemma 4.1. *Let Assumptions 3.1 and 3.2 hold and consider some ξ in $\mathrm{UC_b}(\Omega)$. Then for all $(t, \omega) \in [0, T] \times \Omega$ we have $|V_t(\omega)| \leq C\Lambda_t(\omega)$. Moreover, for all $(t, \omega, \omega') \in [0, T] \times \Omega^2$, $|V_t(\omega) - V_t(\omega')| \leq C\rho(\|\omega - \omega'\|_t)$. Consequently, V_t is $\mathcal{F}_t$-measurable for every $t \in [0, T]$.*

Proof. (i) For each $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}^{t,\kappa}_H$, let α be some positive constant which will be fixed later and let $\eta \in (0, 1)$. Since F is uniformly Lipschitz in (y, z) and satisfies Assumption 3.1(iv), we have

$$\left|\widehat{F}^{t,\omega}_s(y, z, u)\right| \leq \left|\widehat{F}^{t,\omega}_s(0,0,0)\right| + C\left(|y| + |\left(\widehat{a}^t_s\right)^{1/2} z| + \left(\int_E |u(x)|^2 \widehat{\nu}^t_s(dx)\right)^{1/2}\right).$$

Now apply Itô's formula. We obtain

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2+\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2 ds+\int_t^T\int_E e^{\alpha s}\left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\widehat{\nu}_s^t(dx)ds\\
&=e^{\alpha T}\left|\xi^{t,\omega}\right|^2+2\int_t^T e^{\alpha s}y_s^{\mathbb{P},t,\omega}\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega},u_s^{\mathbb{P},t,\omega})ds\\
&\quad-\alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds-2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^{t,c}\\
&\quad-\int_t^T\int_E e^{\alpha s}\left(2y_{s^-}^{\mathbb{P},t,\omega}u_s^{\mathbb{P},t,\omega}(x)+\left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\right)\widetilde{\mu}_{B^{t,d}}(ds,dx)\\
&\leq e^{\alpha T}\left|\xi^{t,\omega}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0,0,0)\right|^2 ds+\left(1+2C+\frac{2C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds\\
&\quad+\eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2 ds+\eta\int_t^T\int_E e^{\alpha s}\left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\widehat{\nu}_s^t(dx)ds\\
&\quad-2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^{t,c}-\int_t^T\int_E e^{\alpha s}\left(2y_{s^-}^{\mathbb{P},t,\omega}u_s^{\mathbb{P},t,\omega}(x)+\left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\right)\widetilde{\mu}_{B^{t,d}}(ds,dx).
\end{aligned}
$$

Now choose $\eta=1/2$ for instance and α large enough. By taking expectation we obtain easily

$$
\left|y_t^{\mathbb{P},t,\omega}\right|^2\leq C\left|\Lambda_t(\omega)\right|^2.
$$

The result then follows from the arbitrariness of $\mathbb{P}$.

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in ω of $\xi^{t,\omega}$ and $\widehat{F}^{t,\omega}$. Indeed, for each $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}_H^t$, let α be some positive constant which will be fixed later and let $\eta\in(0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left(\left|(\widehat{a}_s^t)^{\frac12}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2+\int_E e^{\alpha s}(u_s^{\mathbb{P},t,\omega}-u_s^{\mathbb{P},t,\omega'})^2(x)\widehat{\nu}_s^t(dx)\right)ds\\
&\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds\\
&\quad+2C\int_t^T\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left|(\widehat{a}_s^t)^{\frac12}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|ds\\
&\quad+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left(\int_{\mathbb{R}^d}\left|u_s^{\mathbb{P},t,\omega}(x)-u_s^{\mathbb{P},t,\omega'}(x)\right|^2\widehat{\nu}_s^t(dx)\right)^{1/2}ds\\
&\quad+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega},u_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega},u_s^{\mathbb{P},t,\omega})\right|ds\\
&\quad-\alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^{t,c}\\
&\quad-\int_t^T\int_E e^{\alpha s}\left(2(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(u_s^{\mathbb{P},t,\omega}-u_s^{\mathbb{P},t,\omega'})+(u_s^{\mathbb{P},t,\omega}-u_s^{\mathbb{P},t,\omega'})^2\right)(x)\widetilde{\mu}_{B^{t,d}}(ds,dx).
\end{aligned}
$$

We then deduce

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left(\left|(\widehat{a}_s^t)^{\frac{1}{2}}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2+\int_E e^{\alpha s}(u_s^{\mathbb{P},t,\omega}-u_s^{\mathbb{P},t,\omega'})^2(x)\widehat{\nu}_s^t(dx)\right)ds\\
&\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega},u_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega},u_s^{\mathbb{P},t,\omega})\right|^2 ds\\
&+\eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds+\eta\int_t^T\int_E e^{\alpha s}\left|u_s^{\mathbb{P},t,\omega}(x)-u_s^{\mathbb{P},t,\omega'}(x)\right|^2\widehat{\nu}_s^t(dx)ds\\
&+\left(2C+C^2+\frac{2C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds\\
&-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^{t,c}\\
&-\int_t^T\int_E e^{\alpha s}\left(2(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(u_s^{\mathbb{P},t,\omega}-u_s^{\mathbb{P},t,\omega'})+(u_s^{\mathbb{P},t,\omega}-u_s^{\mathbb{P},t,\omega'})^2\right)(x)\widetilde{\mu}_{B^{t,d}}(ds,dx).
\end{aligned}
$$

Now choose $\eta=1/2$ and α such that $\nu:=\alpha-2C-C^2-\frac{2C^2}{\eta}\geq 0$. We obtain the desired result by taking expectation and using the uniform continuity in ω of ξ and F. □

The next proposition is a dynamic programming property verified by the value process, which will prove useful when proving that V provides a solution to the 2BSDEJ with generator F and terminal condition ξ. The result and its proof are intimately connected and use the same arguments as the proof of Proposition 4.7 in [27].

Proposition 4.2. *Under Assumptions 3.1, 3.2 and for $\xi\in \mathrm{UC_b}(\Omega)$, we have for all $0\leq t_1<t_2\leq T$ and for all $\omega\in\Omega$*

$$V_{t_1}(\omega)=\sup_{\mathbb{P}\in\mathcal{P}_H^{t_1,\kappa}}\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,V_{t_2}^{t_1,\omega}).$$

The proof is almost the same as the proof in [27], with minor modifications due to the introduction of jumps.

Proof. Without loss of generality, we assume that $t_1=0$ and $t_2=t$. Thus, we have to prove

$$V_0(\omega)=\sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}}\mathcal{Y}_0^{\mathbb{P}}(t,V_t).$$

Denote $(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}}):=(\mathcal{Y}^{\mathbb{P}}(T,\xi),\mathcal{Z}^{\mathbb{P}}(T,\xi),\mathcal{U}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P}\in\mathcal{P}_H^{\kappa}$, we know by Lemma A.1 in the Appendix, that for $\mathbb{P}-a.e.\ \omega\in\Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega}\in\mathcal{P}_H^{t,\kappa}$. Now thanks to the paper of Li and Tang [30], we know that the solution of BSDEs on the Wiener-Poisson space with Lipschitz generator can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. By the properties of the r.p.c.d., this entails that

$$y_t^{\mathbb{P}}(\omega)=\mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi),\ \text{for}\ \mathbb{P}-a.e.\ \omega\in\Omega. \tag{4.8}$$

Hence, by definition of V_t and the comparison principle for BSDEs with jumps, we get that $y_0^{\mathbb{P}} \leq \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$. By arbitrariness of $\mathbb{P}$, this leads to

$$V_0(\omega) \leq \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [27]. Let $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $\epsilon > 0$. By separability of Ω, there exists a partition $(E_t^i)_{i\geq 1} \subset \mathcal{F}_t$ such that $d_S(\omega, \omega')_t \leq \epsilon/2$ for any i and any $\omega, \omega' \in E_t^i$. Now by Billinsgley [3], we know that the distance for the uniform topology is dominated by the Skorohod metric in the sense that

$$\left\|\omega - \omega'\right\|_t \leq 2d_S(\omega, \omega')_t \leq \epsilon, \text{ for any } i \text{ and any } \omega, \omega' \in E_t^i. \tag{4.9}$$

Now for each i, fix a $\widehat{\omega}_i \in E_t^i$ and let $\mathbb{P}_t^i$ be an $\epsilon-$optimizer of $V_t(\widehat{\omega}_i)$. If we define for each $n \geq 1$, $\mathbb{P}^n := \mathbb{P}^{n,\epsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i, \tag{4.10}$$

then, by Lemma A.2, we know that $\mathbb{P}^n \in \mathcal{P}_H^\kappa$. Besides, by Lemma 4.1 and its proof, we have for any i and any $\omega \in E_t^i$

$$\begin{aligned}V_t(\omega) &\leq V_t(\widehat{\omega}_i) + C\rho(\epsilon) \leq \mathcal{Y}_t^{\mathbb{P}_t^i, t, \widehat{\omega}_i}(T, \xi) + \epsilon + C\rho(\epsilon)\\ &\leq \mathcal{Y}_t^{\mathbb{P}_t^i, t, \omega}(T, \xi) + \epsilon + C\rho(\epsilon) = \mathcal{Y}_t^{(\mathbb{P}^n)^{t,\omega}, t, \omega}(T, \xi) + \epsilon + C\rho(\epsilon),\end{aligned}$$

where we used successively the uniform continuity of V in ω and (4.9), the definition of $\mathbb{P}_t^i$, the uniform continuity of $\mathcal{Y}_t^{\mathbb{P},t,\omega}$ in ω and finally the definition of $\mathbb{P}^n$.

