High-Water Marks and Separation of Private Investments

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Abstract

A fund manager is paid performance fees with a high-water mark provision, and invests both fund's assets and private wealth in separate but potentially correlated risky assets, aiming to maximize expected utility from private wealth in the long run. Relative risk aversion and investment opportunities are constant. We find that the fund's portfolio depends only on the fund's investment opportunities, and the private portfolio only on private opportunities. The manager invests earned fees in the safe asset, allocating remaining private wealth in a constant-proportion portfolio, while the fund is managed as another constant-proportion portfolio, with risk aversion shifted towards one. The optimal welfare is the maximum between the optimal welfare of each investment opportunity, with no diversification gain. In particular, the manager does not hedge fund's exposure with private investments.

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1 Introduction

Performance fees are a hedge fund manager's main source of income. These fees are typically 20% of a fund's total profits, and are subject to a high-water mark provision, which requires that past losses are recovered before additional fees are paid. A manager's large exposure to fund's performance is a powerful incentive to deliver superior returns, but it is also a potential source of moral hazard, as the manager may use private wealth to hedge such exposure. Extant models (Ross, 2004; Carpenter, 2000; Panageas and Westerfield, 2009; Guasoni and Obłój, 2012) acknowledge this issue, but avoid it rather than modeling it, by assuming that private wealth, including earned fees, are invested at the risk-free rate. As a result, the literature is virtually silent on the interplay between a manager's personal and professional investments.[1]

This paper begins to fill this gap, focusing on a model with two investment opportunities, one accessible to the fund, the other accessible to the manager's private account. Investment opportunities are constant over time, and potentially correlated. To make the model tractable, and consistently with the literature, we consider a fund manager with constant relative risk aversion and a long horizon, who maximizes utility from private wealth. The assumption of a long horizon means, in particular, that the model's conclusions are driven by a stationary risk-return tradeoff, not by the short-term incentives created by finite horizons.

We find the manager's optimal investment policies explicitly. For the fund, the optimal portfolio entails a constant risky proportion, which corresponds to the effective risk aversion identified by Guasoni and Obłój (2012) in the absence of private investments. The optimal policy for private wealth is more complex. The manager leaves earned fees in the safe asset, investing remaining wealth according to an optimal constant-proportion portfolio, which corresponds to the manager's own risk aversion. The result of these policies combined is that the manager obtains the maximum welfare between fees' and private investments' welfare, but not more.

The significance of this result is threefold. First, the model predicts that the fund composition does not affect the manager's private investments, and that such investments also do not affect the fund composition – portfolio separation holds. In particular, even if investment opportunities are highly correlated, the manager does not attempt to hedge fund exposure with a position in the private account. The intuition is that, for a long horizon, the benefits from hedging are surpassed by the costs of holding a short position in an asset with positive return.

Second, the manager does not rebalance all private wealth. Indeed, the optimal policy is to leave earned fees in safe asset, and to rebalance only excess wealth. This policy effectively replicates a pocket of private wealth that grows like the high-water mark of the fund, while leaving the other pocket to grow at the optimal rate for private investments. Over time, the pocket with the higher growth rate dominates the private portfolio, delivering the maximum welfare of the two strategies. In contrast to usual portfolio allocation with multiple assets, private investments can outperform the fund, but cannot augment its Sharpe ratio through diversification, regardless of correlation.

Third, since the manager's welfare is the maximum between the fund's and the private wealth's, our policy is always optimal, but never unique. Indeed, if the fund delivers the optimal welfare, it does not matter how the manager invests private capital in excess of earned fees. By contrast, if private investments deliver optimal welfare, it does not matter how earned fees or even the fund are invested. Although lack of uniqueness is an extreme effect of the long-horizon approximation, it highlights that either the fund, or private investments, become the main focus of a manager, without long-lasting interactions. Further, the model yields the conditions under which the manager focuses on the fund rather than on private wealth.

In summary, we find that high-water marks in performance fees reduce (in the long run, eliminate) a manager's incentive to use private investments either to hedge fund exposure, or to augment the fund's returns. This conclusion remains valid if the manager has private access to the fund's investment

[1] As an exception, Aragon and Qian (2010) restrict earned fees to reinvestment in the fund, which also excludes hedging attempts.

opportunities, a situation that is nested in our model when correlation between investment opportunities is perfect, and Sharpe ratios are equal.

The results can inform the decisions of investors and regulators alike. For investors, the main message is that moral hazard is likely to be higher for managers who face shorter horizons (because, for example, redemptions are allowed more frequently). Also, arrangements that increase a manager's horizon, such as longer lock-up periods, reduce the potential for moral-hazard, as do high-water mark provisions. These observations are broadly consistent with those of Aragon and Qian (2010), who find that such contract features help alleviate asymmetric-information issues for hedge funds.