Then, it follows from (4.8) that

$$V_t \leq y_t^{\mathbb{P}^n} + \epsilon + C\rho(\epsilon), \ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i. \tag{4.11}$$

Let now $(y^n, z^n, u^n) := (y^{n,\epsilon}, z^{n,\epsilon}, u^{n,\epsilon})$ be the solution of the following BSDE on $[0, t]$

$$\begin{aligned}y_s^n = &\left[y_t^{\mathbb{P}^n} + \epsilon + C\rho(\epsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} + \int_s^t \widehat{F}_r(y_r^n, z_r^n, u_r^n)dr - \int_s^t z_r^n dB_r^c\\ &- \int_s^t \int_E u_r^n(x)\widetilde{\mu}_{B^d}(dr, dx), \ \mathbb{P}^n - a.s.\end{aligned} \tag{4.12}$$

By the comparison principle for BSDEs with jumps, we know that $\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leq y_0^n$. Then since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (4.12) also holds $\mathbb{P} - a.s.$ Using the same arguments and notations as in the proof of Lemma 4.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right|^2 \leq C\mathbb{E}^{\mathbb{P}}\left[\epsilon^2 + \rho(\epsilon)^2 + \left|V_t - y_t^{\mathbb{P}^n}\right|^2 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 4.1, we have

$$\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leq y_0^n \leq y_0^{\mathbb{P}^n} + C\left(\epsilon + \rho(\epsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{\frac{1}{2}}\right) \leq V_0 + C\left(\epsilon + \rho(\epsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{\frac{1}{2}}\right).$$

Then it suffices to let n go to $+\infty$, use the dominated convergence theorem, and let ϵ go to 0. $\square$

Now we are facing the problem of the regularity in t of V. Indeed, if we want to obtain a solution of the 2BSDE, then it has to be at least càdlàg, $\mathcal{P}_H^\kappa - q.s.$ To this end, we define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,T], r\downarrow t}} V_r.$$

Lemma 4.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r\in\mathbb{Q}\cap(t,T], r\downarrow t} V_r,\ \mathcal{P}_H^\kappa - q.s.$$

and thus V^+ is càdlàg $\mathcal{P}_H^\kappa - q.s.$

Proof. For each $\mathbb{P}$, we define $\widetilde{V}^{\mathbb{P}} := V - \mathcal{Y}^{\mathbb{P}}(T,\xi)$. Then, we recall that we have

$$\widetilde{V}^{\mathbb{P}} \geq 0,\ \mathbb{P} - a.s.$$

Now for any $0 \leq t_1 < t_2 \leq T$, let $(y^{\mathbb{P},t_2}, z^{\mathbb{P},t_2}, u^{\mathbb{P},t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{U}^{\mathbb{P}}(t_2, V_{t_2}))$. Once more, we remind that since solutions of BSDEs can be defined by Picard iterations, we have by the properties of the r.p.c.d. that

$$\mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2})(\omega) = \mathcal{Y}_{t_1}^{\mathbb{P}^{t_1,\omega}, t_1,\omega}(t_2, V_{t_2}^{t_1,\omega}),\ \text{for } \mathbb{P} - a.e.\ \omega.$$

Hence, we conclude from Proposition 4.2 $V_{t_1} \geq y_{t_1}^{\mathbb{P},t_2}$, $\mathbb{P} - a.s.$ Denote

$$\widetilde{y}_t^{\mathbb{P},t_2} := y_t^{\mathbb{P},t_2} - \mathcal{Y}_t^{\mathbb{P}}(T,\xi),\ \widetilde{z}_t^{\mathbb{P},t_2} := \widehat{a}_t^{-1/2}(z_t^{\mathbb{P},t_2} - \mathcal{Z}_t^{\mathbb{P}}(T,\xi)),\ \widetilde{u}_t^{\mathbb{P},t_2} := u_t^{\mathbb{P},t_2} - \mathcal{U}_t^{\mathbb{P}}(T,\xi).$$

Then $\widetilde{V}_{t_1}^{\mathbb{P}} \geq \widetilde{y}_{t_1}^{\mathbb{P},t_2}$ and $(\widetilde{y}^{\mathbb{P},t_2}, \widetilde{z}^{\mathbb{P},t_2}, \widetilde{u}^{\mathbb{P},t_2})$ satisfies the following BSDE on $[0,t_2]$

$$\widetilde{y}_t^{\mathbb{P},t_2} = \widetilde{V}_{t_2}^{\mathbb{P}} + \int_t^{t_2} f_s^{\mathbb{P}}(\widetilde{y}_s^{\mathbb{P},t_2}, \widetilde{z}_s^{\mathbb{P},t_2}, \widetilde{u}_s^{\mathbb{P},t_2})ds - \int_t^{t_2} \widetilde{z}_s^{\mathbb{P},t_2} dW_s^{\mathbb{P}} - \int_t^{t_2}\int_{\mathbb{R}^d} \widetilde{u}_s^{\mathbb{P},t_2}(x)\widetilde{\mu}_{B^d}(ds,dx),$$

where

$$\begin{aligned} f_t^{\mathbb{P}}(\omega,y,z,u) : = &\widehat{F}_t(\omega, y + \mathcal{Y}_t^{\mathbb{P}}(\omega), \widehat{a}_t^{-\frac{1}{2}}(\omega)(z + \mathcal{Z}_t^{\mathbb{P}}(\omega)), u + \bar{\mathcal{U}}_t^{\mathbb{P}}(\omega)) \\ &- \widehat{F}_t(\omega, \mathcal{Y}_t^{\mathbb{P}}(\omega), \mathcal{Z}_t^{\mathbb{P}}(\omega), \mathcal{U}_t^{\mathbb{P}}(\omega)). \end{aligned}$$

By the definition given in Royer [25], we conclude from the above that $\widetilde{V}^{\mathbb{P}}$ is a positive $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. Since $f^{\mathbb{P}}(0,0,0) = 0$, we can apply the downcrossing inequality proved in [25] to obtain classically that for $\mathbb{P} - a.e.\ \omega$, the limit

$$\lim_{r\in\mathbb{Q}\cup(t,T], r\downarrow t} \widetilde{V}_r^{\mathbb{P}}(\omega)$$

exists for all t. Finally, since $\bar{\mathcal{Y}}^{\mathbb{P}}$ is càdlàg, we obtain the desired result. $\square$

We follow now Remark 4.9 in [27], and for a fixed $\mathbb{P} \in \mathcal{P}_H^\kappa$, we introduce the following RBSDE with jumps and with lower obstacle V^+ under $\mathbb{P}$

$$\widetilde{Y}_t^{\mathbb{P}} = \xi + \int_t^T \widehat{F}_s(\widetilde{Y}_s^{\mathbb{P}}, \widetilde{Z}_s^{\mathbb{P}}, \widetilde{U}_s^{\mathbb{P}}, \nu)ds - \int_t^T \widetilde{Z}_s^{\mathbb{P}} dB_s^c - \int_t^T \int_E \widetilde{U}_s^{\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds, dx) + \widetilde{K}_T^{\mathbb{P}} - \widetilde{K}_t^{\mathbb{P}}$$
$$\widetilde{Y}_t^{\mathbb{P}} \geq V_t^+,\ 0 \leq t \leq T,\ \mathbb{P} - a.s.$$
$$\int_0^T \left(\widetilde{Y}_{s^-}^{\mathbb{P}} - V_{s^-}^+\right) d\widetilde{K}_s^{\mathbb{P}} = 0,\ \mathbb{P} - a.s.,$$

where we emphasize that the process $\widetilde{K}^{\mathbb{P}}$ is predictable.

Remark 4.2. *Existence and uniqueness of the above RBSDE under our Assumptions, with the restrictions that the compensator is not random, have been proved by Hamadène and Ouknine [11] or Essaky [10]. However, their proofs can be easily generalized to our context.*

Let us now argue by contradiction and suppose that $\widetilde{Y}^{\mathbb{P}}$ is not equal $\mathbb{P} - a.s.$ to V^+. Then we can assume without loss of generality that $\widetilde{Y}_0^{\mathbb{P}} > V_0^+$, $\mathbb{P} - a.s.$ fix now some $\varepsilon > 0$ and define the following stopping-time

$$\tau^\varepsilon := \inf\left\{t \geq 0,\ \widetilde{Y}_t^{\mathbb{P}} \leq V_t^+ + \varepsilon\right\}.$$

Then $\widetilde{Y}^{\mathbb{P}}$ is strictly above the obstacle before τ^ε, and therefore $\widetilde{K}^{\mathbb{P}}$ is identically equal to 0 in $[0, \tau^\varepsilon]$. Hence, we have

$$\widetilde{Y}_t^{\mathbb{P}} = \widetilde{Y}_{\tau^\varepsilon}^{\mathbb{P}} + \int_t^{\tau^\varepsilon} \widehat{F}_s(\widetilde{Y}_s^{\mathbb{P}}, \widetilde{Z}_s^{\mathbb{P}}, \widetilde{U}_s^{\mathbb{P}})ds - \int_t^{\tau^\varepsilon} \widetilde{Z}_s^{\mathbb{P}} dB_s^c - \int_t^{\tau^\varepsilon} \int_E \widetilde{U}_s^{\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds, dx),\ \mathbb{P} - a.s.$$

Let us now define the following BSDE on $[0, \tau^\varepsilon]$

$$y_t^{+,\mathbb{P}} = V_{\tau^\varepsilon}^+ + \int_t^{\tau^\varepsilon} \widehat{F}_s(y_s^{+,\mathbb{P}}, z_s^{+,\mathbb{P}}, u_s^{+,\mathbb{P}})ds - \int_t^{\tau^\varepsilon} z_s^{+,\mathbb{P}} dB_s^c - \int_t^{\tau^\varepsilon} \int_E u_s^{+,\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds, dx),\ \mathbb{P} - a.s.$$

By the standard a priori estimates already used in this paper, we obtain that

$$\widetilde{Y}_0^{\mathbb{P}} \leq y_0^{+,\mathbb{P}} + C\left|V_{\tau^\varepsilon}^+ - \widetilde{Y}_{\tau^\varepsilon}^{\mathbb{P}}\right| \leq y_0^{+,\mathbb{P}} + C\varepsilon,$$

by definition of τ^ε. Following the arguments in Step 1 of the proof of Theorem 4.5 in [27], we can show that $y_0^{+,\mathbb{P}} \leq V_0^+$ which in turn implies

$$\widetilde{Y}_0^{\mathbb{P}} \leq V_0^+ + C\varepsilon,$$

hence a contradiction by arbitrariness of ε. Therefore, we have obtained the following decomposition

$$V_t^+ = \xi + \int_t^T \widehat{F}_s(V_s^+, \widetilde{Z}_s^{\mathbb{P}}, \widetilde{U}_s^{\mathbb{P}})ds - \int_t^T \widetilde{Z}_s^{\mathbb{P}} dB_s^c - \int_t^T \int_E \widetilde{U}_s^{\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds, dx) + \widetilde{K}_T^{\mathbb{P}} - \widetilde{K}_t^{\mathbb{P}},\ \mathbb{P} - a.s.$$

Finally, we can use the result of Nutz [22] to aggregate the families

$$\left\{\widetilde{Z}^{\mathbb{P}},\ \mathbb{P} \in \mathcal{P}_H^\kappa\right\} \text{ and } \left\{\widetilde{U}^{\mathbb{P}},\ \mathbb{P} \in \mathcal{P}_H^\kappa\right\},$$

into universal processes $\widetilde{Z}$ and $\widetilde{U}$.

We next prove the representation (3.7) for V and V^+, and that, as shown in Proposition 4.11 of [27], we actually have $V = V^+$, $\mathcal{P}_H - q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2RBSDE is actually $\mathbb{F}$-progressively measurable.

Proposition 4.3. *Assume that $\xi \in UC_b(\Omega)$ and that Assumptions 3.1 and 3.2 hold. Then we have*

$$V_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } V_t^+ = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Besides, we also have for all t, $V_t = V_t^+$, $\mathcal{P}_H^\kappa - q.s.$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [27], since we also have a stability result for BSDEs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [27]. □

Therefore, in the sequel we will use V instead of V^+. Finally, we have to check that the minimum condition (3.5) holds. Fix $\mathbb{P}$ in $\mathcal{P}_H^\kappa$ and $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})$. Then, proceeding exactly as in Step 2 of the proof of Theorem 4.1 in [17], but introducing the process γ' of Assumption 3.1(iv) instead of γ, we can similarly obtain

$$V_t - y_t^{\mathbb{P}'} \geq \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_s' d\widetilde{K}_s^{\mathbb{P}'}\right] \geq \mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leq s\leq T} M_s'\left(\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right)\right],$$

where M' is defined as M but with γ' instead of γ. Now let us prove that for any $n > 1$

$$\mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leq s\leq T} M_s'\right)^{-n}\right] < +\infty,\ \mathbb{P}' - a.s. \tag{4.13}$$

First we have

$$\begin{aligned} M_s' = \exp&\left(\int_t^s \lambda_r dr + \int_t^s \eta_r \widehat{a}_r^{-1/2} dB_s^c - \frac{1}{2}\int_t^s |\eta_r|^2 dr + \int_t^s \int_E \gamma_r'(x)\widetilde{\mu}_{B^d}(dr,dx)\right) \\ &\times \prod_{t\leq r\leq s}(1+\gamma_r'(\Delta B_r))e^{-\gamma_r'(\Delta B_r)}. \end{aligned}$$

Define, then $A_s = \mathcal{E}\left(\int_t^s \eta_r \widehat{a}_r^{-1/2} dB_s^c\right)$ and $C_s = \mathcal{E}\left(\int_t^s \int_E \gamma_r'(x)\widetilde{\mu}_{B^d}(dr,dx)\right)$. Notice that both this processes are strictly positive martingales, since η and γ' are bounded and we have assumed that γ' is strictly greater than -1. We have

$$M_s' = \exp\left(\int_t^s \lambda_r dr\right) A_s C_s.$$

Since the process λ is bounded, we have

$$\left(\inf_{t\leq s\leq T} M_s'\right)^{-n} \leq C\left(\inf_{t\leq s\leq T} A_s C_s\right)^{-n} = C\left(\sup_{t\leq s\leq T}(A_s C_s)^{-1}\right)^n.$$

Using the Doob inequality for the submartingale $(A_sC_s)^{-1}$, we obtain

$$\mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leq s\leq T} M'_s\right)^{-n}\right] \leq C\mathbb{E}_t^{\mathbb{P}'}\left[(C_TA_T)^{-n}\right] \leq C\left(\mathbb{E}_t^{\mathbb{P}'}\left[(C_T)^{-2n}\right]\mathbb{E}_t^{\mathbb{P}'}\left[(A_T)^{-2n}\right]\right)^{1/2} < +\infty,$$

where we used the fact that since η is bounded, the continuous stochastic exponential A has negative moments of any order, and where the same result holds for the purely discontinuous stochastic exponential C by Lemma A.4 in [17].

Then, we have for any $p > 1$

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}'}\left[\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right] \\
&= \mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leq s\leq T} M'_s\right)^{1/p}\left(\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right)\left(\inf_{t\leq s\leq T} M'_s\right)^{-1/p}\right] \\
&\leq \left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leq s\leq T} M'_s\left(\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right)\right]\right)^{1/p}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leq s\leq T} M_s^{'-\frac{2}{p-1}}\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right)^2\right]\right)^{\frac{p-1}{2p}} \\
&\leq C\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[\left(\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right)^2\right]\right)^{\frac{p-1}{2p}}\left(V_t - y_t^{\mathbb{P}'}\right)^{1/p},
\end{aligned}$$

where we used (4.13). Arguing as in Step (iii) of the proof of Theorem 3.1, the above inequality along with Proposition 4.3 shows that we have

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right] = 0,$$

that is to say that the minimum condition 3.5 is satisfied. This implies that the family $\left\{\widetilde{K}^{\mathbb{P}}\right\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ satisfies the consistency condition (i) of Theorem 2.1 in [17] and therefore can be aggregated by this Theorem.

4.3 Main result

We are now in position to state the main result of this section

Theorem 4.1. *Let $\xi \in \mathcal{L}_H^{2,\kappa}$. Under Assumptions 3.1 and 3.2, there exists a unique solution $(Y, Z, U) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2 \times \mathbb{J}_H^{2,\kappa}$ of the* 2BSDEJ (3.3).

Proof. The proof follow the lines of the proof of Theorem 4.7 in [26], using the a priori estimates of Theorem 4.4 in [17], therefore we omit it. □

4.4 An extension of the representation formula

So far, we managed to provide wellposedness results for 2BSDEs with jumps, by working under a set a probability measures which, if restricted to the ones for which the canonical process is a continuous local martingale is strictly smaller than the one considered in [26] or [24]. This is due mainly to the fact that we had to restrict ourselves to processes α satisfying a separability condition (see Definition 2.5 in [17] for more details) in order to retrieve the aggregation result of Theorem 2.1 of [17], which was crucial to our analysis since it allowed us to define an aggregator for the family of predictable compensators.

This is clearly not very satisfying, not only from the theoretical point of view, but also from the practical one. Indeed, the set from which the processes α are allowed to be chosen corresponds in financial applications to the set of possible volatility processes for the market considered. It is therefore desirable to have the greatest possible generality. However, we emphasize that the restrictions we put on the predictable compensators ν are clearly not a problem from the point of view of the applications. Indeed, our set of compensators is strictly greater than the one associated to pure jump additive processes. Those processes, and more precisely the Lévy processes, being the most widely used in applications, our set is not really restrictive.

The aim of this section is to show that under additional assumptions, we can show that the representation formula (3.7) also holds for a larger set of probability measures for which there is no longer any restrictions on the processes α. In this regard, we recall the set of probability measures $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}}$ defined in Remark 2.2. We recall that every probability measure in this set satisfy the Blumenthal $0-1$ law and the martingale representation property. Moreover, exactly as in Definition 3.1, we define and restrict ourselves to the subset $\overline{\mathcal{P}}^\kappa_H$ of $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}}$. We define the following space for each $p \geq \kappa$,

$$\overline{\mathbb{L}}^{p,\kappa}_H := \left\{\xi,\ \|\xi\|_{\overline{\mathbb{L}}^{p,\kappa}_H} < +\infty\right\} \text{ where } \|\xi\|^p_{\overline{\mathbb{L}}^{p,\kappa}_H} := \sup_{\mathbb{P}\in\overline{\mathcal{P}}^\kappa_H} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}} \left(\mathbb{E}^{H,\mathbb{P}}_t[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right],$$

and we let

$$\overline{\mathcal{L}}^{p,\kappa}_H := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\overline{\mathbb{L}}^{p,\kappa}_H}, \text{ for every } 1<\kappa\leq p.$$

We then have the following result, which is similar to Theorem 5.3 in [27]

Theorem 4.2. *Let $\xi \in \overline{\mathcal{L}}^{2\kappa}_H$ and in addition to Assumptions 3.1 and 3.2, assume that*

- *F is uniformly continuous in a for $a \in D^1_{F_t}$, and for all (t,ω,y,z,u,a,ν)*

$$|F_t(\omega,y,z,u,a,\nu)| \leq C\left(1+\|\omega\|_t+|y|+|z|+\left|a^{1/2}\right|\right). \tag{4.14}$$

- *$\mathcal{P}^\kappa_H$ is dense in $\overline{\mathcal{P}}^\kappa_H$ in the sense that for any $\mathbb{P}^{\alpha,\nu}\in\overline{\mathcal{P}}^\kappa_H$ and for any $\varepsilon>0$, there exists $\mathbb{P}^{\alpha^\varepsilon,\nu}\in\mathcal{P}^\kappa_H$ such that*

$$\mathbb{E}^{\mathbb{P}_\nu}\left[\int_0^T |(\alpha^\varepsilon_t)^{1/2}-\alpha^{1/2}_t|^2dt\right]\leq\varepsilon. \tag{4.15}$$

Then, we have

$$Y_0 = \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} y^{\mathbb{P}}_0 = \sup_{\mathbb{P}\in\overline{\mathcal{P}}^\kappa_H} y^{\mathbb{P}}_0,$$

where under any $\mathbb{P}:=\mathbb{P}^{\alpha,\nu}\in\overline{\mathcal{P}}^\kappa_H$, $(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}})$ is the unique solution of the BSDEJ

$$y^{\mathbb{P}}_t = \xi + \int_t^T F_s(y^{\mathbb{P}}_s,z^{\mathbb{P}}_s,u^{\mathbb{P}}_s,\widehat{a}_s,\nu_s)ds - \int_t^T Z_s dB^c_s - \int_t^T\int_E u^{\mathbb{P}}_s(x)\widetilde{\mu}^{\mathbb{P}}_{B^d}(ds,dx),\ \mathbb{P}-a.s.$$

Proof. First, we remind that Remark 3.3 ensures existence and uniqueness of the solutions of our BSDEs under any $\mathbb{P} \in \overline{\mathcal{P}}^{\kappa}_H$. We will proceed in two steps.

(i) $\xi \in \mathrm{UC_b}(\Omega)$

For any $\mathbb{P} := \mathbb{P}^{\alpha,\nu} \in \overline{\mathcal{P}}^{\kappa}_H$ and any $\varepsilon > 0$, let $\mathbb{P}^{\varepsilon} := \mathbb{P}^{\alpha^{\varepsilon},\nu} \in \mathcal{P}^{\kappa}_H$ be given by (4.15). Using the process $L^{\mathbb{P}}$ defined in (2.4), we have $\mathbb{P} - a.s.$

$$y_t^{\mathbb{P}} = \xi(B_.) + \int_t^T F_s(B_., y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, u_s^{\mathbb{P}}, \widehat{a}_s, \nu_s)ds - \int_t^T \widehat{a}_s^{1/2} z_s^{\mathbb{P}} dL_s^{\mathbb{P},c}$$
$$- \int_t^T \int_E u_s^{\mathbb{P}}(x) \left(\mu_{L^{\mathbb{P},d}}(ds,dx) - \nu_s(dx)ds\right).$$

Let now $(\overline{y}^{\mathbb{P}}, \overline{z}^{\mathbb{P}}, \overline{u}^{\mathbb{P}})$ denote the unique solution of the following BSDEJ under $\mathbb{P}_\nu$

$$\overline{y}_t^{\mathbb{P}} = \xi(X_.^{\alpha,\nu}) + \int_t^T F_s(X_.^{\alpha,\nu}, \overline{y}_s^{\mathbb{P}}, \overline{z}_s^{\mathbb{P}}, \overline{u}_s^{\mathbb{P}}, \alpha_s, \nu_s)ds - \int_t^T \alpha_s^{1/2} \overline{z}_s^{\mathbb{P}} dB_s^c$$
$$- \int_t^T \int_E \overline{u}_s^{\mathbb{P}}(x) \left(\mu_{B^d}(ds,dx) - \nu_s(dx)ds\right).$$

By definition of $\mathbb{P}^{\alpha,\nu}$, we know that the distribution of $y^{\mathbb{P}}$ under $\mathbb{P}$ is equal to the distribution of $\overline{y}^{\mathbb{P}}$ under $\mathbb{P}_\nu$. Since the Blumenthal $0-1$ law also holds, this implies clearly that we have

$$y_0^{\mathbb{P}} = \overline{y}_0^{\mathbb{P}}.$$

Similarly, we define $y^{\mathbb{P}^\varepsilon}$ and $\overline{y}^{\mathbb{P}^\varepsilon}$. Then, using classical estimates from the BSDEJ theory (see [1] for instance) we have

$$\begin{aligned}
&|y_0^{\mathbb{P}} - y_0^{\mathbb{P}^\varepsilon}|^2 = |\overline{y}_0^{\mathbb{P}} - \overline{y}_0^{\mathbb{P}^\varepsilon}|^2 \\
&\leq C\mathbb{E}^{\mathbb{P}_\nu}\left[|\xi(X_.^{\alpha,\nu}) - \xi(X_.^{\alpha^\varepsilon,\nu})|^2 + \int_0^T \left|F_t(X_.^{\alpha}, \overline{y}_t^{\mathbb{P}}, \overline{z}_t^{\mathbb{P}}, \alpha_t, \nu_t) - F_t(X_.^{\alpha^\varepsilon}, \overline{y}_t^{\mathbb{P}}, \overline{z}_t^{\mathbb{P}}, \alpha_t^{\varepsilon}, \nu_t)\right|^2 dt\right].
\end{aligned}$$