From a regulatory viewpoint, our results suggest that restrictions on managers' private investments may be of secondary importance, if funds' contracts lead managers to act with a long horizon perspective, because high-water marks, combined with long horizons, reduce managers' incentives to privately trade against investors' interests.

The paper is most closely related to the literature in portfolio choice with hedge funds (Detemple et al., 2010) and high-water marks (Panageas and Westerfield, 2009; Janecek and Sırbu, 2010), and drawdown constraints (Grossman and Zhou, 1993; Cvitanic and Karatzas, 1995; Elie and Touzi, 2008), and is the first one to consider a manager who simultanously trades in the fund and in private wealth as to maximize personal welfare. The rest of the paper is organized as follows: the next section presents the model, its solution, and discusses the main implications. Section 3 offers a heuristic derivation of the main result, using informal arguments of stochastic control, and section 4 contains the formal verification theorem. Section 5 concludes.

2 Main Result

2.1 Model

A fund manager aims at maximizing utility from private wealth at a long horizon. (For brevity, henceforth 'private wealth' is simply 'wealth', unless ambiguity arises.) To achieve this goal, the manager has two tools: allocating the fund's assets X between a safe asset and a risky asset S^X, and allocating wealth F, including performance fees earned from the fund, between the safe asset and another risky asset S^F. The interpretation is that the fund has access to investment opportunities that, because of scale, regulation, or technology, are restricted to institutional investors. Examples of such investments are institutional funds, restricted shares, such as Rule 144a securities, or high-frequency trading strategies. By contrast, the manager's wealth is invested in securities available to individual investors.

The fund's (S^X) and private (S^F) risky assets follow two correlated geometric Brownian motions, with expected returns and volatilities μ^X, σ^X and μ^F, σ^F respectively. Formally, consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$ equipped with the Brownian motions $(W_t^X)_{t\geq 0}$ and $(W_t^F)_{t\geq 0}$, with correlation ρ (i.e., $\langle W^X, W^F\rangle_t = \rho t$). Define the risky assets as

$$\frac{dS_t^X}{S_t^X} = \mu^X dt + \sigma^X dW_t^X, \tag{1}$$

$$\frac{dS_t^F}{S_t^F} = \mu^F dt + \sigma^F dW_t^F. \tag{2}$$

The manager chooses the proportion of the fund π^X to invest in the asset S^X, and the proportion of wealth π^F to invest in the asset S^F. The strategies π^X and π^F are square-integrable processes, adapted to $\mathcal{F}_t$, defined as the augmentation of the filtration generated by W^X and W^F.

The high-water mark X_t^* is the running maximum $X_t^* = \max_{0\leq s\leq t} X_s$ of the net value of the fund. To ease notation, we assume a zero safe rate.[2] Then, the net fund return equals the gross return on the

[2]Guasoni and Obłój (2012) consider a constant safe rate, and find that its value does not affect the optimal fund's policy, suggesting that the assumption of a zero safe rate is inconsequential.

amount invested $X_t^{\pi^X}\pi_t^X$, minus performance fees, which are a fraction of the increase in the high-water mark. Thus,

$$dX_t^{\pi^X} = X_t^{\pi^X}\pi_t^X\left(\mu^X dt + \sigma^X dW_t^X\right) - \frac{\alpha}{1-\alpha} dX_t^{\pi^X *}. \tag{3}$$

In this equation, the last term reflects the fact that each dollar of gross profit is split into α as fees, plus $1-\alpha$ as net profit, whence performance fees are $\alpha/(1-\alpha)$ times net profits.

Similarly, the return on the manager's wealth equals the return on the risky wealth $F_t^{\pi^X,\pi^F}\pi_t^F$, plus the fees earned from the fund, i.e.

$$dF_t^{\pi^X,\pi^F} = F_t^{\pi^X,\pi^F}\pi_t^F\left(\mu^F dt + \sigma^F dW_t^F\right) + \frac{\alpha}{1-\alpha} dX_t^{\pi^X *}. \tag{4}$$

Note, in particular, that while the fund evolution depends only on its policy π^X, the evolution of wealth $F_t^{\pi^X,\pi^F}$ depends both on π^F and on π^X, as the latter drives earned fees.

The fund manager chooses $\hat{\pi}_t^X$ and $\hat{\pi}_t^F$ as to maximize expected utility from fees in the long run, that is, the equivalent safe rate (ESR) of wealth (cf. Grossman and Zhou (1993); Dumas and Luciano (1991); Cvitanic and Karatzas (1995)):

$$\mathrm{ESR}_\gamma(\pi^X,\pi^F) = \begin{cases} \lim\limits_{T\to\infty} \dfrac{1}{T}\ln \mathbb{E}\left[\left(F_T^{\pi^X,\pi^F}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, & 0<\gamma\neq 1, \\ \lim\limits_{T\to\infty} \dfrac{1}{T}\mathbb{E}\left[\ln F_T^{\pi^X,\pi^F}\right], & \gamma = 1. \end{cases} \tag{5}$$

This equivalent safe rate measures the manager's welfare, and has the dimension of an interest rate. It corresponds to the hypothetical safe rate which would make the manager indifferent between (i) actively managing fund and wealth, and (ii) retiring from the fund, investing all wealth at this riskless rate.