Then, we have by (4.14)

$$\begin{aligned}
\left|F_t(X_.^{\alpha^\varepsilon}, \overline{y}_t^{\mathbb{P}}, \overline{z}_t^{\mathbb{P}}, \alpha_t^{\varepsilon}, \nu_t)\right| &\leq C\left(1 + \left\|X^{\alpha^\varepsilon,\nu}\right\|_t + |\overline{y}_t^{\mathbb{P}}| + |\overline{z}_t^{\mathbb{P}}| + |\alpha_t^{\varepsilon}|^{1/2}\right) \\
&\leq C\left(1 + \|X^{\alpha,\nu}\| + |\overline{y}_t^{\mathbb{P}}| + |\overline{z}_t^{\mathbb{P}}| + |\alpha_t|^{1/2}\right) \\
&\quad + C\left(\left\|X^{\alpha^\varepsilon,\nu} - X^{\alpha,\nu}\right\| + |\alpha_t^{\varepsilon} - \alpha_t|^{\mathbf{1}/2}\right).
\end{aligned} \tag{4.16}$$

Using Doob's inequality and Itô's isometry, it is easy to see that (4.15) implies that

$$\mathbb{E}^{\mathbb{P}_\nu}\left[\sup_{0\leq t\leq T}\left|X_t^{\alpha^\varepsilon,\nu} - X_t^{\alpha,\nu}\right|^2\right] \leq \varepsilon.$$

Since ξ is also uniformly continuous and bounded in ω, we can apply the dominated convergence Theorem in (4.16) to obtain

$$\lim_{\varepsilon\to 0}|y_0^{\mathbb{P}} - y_0^{\mathbb{P}^\varepsilon}| = 0.$$

This clearly implies the result in that case.

(ii) $\xi \in \overline{\mathcal{L}}_H^{2,\kappa}$

In that case, with the same ntations as above, let $\xi_n \in \mathrm{UC_b}(\Omega)$ such that $\|\xi - \xi_n\|_{\overline{\mathbb{L}}_H^{2,\kappa}} \underset{n\to+\infty}{\longrightarrow} 0$. Then, we define $y^{\mathbb{P},n}$ the solution of the BSDEJ with terminal condition ξ_n and generator $F_t(\cdot, \widehat{a}_s, \nu_s)$ under $\mathbb{P}$. Then, we have

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} y_0^{\mathbb{P},n} = \sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa} y_0^{\mathbb{P},n}. \tag{4.17}$$

Moreover, using classical estimates for BSDEs, we can show that

$$\left| y_0^{\mathbb{P},n} - y_0^{\mathbb{P}} \right|^2 \leq C \left\| \xi_n - \xi \right\|^2_{\overline{\mathbb{L}}_H^{2,\kappa}}.$$

This shows that the convergence of $y_0^{\mathbb{P},n}$ to $y_0^{\mathbb{P}}$ is uniform with respect to $\mathbb{P} \in \overline{\mathcal{P}}_H^\kappa$. Hence we can pass to the limit in (4.17) and exchange the limit and the suprema to obtain the desired result. □

We finish this Section by recalling a result from [27] (see Proposition 5.4) which gives a sufficient condition for the density condition (4.15)

Lemma 4.3. *Assume that the domain of F does not depend on t and that D_F^1 contains a countable dense subset. Then* (4.15) *holds.*

Proof. It suffices to notice that in our framework, all the constant mappings belong to $\widetilde{A}_0$. Then Proposition 5.4 in [27] applies. □

5 Application to a robust utility maximization problem

We study in this part a robust exponential utility maximization problem in the spirit of [12] and [20]. We consider the case of an agent maximizing the utility of his terminal wealth, over all possible trading strategies. The agent also have at the terminal date T a claim ξ, which is a $\mathcal{F}_T$-measurable random variable. We work in a setting with model ambiguity, due to the presence of the set $\mathcal{P}_H^\kappa$, composed of non dominated probability measures, accounting for the volatility uncertainty and jump measure uncertainty (see also remark 5.2). In [9] and [12], the utility maximization problems are solved by means of stochastic control techniques, and the value function is linked with some particular BSDEs. In our case, since we work with a whole set of non dominated probability measures, our value function will be linked to some second order BSDE with jumps, and the general martingale optimality principle described in [9] is treated via the use of non-linear martingales.

5.1 The Market

In this section, we will always assume that the matrices $\underline{a} := \underline{a}^{\mathbb{P}}$ and $\overline{a} := \overline{a}^{\mathbb{P}}$ are uniformly bounded in $\mathbb{P}$. In particular, this implies that we can restrict ourselves to the case where the parameter a in the definition of a generator F is bounded. We consider a financial market consisting of one riskless asset, whose price is assumed to be equal to 1 for simplicity, and one risky asset whose price process $(S_t)_{0\leq t\leq T}$ is assumed to follow a mixed-diffusion

$$\frac{dS_t}{S_{t^-}} = b_t dt + dB_t^c + \int_E \beta_t(x)\mu_{B^d}(dt,dx),\ \overline{\mathcal{P}}_H^\kappa - q.s., \tag{5.1}$$

where we assume that

Assumption 5.1. (i) (b_t) *is a bounded* $\mathbb{F}$*-predictable process uniformly continuous in* ω.

(ii) (β_t) *is a bounded* $\mathbb{F}$*-predictable process which is uniformly continuous in* ω*, verifies*

$$\sup_{\nu\in\mathcal{N}} \int_0^T \int_E |\beta_t(x)|\,\nu_t(dx)dt < +\infty,\ \overline{\mathcal{P}}_H^\kappa - q.s.,$$

and satisfies

$$C_1(1\wedge|x|) \leq \beta_t(x) \leq C_2(1\wedge|x|),\ \overline{\mathcal{P}}_H^\kappa - q.s., \text{ for all } (t,x)\in[0,T]\times E,$$

where $C_2 \geq 0 \geq C_1 > -1+\delta$*, where* $\delta > 0$ *is fixed.*

Remark 5.1. *The uniform continuity assumption on* ω *is here to ensure that the 2BSDEs we will encounter in the sequel indeed have solutions. The assumption on* β *is classical [19] and implies that the price process* S *is positive.*

Remark 5.2. *The volatility is implicitly embedded in the model. Indeed, under each* $\mathbb{P}\in\mathcal{P}_H^\kappa$*, we have* $dB_s^c \equiv \widehat{a}_t^{1/2} dW_t^{\mathbb{P}}$ *where* $W^{\mathbb{P}}$ *is a Brownian motion under* $\mathbb{P}$*. Therefore,* $\widehat{a}^{1/2}$ *plays the role of volatility under each* $\mathbb{P}$ *and thus allows us to model the volatility uncertainty. Similarly, we have incertitude on the jumps of our price process, since the predictable compensator associated to the jumps of the discontinuous part of the canonical process changes with the probability considered. This allows us to have incertitude not only about the size of the jumps but also about their laws.*

We then denote $\pi = (\pi_t)_{0\leq t\leq T}$ a trading strategy, which is a 1-dimensional $\mathbb{F}$-predictable process, supposed to take its value in some compact set C. The process π_t describes the amount of money invested in the stock at time t. The number of shares is $\frac{\pi_t}{S_{t^-}}$. So the liquidation value of a trading strategy π with positive initial capital x is given by the following wealth process:

$$X_t^\pi = x + \int_0^t \pi_s \left(dB_s^c + b_s ds + \int_E \beta_s(x)\mu_{B^d}(ds,dx)\right),\ 0\leq t\leq T,\ \overline{\mathcal{P}}_H^\kappa - q.s.$$

The problem of the investor in this financial market is to maximize his expected exponential utility under model uncertainty from his total wealth $X_T^\pi - \xi$ where ξ is a liability at time T which is a random variable assumed to be $\mathcal{F}_T$-measurable. Then the value function V of the maximization problem can be written as

$$V^\xi(x) := \sup_{\pi\in\mathcal{C}} \inf_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-\eta\left(X_T^\pi - \xi\right)\right)\right] = -\inf_{\pi\in\mathcal{C}} \sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\eta\left(X_T^\pi - \xi\right)\right)\right]. \tag{5.2}$$

where

$$\mathcal{C} := \{(\pi_t) \text{ which are predictable and take values in } C\},$$

is our set of admissible strategies.

Before going on, we emphasize immediately, that in the sequel we will limit ourselves to probability measures in $\mathcal{P}_H^\kappa$. We will recover the supremum over all probability measures in $\overline{\mathcal{P}}_H^\kappa$ at the end by showing that Theorem 4.2 applies.

To find the value function V^ξ and an optimal trading strategy π^*, we follow the ideas of the general *martingale optimality principle* approach as in [9] and [12] but adapt it here to a nonlinear framework. This is the same approach as in [20].

Let $\{R^\pi\}$ be a family of processes which satisfy the following properties

Properties 5.1. (i) $R_T^\pi = \exp(-\eta(X_T^\pi - \xi))$ *for all* $\pi \in \mathcal{C}$.

(ii) $R_0^\pi = R_0$ *is constant for all* $\pi \in \mathcal{C}$.

(iii) *We have*

$$R_t^\pi \leq \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \ \mathbb{E}_t^{\mathbb{P}'}[R_T^\pi], \ \forall \pi \in \mathcal{C}$$

$$R_t^{\pi^*} = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \ \mathbb{E}_t^{\mathbb{P}'}[R_T^{\pi^*}] \text{ for some } \pi^* \in \mathcal{C}, \ \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Then it follows

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[U(X_T^\pi - \xi)] \geq R_0 = \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[U(X_T^{\pi^*} - \xi)] = -V^\xi(x). \tag{5.3}$$

5.2 Solving the optimization problem with a Lipschitz 2BSDEJ

To construct R^π, we set

$$R_t^\pi = \exp(-\eta X_t^\pi)Y_t, \ t \in [0,T], \ \pi \in \mathcal{C},$$

where $(Y,Z,U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ is the unique solution of the following 2BSDEJ

$$Y_t = e^{\eta\xi} + \int_t^T \widehat{F}_s(Y_s,Z_s,U_s)ds - \int_t^T Z_s dB_s^c - \int_t^T \int_E U_s(x)\widetilde{\mu}_{B^d}(ds,dx) + K_T - K_t. \tag{5.4}$$

The generator $\widehat{F}$ is chosen so that R^π satisfies the Properties 5.1. Let us apply Itô's formula to $\exp(-\eta X_t^\pi)Y_t$ under some $\mathbb{P} \in \mathcal{P}_H^\kappa$. We obtain after some calculations

$$\begin{aligned} d\left(e^{-\eta X_t^\pi}Y_t\right) = e^{-\eta X_{t^-}^\pi} \Big[& -\eta\pi_s b_s Y_s ds + \frac{\eta^2}{2}\pi_s^2 \widehat{a}_s Y_s ds - \eta\pi_s\widehat{a}_s Z_s ds - \widehat{F}_s(Y_s,Z_s,U_s)ds \\ & + \int_E \left(e^{-\eta\pi_s\beta_s(x)} - 1\right)(Y_s + U_s(x))\widehat{\nu}_s(dx)ds + (Z_s - \eta\pi_s Y_s)\, dB_s^c \\ & + \int_E \left(e^{-\eta\pi_s\beta_s(x)} - 1\right)(Y_{s^-} + U_s(x)) + U_s(x)\widetilde{\mu}_{B^d}(ds,dx) - dK_t \Big]. \end{aligned} \tag{5.5}$$

Hence the appropriate choice for F

$$F_s(y,z,u,a,\nu) := \inf_{\pi \in C} \left\{ (-\eta b_s + \frac{\eta^2}{2}\pi a)\pi y - \eta \pi a z + \int_E \left(e^{-\eta\pi\beta_s(x)} - 1 \right)(y + u(x))\nu(dx) \right\}.$$

First, because of Assumption 5.1, F is uniformly Lipschitz in (y,z), uniformly continuous in ω. It is also continuous in a and since $D_F^1 = [\underline{a}, \overline{a}]$, it is even uniformly continuous in a. Besides, it is convex in a and ν (since it is the infimum of a family of linear functions) and hence can be written as a Fenchel-Legendre transform. Moreover, its domain clearly does not depend on (ω, t, y, z, u) by our boundedness assumptions. Besides, D_F^1 clearly contains a countable dense subset. This in particular shows that Theorem 4.2 applies here. Finally,

$$\inf_{\pi \in C} \int_E \left(e^{-\eta\pi\beta_s(\omega,x)} - 1 \right)(u(x) - u'(x))\nu(dx) \leq F_s(\omega,y,z,u,a,\nu) - F_s(\omega,y,z,u',a,\nu)$$
$$F_s(\omega,y,z,u,a,\nu) - F_s(\omega,y,z,u',a,\nu) \leq \sup_{\pi \in C} \int_E \left(e^{-\eta\pi\beta_s(\omega,x)} - 1 \right)(u(x) - u'(x))\nu(dx).$$

Since C is compact and β is bounded, it is therefore clear from the above inequalities that Assumption 3.1(iv) is satisfied. Therefore, if we assume that $e^{\eta\xi} \in \mathcal{L}_H^{2,\kappa}$ (for instance if $\xi \in \mathcal{L}_H^{\infty,\kappa}$), the 2BSDEJ (5.4) indeed has a unique solution and R^π is well defined. Let us now prove that it satisfies the properties 5.1. The property (i) is clear by definition and (ii) holds because of Proposition 4.3 together with the Bluementhal $0-1$ law. Now for any $0 \leq t \leq T$, any $\pi \in \mathcal{C}$, any $\mathbb{P} \in \mathcal{P}_H^\kappa$ and any $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$, we have from (5.5)

$$\mathbb{E}_t^{\mathbb{P}'}[R_T^\pi] - R_t^\pi \geq -\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{-\eta X_{s^-}^\pi} dK_s\right]. \tag{5.6}$$

Let us now prove that for any $\pi \in \mathcal{C}$ and for any $\mathbb{P}$, we have

$$\underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{-\eta X_{s^-}^\pi} dK_s\right] = 0. \tag{5.7}$$

This is similar to what we did in the proof of Theorem 3.1, and therefore we know that it is sufficient to prove that for any $p > 1$

$$\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{t \leq s \leq T} e^{-pX_s^\pi}\right] \leq C_p, \tag{5.8}$$

for some positive constant C_p depending only on p and the bounds for π, b and β. Let M be such that $-M \leq \pi \leq M$, we have

$$\begin{aligned} e^{-X_s^\pi} &\leq e^{TM\|b\|_\infty - \int_0^t \pi_s dB_s^c + M\int_0^t \int_E |\beta_s(x)| \mu_{B^d}(ds,dx)} \\ &= e^{TM\|b\|_\infty + \frac{1}{2}\int_0^t \pi_s^2 \widehat{a}_s ds + M\int_0^T \int_E |\beta_s(x)| \widehat{\nu}_s(dx)ds + M\int_0^T \int_E |\beta_s(x)| - \ln(1+|\beta_s(x)|)\widehat{\nu}_s(dx)ds} \\ &\quad \times \mathcal{E}\left(\int_0^t \pi_s dB_s^c\right) \mathcal{E}\left(\int_0^t |\beta_s(x)| \widetilde{\mu}_{B^d}(ds,dx)\right)^M \\ &\leq C\mathcal{E}\left(-\int_0^t \pi_s dB_s^c\right) \mathcal{E}\left(\int_0^t |\beta_s(x)| \widetilde{\mu}_{B^d}(ds,dx)\right)^M, \end{aligned}$$

where we used in the last inequality the fact that π, $\widehat{a}$ and β are uniformly bounded and that

$$\sup_{\nu\in\mathcal{N}}\int_0^T\int_E |\beta_t(x)|\,\nu_t(dx)dt < +\infty \text{ and } |\beta_s(x)-\ln(1+\beta_s(x))| \leq C|\beta_s(x)|^2.$$

Then (5.8) comes from the fact that above Doléans-Dade exponentials have moments of any order (it is clear for the continuous one, and we refer again to Lemma A.4 in [17] for the discontinuous one). Using (5.7) in (5.6), we obtain

$$R_t^\pi \leq \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}[R_T^\pi].$$

Now, using a classical mesurable selection argument (see [6] (chapitre III) or [7] or Lemma 3.1 in [8]) we can define a predictable process $\pi^*\in\mathcal{C}$ such that

$$\widehat{F}_s(Y_s,Z_s,U_s) = (-\eta b_s + \frac{\eta^2}{2}\pi_s^*\widehat{a}_s)\pi_s^* Y_s - \eta\pi_s^*\widehat{a}_s + \int_E\left(e^{-\eta\pi_s^*\beta_s(x)}-1\right)(Y_s+U_s(x))\widehat{\nu}_s(dx).$$

Using the same arguments as above, we obtain

$$R_t^{\pi^*} = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}[R_T^{\pi^*}],$$

which proves (iii) of Property 5.1 holds.

We summarize everything in the following proposition

Proposition 5.1. *Assume that* $\exp(\eta\xi)\in\overline{\mathcal{L}}_H^{2,\kappa}$. *Then, under Assumption 5.1, the value function of the optimization problem* (5.2) *is given by*

$$V^\xi(x) = -e^{-\eta x}Y_0,$$

where Y_0 *is defined as the initial value of the unique solution* $(Y,Z,U)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}\times\mathbb{J}_H^{2,\kappa}$ *of the following 2BSDEJ*

$$Y_t = e^{\eta\xi} + \int_t^T \widehat{F}_s(Y_s,Z_s,U_s)ds - \int_t^T Z_s dB_s^c - \int_t^T\int_E U_s(x)\widetilde{\mu}_{B^d}(ds,dx) + K_T - K_t, \quad (5.9)$$

where the generator is defined as follows

$$\widehat{F}_t(\omega,y,z,u) := F_t(\omega,y,z,u,\widehat{a}_t,\widehat{\nu}_t),$$

where

$$F_t(y,z,u,a,\nu) := \inf_{\pi\in C}\left\{(-\eta b_t + \frac{\eta^2}{2}\pi a)\pi y - \eta\pi a z + \int_E\left(e^{-\eta\pi\beta_t(x)}-1\right)(y+u(x))\nu(dx)\right\}.$$

Moreover, there exists an optimal trading strategy π^* *realizing the infimum above.*

5.3 A link with a particular quadratic 2BSDEJ

In the case without uncertainty on the parameters, coming back to the paper of El Karoui and Rouge [9], utility maximization problems in a continuous framework with constrained strategies have usually been linked to BSDEs with a quadratic growth generator. The same type of results were later proved by Morlais [21] in a discontinuous setting, that is to say when the assets in the market are assumed to have jumps. More recently, Lim and Quenez [19] showed that when the strategies are constrained in a compact set, then one only needed to consider a BSDE with a Lipschitz generator, thus simplifying the approach of Morlais. Since we provided in this paper a wellposedness theory for Lipscitz 2BSDEs with jumps, we used the ideas of [19] in the previous section. Our aim in this section is to show that with the result of Proposition 5.1 and by making a change of variables, we can prove the existence of a solution to a particular 2BSDEJ, similar to the one in [21], whose generator satisfies a quadratic growth condition.

Before proceeding with the proof, we recall that as for quadratic BSDEs and 2BSDEs, we always have a deep link between the martingale part of the solution and the BMO spaces. So we need to introduce the following spaces.

$\mathbb{J}^{2,\kappa,H}_{BMO}$ denotes the space of predictable and $\mathcal{E}$-mesurable applications $U:\Omega\times[0,T]\times E$ such that

$$\|U\|^2_{\mathbb{J}^{2,\kappa,H}_{BMO}} := \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H}\left\|\int_0^\cdot\int_E U_s(x)\widetilde{\mu}_{B^d}(ds,dx)\right\|_{\mathrm{BMO}(\mathbb{P})} < +\infty,$$

where $\|\cdot\|_{\mathrm{BMO}(\mathbb{P})}$ is the usual $\mathrm{BMO}(\mathbb{P})$ norm under $\mathbb{P}$.

$\mathbb{H}^{2,\kappa,H}_{BMO}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{H^{2,\kappa,H}_{BMO}} := \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H}\left\|\int_0^\cdot Z_s dB^c_s\right\|_{\mathrm{BMO}(\mathbb{P})} < +\infty.$$

We refer the reader to the book by Kazamaki [15] and the references therein for more information on the BMO spaces.

Proposition 5.2. *Assume that* $\exp(\eta\xi)\in\mathcal{L}^{2,\kappa}_H$ *and that* ξ *is bounded quasi-surely. Then, there exists a solution* $(Y',Z',U')\in\mathbb{D}^{2,\kappa}_H\times\mathbb{H}^{2,\kappa}_H\times\mathbb{J}^{2,\kappa}_H$ *to the quadratic 2BSDEJ*

$$Y'_t = \xi + \int_t^T \widehat{F}'_s(Z'_s,U'_s)ds - \int_t^T Z'_s dB^c_s - \int_t^T\int_E U'_s(x)\widetilde{\mu}_{B^d}(ds,dx) + K'_T - K'_t, \quad (5.10)$$

where the generator is defined as follows

$$\widehat{F}'_t(\omega,z,u) := F'_t(\omega,z,u,\widehat{a}_t,\widehat{\nu}_t), \quad (5.11)$$

where

$$F'_t(z,u,a,\nu) := \inf_{\pi\in C}\left\{\frac{\eta}{2}\left|\pi a^{1/2} - \left(a^{1/2}z + \frac{b_t+\int_E\beta_t(x)\nu(dx)}{a^{1/2}\eta}\right)\right|^2 + \frac{1}{\eta}j\left(\eta(u-\pi\beta_t)\right)\right\}$$
$$-\left(b_t+\int_E\beta_t(x)\nu(dx)\right)a^{1/2}z - \frac{|b_t+\int_E\beta_t(x)\nu(dx)|^2}{2a\eta},$$

where $j(u,\nu) := \int_E \left(e^{u(x)} - 1 - u(x)\right)\nu(dx)$. *Moreover,*

$$Y'_t = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\; y'^{\mathbb{P}}_t,$$

where $y'^{\mathbb{P}}$ *is the solution to the quadratic BSDE with the same terminal condition* ξ *and generator* $\widehat{F}'$. *Besides,* Y' *is also bounded quasi-surely.*

Proof.
Step 1: As in Morlais [21], we can verify that the generator $\widehat{F}'$ satisfies the following conditions.

(i) $\widehat{F}'$ has the quadratic growth property. There exists $(\alpha,\delta)\in\mathbb{R}_+\times\mathbb{R}_+^*$ such that for all (t,z,u), $\mathcal{P}_H^\kappa - q.s.$ $\left|\widehat{F}'_t(0,0)\right|\leq\alpha$ and

$$-\left|\widehat{F}'_t(0,0)\right| - \frac{\delta}{2}\left|\widehat{a}_t^{1/2}z\right|^2 - \frac{1}{\delta}\widehat{j}_t(-\eta u)\leq \widehat{F}'_t(z,u)\leq\left|\widehat{F}'_t(0,0)\right| + \frac{\delta}{2}\left|\widehat{a}_t^{1/2}z\right|^2 + \frac{1}{\delta}\widehat{j}_t(\gamma u),$$

where $\widehat{j}_t(u) := \int_E\left(e^{u(x)} - 1 - u(x)\right)\widehat{\nu}_t(dx)$.

(ii) We have a "local Lipschitz" condition in z, $\exists\mu>0$ and a progressively measurable process $\phi\in\mathbb{H}^{2,\kappa,H}_{BMO}$ such that for all (t,z,z',u), $\mathcal{P}_H^\kappa - q.s.$

$$\left|\widehat{F}'_t(z,u) - \widehat{F}'_t(z',u) - \phi_t.(\widehat{a}_t^{1/2}z - \widehat{a}_t^{1/2}z')\right|\leq\mu\left|\widehat{a}_t^{1/2}z - \widehat{a}_t^{1/2}z'\right|\left(\left|\widehat{a}_t^{1/2}z\right| + \left|\widehat{a}_t^{1/2}z'\right|\right).$$

(iii) For every (z,u,u') there exists a predictable and $\mathcal{E}$-mesurable process (γ_t) such that $\mathcal{P}_H^\kappa - q.s.$,

$$\widehat{F}'_t(z,u) - \widehat{F}'_t(z,u')\leq\int_0^t\int_E\gamma_s(x)\left(u(x)-u'(x)\right)\widehat{\nu}_t(dx)ds$$

$$\int_0^t\int_E\gamma'_s(x)\left(u(x)-u'(x)\right)\widehat{\nu}_t(dx)ds\leq\widehat{F}'_t(z,u) - \widehat{F}'_t(z,u'),$$

where there exists constants $C_1, C'_1 > -1$ and $C_2, C'_2 > 0$, independent of (z,u,u') such that

$$C_1(1\wedge|x|)\leq\gamma_t(x)\leq C_2(1\wedge|x|),\text{ and } C'_1(1\wedge|x|)\leq\gamma'_t(x)\leq C'_2(1\wedge|x|).$$

In particular, γ and γ' are in $\mathbb{J}^{2,\kappa,H}_{BMO}$.

We know from [21], that under each $\mathbb{P}$, the BSDEJ with terminal condition ξ and generator $\widehat{F}'$ has a solution, that we denote $(y'^{\mathbb{P}}, z'^{\mathbb{P}}, u'^{\mathbb{P}})$. Moreover, $y'^{\mathbb{P}}$ is bounded and there exists a bounded version of $u'^{\mathbb{P}}$ (that we still denote $u'^{\mathbb{P}}$) , with the following estimates

$$\left\|y'^{\mathbb{P}}\right\|_\infty\leq C(1+\|\xi\|_\infty)\text{ and } |u'^{\mathbb{P}}|\leq 2\left\|y'^{\mathbb{P}}\right\|_\infty,$$

where $C>0$. Next by applying Itô's formula, we have $y_t^{\mathbb{P}} = e^{\eta y'^{\mathbb{P}}}$, $t\in\mathbb{R}^+$, where $(y^{\mathbb{P}}, z^{\mathbb{P}}, u^{\mathbb{P}})$ is the solution under $\mathbb{P}$ of equation (5.9). This gives that

$$e^{-C'}\leq y_t^{\mathbb{P}}\leq e^{C'},\ t\in\mathbb{R}^+,\ C'\in\mathbb{R}.$$

In particular $y^{\mathbb{P}}$ is strictly positive and bounded away from 0, and by representation (3.7), so is Y solving (5.9). Thus, we can make the following change of variables

$$Y'_t := \frac{1}{\eta}\log(Y_t).$$

Then by Itô's formula and the fact that K has only predictable jumps, we can verify that the triple (Y', Z', U') satisties (5.10) where

$$Z'_t := \frac{1}{\eta}\frac{Z_t}{Y_t},\ U'_t := \frac{1}{\eta}\log\left(1+\frac{U_t}{Y_{t-}}\right),\ K'_t := \int_0^t \frac{1}{\eta Y_s}dK^c_s - \sum_{0<s\leq t}\frac{1}{\eta}\log\left(1-\frac{\Delta K^d_s}{Y_{s-}}\right).$$

In particular, K' is predictable and nondecreasing with ${K'}_0 = 0$. Finally, due to the monotonicity of the function $\frac{1}{\eta}\log(x)$, we have the following representation for Y'

$$Y'_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^\kappa_H(t^+,\mathbb{P})} y'^{\mathbb{P}}_t.$$

<u>**Step**</u> 2: Next, we will prove the minimum condition for K'. As in [24] for 2BSDE with quadratic growth generator, we use the above representation of Y' and the properties verified by $\widehat{F}'$ in z and u.

Fix $\mathbb{P}$ in $\mathcal{P}^\kappa_H$ and $\mathbb{P}' \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})$, denote

$$\delta Y' := Y' - y'^{\mathbb{P}'},\ \delta Z' := Z' - z'^{\mathbb{P}'} \text{ and } \delta U' := U' - u'^{\mathbb{P}'}.$$

By the "local Lipschitz" condition (ii) of $\widehat{F}'$ in z, there exist a process η with

$$|\eta_t| \leq \mu\left(\left|\widehat{a}_t^{1/2}Z_t\right| + \left|\widehat{a}_t^{1/2}z_t^{\mathbb{P}'}\right|\right),\ \mathbb{P}' - a.s.$$

such that we have for all $0\leq t\leq T$, $\mathbb{P}' - a.s.$

$$\begin{aligned}\delta Y'_t =& \int_t^T (\eta_s+\phi_s)\widehat{a}_s^{\frac{1}{2}}\delta Z'_s ds - \int_t^T \delta Z'_s dB^c_s - \int_t^T\int_E \delta U'_s(x)\left[\widetilde{\mu}_{B^d}(ds,dx) - \gamma'_s(x)\widehat{\nu}(dx)ds\right] \\ &+ \int_t^T \left[\widehat{F}'_s(z'_s,U'_s) - \widehat{F}'_s(z'_s,u'_s)\right]ds - \int_t^T\int_E \gamma'_s(x)\delta U'_s(x)\widehat{\nu}(dx)ds + K'_T - K'_t. \quad (5.12)\end{aligned}$$

Let us now prove that we have $Z' \in \mathbb{H}^{2,\kappa,H}_{BMO}$, $U' \in \mathbb{J}^{2,\kappa,H}_{BMO}$ and that U' (or more precisely a version of U') is uniformly bounded. Actually, this can be proved using exactly the same arguments as in the proof of Lemma 3.3.1 in [24] on the one hand, that is to say applying Itô's formula to $e^{-\nu Y'_t}$ for a well chosen $\nu > 0$. This allows then to get rid of the non-decreasing process K' in the estimates. Then on the other hand, one can use the same calculations as in the proof of Lemma 2.2 in [16] to obtain that

$$Z' \in \mathbb{H}^{2,\kappa,H}_{BMO},\ \left(e^{-\delta U'} - 1\right) \in \mathbb{J}^{2,\kappa,H}_{BMO} \text{ and that } U' \text{ is uniformly bounded.}$$

Then, using the mean value theorem, it is easy to show that for every $t\in[0,T]$ and every $x\in E$, there exists some $c\in[-(U_t(x))^-, (U_t(x))^+]$ such that

$$U'_t(x) = \frac{1-e^{-\delta U'_t(x)}}{\delta}e^{\delta c}.$$

Since c is bounded, this clearly implies that $U' \in \mathbb{J}^{2,\kappa,H}_{BMO}$. Then the process η defined above is also in $\mathbb{H}^{2,\kappa,H}_{BMO}$. Since the process γ' is bounded and has jumps greater than $-1+\delta$, Kazamaki criterion (see [14]) implies that we can define an equivalent probability measure $\mathbb{Q}'$ such that

$$\frac{d\mathbb{Q}'}{d\mathbb{P}'} = \mathcal{E}\left(\int_0^{\cdot} (\eta_s + \phi_s)\, \widehat{a}_s^{-1/2} dB^c_s + \int_0^{\cdot}\int_E \gamma'_s(x)\widetilde{\mu}_{B^d}(ds,dx)\right).$$

By taking conditional expectations in (5.12) and using property (iii) satisfied by F', we obtain

$$Y'_t - {y'}_t^{\mathbb{P}'} \geq \mathbb{E}_t^{\mathbb{Q}'}\left[K'_T - K'_t\right].$$

For notational convenience, denote

$$\mathcal{E}^1_t := \mathcal{E}\left(\int_0^t (\phi_s+\eta_s)\widehat{a}_s^{-1/2} dB^c_s\right) \text{ and } \mathcal{E}^2_t := \mathcal{E}\left(\int_0^t \int_E \gamma'_s(x)\widetilde{\mu}_{B^d}(ds,dx)\right).$$

Let $r>1$ be the number given by Lemma 3.2.2 in [24] applied to $\mathcal{E}^1$. Then we estimate

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}'}\left[{K'}_T - K'_t\right]\\
&\leq \mathbb{E}_t^{\mathbb{P}'}\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}\left(K'_T - K'_t\right)\right]^{\frac{1}{2r-1}} \mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_t}{\mathcal{E}_T}\right)^{\frac{1}{2(r-1)}}\left(K'_T-K'_t\right)\right]^{\frac{2(r-1)}{2r-1}}\\
&\leq \left(\delta Y'_t\right)^{\frac{1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}^1_t}{\mathcal{E}^1_T}\right)^{\frac{1}{r-1}}\right]\right)^{\frac{r-1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}^2_t}{\mathcal{E}^2_T}\right)^{\frac{2}{r-1}}\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(K'_T-K'_t\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}\\
&\leq C\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K'_T\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}\left(\delta Y'_t\right)^{\frac{1}{2r-1}},
\end{aligned}$$

where we used in the last inequality Lemma 3.2.2 in [24] for the term involving $\mathcal{E}^1$ and where we used Lemma A.4 of [17] for the term involving $\mathcal{E}^2$ (this is possible because of the properties verified by γ').

With the same argument as in Step (iii) of the proof of Theorem 3.1, the above inequality along with the representation for Y' shows that we have

$$\underset{\mathbb{P}'\in\mathcal{P}^\kappa_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}^{\mathbb{P}'}\left[K'_T - K'_t\right] = 0,$$

that is to say that the minimum condition 3.5 is verified. Indeed, the only thing to verify is that

$$\underset{\mathbb{P}'\in\mathcal{P}^\kappa_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\left(K'_T\right)^4\right] < +\infty. \tag{5.13}$$

With the BMO properties satisfied by Z' and U' we can follow the proof of Theorem 3.2 in [24] to show that

$$\sup_{\mathbb{P}\in\mathcal{P}^\kappa_H}\mathbb{E}^{\mathbb{P}}\left[\left(K'_T\right)^4\right] < +\infty.$$

Then (5.13) can be obtained exactly as in the proof of Step (iii) of Theorem 4.1 in [17]. □

A Appendix

A.1 The measures $\mathbb{P}^{\alpha,\nu}$

Lemma A.1. *Let $\mathbb{P} \in \mathcal{P}_{\tilde{A}}$ and τ be an $\mathbb{F}^B$-stopping time. Then*

$$\mathbb{P}^{\tau,\omega} \in \mathcal{P}_{\tilde{A}}^{\tau(\omega)}$$

Proof.
Step 1: Let us first prove that

$$(\mathbb{P}_\nu)^{\tau,\omega} = \mathbb{P}^{\tau(\omega)}_{\nu^{\tau,\omega}}, \ \ \mathbb{P}_\nu\text{-a.s. on } \Omega, \tag{A.1}$$

where $(\mathbb{P}_\nu)^{\tau,\omega}$ denotes the probability measure on Ω^τ, constructed from the regular conditional probability distribution (r.c.p.d.) of $\mathbb{P}_\nu$ for the stopping time τ, evaluated at ω, and $\mathbb{P}^{\tau(\omega)}_{\nu^{\tau,\omega}}$ is the unique solution of the martingale problem $(\mathbb{P}^1, \tau(\omega), T, Id, \nu^{\tau,\omega})$, where $\mathbb{P}^1$ is such that $\mathbb{P}^1(B^\tau_\tau = 0) = 1$.

It is enough to show that the shifted processes M^τ, J^τ, Q^τ are $(\mathbb{P}_\nu)^{\tau,\omega}$-local martingales, where M, J and Q are defined in Remark 2.1. For this, take a bounded $\mathbb{F}^\tau$-stopping time S. Observe that it is then clear that there exists a bounded $\mathbb{F}$-stopping time $\tilde{S}$ such that $S = \tilde{S}^{\tau,\omega}$. Then, following the definitions in Subsection 4.1,

$$\Delta B_S^{\tau,\omega}(\widetilde{\omega}) = \Delta B_S(\omega \otimes_\tau \widetilde{\omega}) = \Delta(\omega \otimes_\tau \widetilde{\omega})(S) = \Delta\omega_S \mathbf{1}_{\{S\leq\tau\}} + \Delta\widetilde{\omega}_S \mathbf{1}_{\{S>\tau\}},$$

and that for $S \geq \tau$

$$B_S(\omega \otimes_\tau \widetilde{\omega}) = (\omega \otimes_\tau \widetilde{\omega})(S) = \omega_\tau + \widetilde{\omega}_S = B_\tau(\omega) + B^\tau_S(\widetilde{\omega}).$$

From this we get

$$\begin{aligned}
M_S^{\tau,\omega}(\widetilde{\omega}) = M_S(\omega \otimes_\tau \widetilde{\omega}) =& B_S(\omega \otimes_\tau \widetilde{\omega}) - \sum_{u\leq S} \mathbf{1}_{|\Delta B_u(\omega\otimes_\tau\widetilde{\omega})|>1} \Delta B_u(\omega \otimes_\tau \widetilde{\omega}) \\
&+ \int_0^S \int_E x\mathbf{1}_{|x|>1} \nu_u(\omega \otimes_\tau \widetilde{\omega})(dx)\, du \\
=& B^\tau_S(\widetilde{\omega}) + B_t(\omega) - \sum_{u\leq\tau} \mathbf{1}_{|\Delta\omega_u|>1}\Delta\omega_u - \sum_{\tau<u\leq S} \mathbf{1}_{|\Delta B^\tau_u(\widetilde{\omega})|>1} \Delta B^\tau_u(\widetilde{\omega}) \\
&+ \int_0^\tau \int_E x\mathbf{1}_{|x|>1}\nu_u(\omega)(dx)\, du + \int_\tau^S \int_E x\mathbf{1}_{|x|>1} \nu_u^{\tau,\omega}(\widetilde{\omega})(dx)\, du \\
=& M^\tau_S(\widetilde{\omega}) + M_\tau(\omega),
\end{aligned}$$

and we can now compute

$$\begin{aligned}
\mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}[M^\tau_S] &= \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[M_S^{\tau,\omega} - M_\tau(\omega)\right] \\
&= \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[M^{\tau,\omega}_{\tilde{S}^{\tau,\omega}}\right] - M_\tau(\omega) \\
&= \mathbb{E}^{\mathbb{P}_\nu}_\tau[M_{\tilde{S}}](\omega) - M_\tau(\omega) = 0, \text{for } \mathbb{P}_\nu\text{-a.e. } \omega.
\end{aligned}$$

Since S is an arbitrary bounded stopping time, we have that M^τ is a $(\mathbb{P}_\nu)^{\tau,\omega}$-local martingale for $\mathbb{P}_\nu$-a.e. ω..

We treat the case of the process J^τ analogously and write

$$\begin{aligned}
J_S^{\tau,\omega}(\widetilde{\omega}) &= \big(M_S^{\tau,\omega}(\widetilde{\omega})\big)^2 - S - \int_0^\tau \int_E x^2 \nu_u(\omega)(dx)du - \int_\tau^S \int_E x^2 \nu_u^{\tau,\omega}(\widetilde{\omega})(dx)du \\
&= (M_S^\tau(\widetilde{\omega}))^2 + (M_\tau(\omega))^2 + 2M_S^\tau(\widetilde{\omega})M_\tau(\omega) - (S-\tau) - \int_\tau^S \int_E x^2 \nu_u^{\tau,\omega}(\widetilde{\omega})(dx)du \\
&\quad - \int_0^\tau \int_E x^2 \nu_u(\omega)(dx)du - \tau \\
&= J_S^\tau(\widetilde{\omega}) + J_\tau(\omega) + 2M_S^\tau(\widetilde{\omega})M_\tau(\omega).
\end{aligned}$$

Then we can compute the expectation

$$\mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[J_S^\tau\right] = \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[J_S^{\tau,\omega} - 2M_S^\tau M_\tau(\omega)\right] - J_\tau(\omega) = 0, \text{for } \mathbb{P}_\nu\text{-a.e. } \omega.$$

J^τ is then a $(\mathbb{P}_\nu)^{\tau,\omega}$-local martingale for $\mathbb{P}_\nu$-a.e. ω.. Finally, we do the same kind of calculation for Q^τ, and we obtain

$$\begin{aligned}
Q_S^{\tau,\omega}(\widetilde{\omega}) &= \int_0^S \int_E g(x)\mu_B(\omega \otimes_\tau \widetilde{\omega}, dx, du) - \int_0^S \int_E g(x)\nu_u^{\tau,\omega}(\widetilde{\omega})(dx)\,du \\
&= \int_0^\tau \int_E g(x)\mu_B(\omega, dx, du) + \int_\tau^S \int_E g(x)\mu_{B^\tau}(\widetilde{\omega}, dx, du) \\
&\quad - \int_0^\tau \int_E g(x)\nu_u(\omega)(dx)du - \int_\tau^S \int_E g(x)\nu_u^{\tau,\omega}(\widetilde{\omega})(dx)\,du \\
&= Q_S^\tau(\widetilde{\omega}) + Q_\tau(\omega).
\end{aligned}$$

And again we compute the expectation over the $\widetilde{\omega} \in \Omega^\tau$, under the measure $(\mathbb{P}_\nu)^{\tau,\omega}$

$$\begin{aligned}
\mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[Q_S^\tau\right] &= \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[Q_S^{\tau,\omega} - Q_\tau(\omega)\right] \\
&= \mathbb{E}_\tau^{\mathbb{P}_\nu}[Q_S](\omega) - Q_\tau(\omega) = 0, \text{for } \mathbb{P}_\nu\text{-a.e. } \omega.
\end{aligned}$$

We have the desired result, and conclude that (A.1) holds true. We can now deduce that for any $(\alpha,\nu) \in \tilde{\mathcal{A}}$

$$\mathbb{P}^{\alpha^{\tau,\omega},\nu^{\tau,\omega}} \in \mathcal{P}_{\tilde{\mathcal{A}}}^{\tau(\omega)} \ \mathbb{P}_\nu\text{-a.s. on } \Omega. \tag{A.2}$$

Indeed, if $(\alpha,\nu) \in \mathcal{D}\times\mathcal{N}$, then $(\alpha^{\tau,\omega},\nu^{\tau,\omega}) \in \mathcal{D}^{\tau(\omega)} \times \mathcal{N}^{\tau(\omega)}$, because

$$\int_{\tau(\omega)}^T \int_E (1 \wedge |x|^2)\nu_s^{\tau,\omega}(\widetilde{\omega})(dx)ds + \int_{\tau(\omega)}^T \int_{|x|>1} x\nu_s^{\tau,\omega}(\widetilde{\omega})(dx)ds + \int_{\tau(\omega)}^T |\alpha_s^{\tau,\omega}(\widetilde{\omega})|\,ds < \infty.$$

Step 2: We define $\tilde{\tau} := \tau \circ X^\alpha$, $\tilde{\alpha}^{\tau,\omega} := \alpha^{\tilde{\tau},\beta_\alpha(\omega)}$ and $\tilde{\nu}^{\tau,\omega} := \nu^{\tilde{\tau},\beta_\alpha(\omega)}$ where β_α is a measurable map such that $B = \beta_\alpha(X^\alpha)$, $\mathbb{P}_\nu$-a.s. Moreover, $\tilde{\tau}$ is a stopping time and we have $\tau = \tilde{\tau} \circ \beta_\alpha$ since

$$\tilde{\tau} \circ \beta_\alpha = \tau \circ \beta_\alpha(X^\alpha) = \tau \circ B = \tau$$

and using (A.2),

$$\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}} \in \mathcal{P}_{\tilde{\mathcal{A}}}^{\tau(\omega)} \;\; \mathbb{P}^{\alpha,\nu}\text{-a.s. on } \Omega.$$

Step 3: We show that

$$\mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\ldots,B_{t_n\wedge\tau}\right)\psi\left(B_{t_1},\ldots,B_{t_n}\right)\right]=\mathbb{E}^{\mathbb{P}_{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\ldots,B_{t_n\wedge\tau}\right)\psi_\tau\right]$$

for every $0<t_1<\cdots<t_n\le T$, every continuous and bounded functions ϕ and ψ and

$$\psi_\tau(\omega)=\mathbb{E}^{\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}}\left[\psi(\omega(t_1),\ldots,\omega(t_k),\omega(t)+B^t_{t_{k+1}},\ldots,\omega(t)+B^t_{t_n})\right],$$

for $t:=\tau(\omega)\in[t_k,t_{k+1})$.

Recall that $\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}$ is defined by $\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}=\mathbb{P}_{\tilde{\nu}^{\tau,\omega}}\circ\left(X^{\tilde{\alpha}^{\tau,\omega}}\right)^{-1}$, then

$$\begin{aligned}\psi_\tau(\omega)=\mathbb{E}^{\mathbb{P}^{\tau(\omega)}_{\tilde{\nu}^{\tau,\omega}}}\Big[\psi\Big(&\omega(t_1),\ldots,\omega(t_k),\omega(t)+\int_t^{t_{k+1}}\left(\alpha_s^{\tilde{\tau},\beta_\alpha(\omega)}\right)^{1/2}d(B^c_s)^{\tau(\omega)}\\&+\int_t^{t_{k+1}}\int_E x(\mu_{B^{\tau(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\beta_\alpha(\omega)}(dx)ds),\ldots,\omega(t)+\int_t^{t_n}\left(\alpha_s^{\tilde{\tau},\beta_\alpha(\omega)}\right)^{1/2}d(B^c_s)^{\tau(\omega)}\\&+\int_t^{t_n}\int_E x(\mu_{B^{\tau(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\beta_\alpha(\omega)}(dx)ds)\Big)\Big].\end{aligned}$$

Then, $\forall\omega\in\Omega$, if $t:=\tilde{\tau}(\omega)=\tau\left(X^\alpha(\omega)\right)\in[t_k,t_{k+1}[$,

$$\begin{aligned}\psi_\tau\left(X^\alpha(\omega)\right)=\mathbb{E}^{\mathbb{P}^{\tilde{\tau}(\omega)}_{\nu^{\tilde{\tau},\omega}}}\Big[\psi\Big(&X^\alpha_{t_1}(\omega),\ldots,X^\alpha_{t_k}(\omega),X^\alpha_t(\omega)+\int_t^{t_{k+1}}\left(\alpha_s^{\tilde{\tau},\omega}\right)^{1/2}d(B_s^{\tilde{\tau}(\omega)})^c\\&+\int_t^{t_{k+1}}\int_E x(\mu_{B^{\tilde{\tau}(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\omega}(dx)ds),\ldots,X^\alpha_t(\omega)+\int_t^{t_n}\left(\alpha_s^{\tilde{\tau},\omega}\right)^{1/2}d(B_s^{\tilde{\tau}(\omega)})^c\\&+\int_t^{t_n}\int_E x(\mu_{B^{\tilde{\tau}(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\omega}(dx)ds)\Big)\Big].\end{aligned}\tag{A.3}$$

We remark that for every $\omega\in\Omega$,

$$\begin{aligned}&\alpha_s(\omega)=\alpha_s\left(\omega\otimes_{\tilde{\tau}(\omega)}\omega^{\tilde{\tau}(\omega)}\right)=\alpha_s^{\tilde{\tau},\omega}\left(\omega^{\tilde{\tau}(\omega)}\right)\\&\text{and }\;\nu_s(\omega)(dx)=\nu_s\left(\omega\otimes_{\tilde{\tau}(\omega)}\omega^{\tilde{\tau}(\omega)}\right)(dx)=\nu_s^{\tilde{\tau},\omega}\left(\omega^{\tilde{\tau}(\omega)}\right)(dx)\end{aligned}$$

By definition, the $(\mathbb{P}_\nu)^{\tilde{\tau},\omega}$-distribution of $B^{\tilde{\tau}(\omega)}$ is equal to the $(\mathbb{P}_\nu)^\omega_{\tilde{\tau}}$-distribution of $(B_\cdot-B_{\tilde{\tau}(\omega)})$. (A.3) then becomes

$$\begin{aligned}\psi_\tau\left(X^\alpha(\omega)\right)=&\mathbb{E}^{(\mathbb{P}_\nu)^\omega_{\tilde{\tau}}}\Big[\psi\Big(X^\alpha_{t_1}(\omega),\ldots,X^\alpha_{t_k}(\omega),X^\alpha_t(\omega)+\int_t^{t_{k+1}}\alpha_s^{1/2}(B^c)d(B^c_s)\\&+\int_t^{t_{k+1}}\int_E x(\mu_B(ds,dx)-\nu_s(dx)ds),\ldots,X^\alpha_t(\omega)+\int_t^{t_n}\alpha_s^{1/2}(B^c)d(B^c_s)\\&+\int_t^{t_n}\int_E x(\mu_B(ds,dx)-\nu_s(dx)ds)\Big)\Big]\\=&\mathbb{E}^{(\mathbb{P}_\nu)^\omega_{\tilde{\tau}}}\left[\psi\Big(X^\alpha_{t_1},\ldots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\ldots,X^\alpha_{t_n}\Big)\right]\\=&\mathbb{E}^{\mathbb{P}_\nu}\left[\psi\Big(X^\alpha_{t_1},\ldots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\ldots,X^\alpha_{t_n}\Big)|\mathcal{F}_{\tilde{\tau}}\right](\omega),\;\;\mathbb{P}_\nu\text{-a.s. on }\Omega.\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\ldots,B_{t_n\wedge\tau}\right)\psi_\tau\right]=\mathbb{E}^{\mathbb{P}_\nu}\left[\phi\Big(X^\alpha_{t_1\wedge\tilde\tau},\ldots,X^\alpha_{t_n\wedge\tilde\tau}\Big)\psi_{\tilde\tau}(X^\alpha)\right]\\
&=\mathbb{E}^{\mathbb{P}_\nu}\left[\phi\Big(X^\alpha_{t_1\wedge\tilde\tau},\ldots,X^\alpha_{t_n\wedge\tilde\tau}\Big)\mathbb{E}^{\mathbb{P}_\nu}\left[\psi\Big(X^\alpha_{t_1},\ldots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\ldots,X^\alpha_{t_n}\Big)|\mathcal{F}_{\tilde\tau}\right]\right]\\
&=\mathbb{E}^{\mathbb{P}_\nu}\left[\phi\Big(X^\alpha_{t_1\wedge\tilde\tau},\ldots,X^\alpha_{t_n\wedge\tilde\tau}\Big)\psi\Big(X^\alpha_{t_1},\ldots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\ldots,X^\alpha_{t_n}\Big)\right]\\
&=\mathbb{E}^{\mathbb{P}_{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\ldots,B_{t_n\wedge\tau}\right)\psi\left(B_{t_1},\ldots,B_{t_n}\right)\right].
\end{aligned}$$

Step 4: Now we prove that $\mathbb{P}^{\tau,\omega}=\mathbb{P}^{\tilde\alpha^{\tau,\omega},\tilde\nu^{\tau,\omega}}$, $\mathbb{P}$-a.s. on Ω.

By definition of the conditional expectation,

$$\psi_\tau(\omega)=\mathbb{E}^{\mathbb{P}^{\tau,\omega}}\left[\psi(\omega(t_1),\ldots,\omega(t_k),\omega(t)+B^t_{t_{k+1}},\ldots,\omega(t)+B^t_{t_n})\right],\ \mathbb{P}^{\alpha,\nu}\text{-a.s.},$$

where $t:=\tau(\omega)\in[t_k,t_{k+1}[$, and where the $\mathbb{P}^{\alpha,\nu}$-null set can depend on $(t_1,\ldots,t_n)$ and ψ, but we can choose a common null set by standard approximation arguments.

Then by a density argument we obtain

$$\mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\eta]=\mathbb{E}^{\mathbb{P}^{\tilde\alpha^{\tau,\omega},\tilde\nu^{\tau,\omega}}}[\eta],\ \text{for } \mathbb{P}^{\alpha,\nu}\text{-a.e. } \omega,$$

for every bounded and $\mathcal{F}^{\tau(\omega)}_T$-mesurable random variable η. This implies $\mathbb{P}^{\tau,\omega}=\mathbb{P}^{\tilde\alpha^{\tau,\omega},\tilde\nu^{\tau,\omega}}$, $\mathbb{P}$-a.s. on Ω. And from the Step 1 we deduce that $\mathbb{P}^{\tau,\omega}\in\overline{\mathcal{P}}^{\tau(\omega)}_{\tilde{\mathcal{A}}}$. □

Lemma A.2. *We have $\mathbb{P}^n\in\mathcal{P}^\kappa_H$, where $\mathbb{P}^n$ is defined by (4.10).*

Proof. Since by definition, $\mathbb{P}^i_t\in\mathcal{P}^t_H$ and $\mathbb{P}\in\mathcal{P}_H$, we have $\mathbb{P}^i_t=\mathbb{P}^{\alpha^i,\nu^i}$ and $\mathbb{P}=\mathbb{P}^{\alpha,\nu}$, for $(\alpha^i,\nu^i)\in\widetilde{\mathcal{A}}^t$ and $(\alpha,\nu)\in\widetilde{\mathcal{A}}$, $i=1,\ldots,n$. Next we define

$$\begin{aligned}
\overline{\alpha}_s &:= \alpha_s\mathbf{1}_{[0,t)}(s)+\left[\sum_{i=1}^n\alpha^i_s\mathbf{1}_{E^i_t}(X^\alpha)+\alpha_s\mathbf{1}_{\hat E^n_t}(X^\alpha)\right]\mathbf{1}_{[t,T]}(s),\ \text{and}\\
\overline{\nu}_s &:= \nu_s\mathbf{1}_{[0,t)}(s)+\left[\sum_{i=1}^n\nu^i_s\mathbf{1}_{E^i_t}(X^\alpha)+\nu_s\mathbf{1}_{\hat E^n_t}(X^\alpha)\right]\mathbf{1}_{[t,T]}(s).
\end{aligned}$$

Now following the arguments in the proof of step 3 of Lemma A.1, we prove that for any $0<t_1<\cdots<t_k=t<t_{k+1}<t_n$ and any continuous and bounded functions ϕ and ψ,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\alpha,\nu}}&\left[\phi(B_{t_1},\ldots,B_{t_k})\sum_{i=1}^n\mathbb{E}^{\mathbb{P}^{\alpha^i_t,\nu^i_t}}\left[\psi(B_{t_1},\ldots,B_{t_k},B_t+B^t_{t_{k+1}},\ldots,B_t+B^t_{t_n})\right]\mathbf{1}_{E^i_t}\right]\\
&=\mathbb{E}^{\mathbb{P}^{\overline{\alpha},\overline{\nu}}}\left[\phi(B_{t_1},\ldots,B_{t_k})\psi(B_{t_1},\ldots,B_{t_n})\right].
\end{aligned}$$

This implies that $\mathbb{P}^n=\mathbb{P}^{\overline{\alpha},\overline{\nu}}\in\mathcal{P}_{\widetilde{\mathcal{A}}}$. And since all the probability measures $\mathbb{P}^i$ satisfy the requirements of Definition 4.1, we have $\mathbb{P}^n=\mathbb{P}^{\overline{\alpha},\overline{\nu}}\in\mathcal{P}^\kappa_H$. □

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