2.2 Solution and Discussion

The main result identifies the manager's optimal policies, and identifies the corresponding welfare. The result below is proved in the case that risk aversion is logarithmic, or lower ($\gamma \leq 1$), hence it allows to understand the risk-neutral limit $\gamma \downarrow 0$. We conjecture that the same result remains valid for risk aversion greater than one, but we cannot offer a formal proof for this case.

Theorem 1. *Let $\gamma \in (0,1]$, and set $\gamma^* = \alpha + (1-\alpha)\gamma$. The investment policies*

$$\hat{\pi}_t^X = \frac{\mu^X}{\gamma^*\left(\sigma^X\right)^2}, \tag{6}$$

$$\hat{\pi}_t^F = \left(1 - \frac{\alpha}{1-\alpha}\frac{\left(X^{\hat{\pi}^X}\right)_t^* - X_0}{F_t}\right)\frac{\mu^F}{\gamma\left(\sigma^F\right)^2}, \tag{7}$$

attain the manager's maximum equivalent safe rate of wealth, which equals

$$\mathrm{ESR}_\gamma(\hat{\pi}^X,\hat{\pi}^F) = \max\left(\frac{\left(\nu^X\right)^2}{2(\gamma + \frac{\alpha}{1-\alpha})}, \frac{\left(\nu^F\right)^2}{2\gamma}\right). \tag{8}$$

The optimal fund policy in (6) shows that the manager invests in the constant-proportion portfolio with risk aversion γ^* between one and the manager's own risk aversion γ. This policy coincides with the one obtained by Guasoni and Obłój (2012) in the absence of private investment opportunities, which corresponds in our model to $\nu^F = 0$. In this case, the private risky opportunity has zero return, hence it is never used.

The private policy in (7) is best understood in terms of the total risky and safe positions, which amount respectively to

$$\hat{F}_t\hat{\pi}_t^F = \left(\hat{F}_t - \frac{\alpha}{1-\alpha}\left(\hat{X}_t^* - X_0\right)\right)\frac{\mu^F}{\gamma\left(\sigma^F\right)^2}, \tag{9}$$

$$\hat{F}_t(1-\hat{\pi}_t^F) = \left(\hat{F}_t - \frac{\alpha}{1-\alpha}\left(\hat{X}_t^* - X_0\right)\right)\left(1 - \frac{\mu^F}{\gamma\left(\sigma^F\right)^2}\right) + \frac{\alpha}{1-\alpha}\left(\hat{X}_t^* - X_0\right). \tag{10}$$

These formulas show that the manager divides wealth into earned fees $\frac{\alpha}{1-\alpha}\left(\hat{X}_t^* - X_0\right)$, which are set aside in the safe asset, and the rest, which is invested in the constant-proportion portfolio with the manager's own risk aversion γ. The manager's welfare in (8) equals the maximum between the welfare of fees and the welfare of private investments.

2.3 Portfolio Separation

A salient feature of this result is that the fund policy is independent of private positions, and vice versa. In other words, $\hat{\pi}^X$ does not depend on μ^F, σ^F, and $\hat{\pi}^F$ does not depend on μ^X, σ^X. Further, neither $\hat{\pi}^X$ nor $\hat{\pi}^F$ depend on the correlation ρ between investment opportunities. We call this property portfolio separation.

Portfolio separation entails that a manager with long horizon has no incentive to hedge the exposure of future performance fees to the fund's investments with private risky assets, regardless of their correlation. In fact, hedging does not take place even in the limit case $\mu^X = \mu^F, \sigma^X = \sigma^F, \rho = 1$, which corresponds to a manager who has unfettered access to the fund's investments with private capital, and hence faces a dynamically complete market. To understand why such hedging is ineffective, suppose that risky assets are positively correlated, and consider a manager with a fund trading well below its high-water mark. In this case, a short position in the private asset is a poor hedge, because the high-water mark (and hence future income) is insensitive to small variations in the fund value, while the short position reduces the long-term growth of wealth.

Conversely, portfolio separation implies that the manager has no incentive to take more or less risk in the fund, in view of private investment opportunities outside the fund. A priori, it may seem plausible that a manager takes more risk in the fund if outside opportunities are attractive, because more risk is likely to lead to earlier performance fees, which could then be invested in outside opportunities. However, this tactic can only generate a one-time transfer of wealth, but not a lasting increase in the growth rate of the manager's wealth, hence it is irrelevant in the long run.

In summary, the message of portfolio separation is largely positive: if horizons are long, then moral hazards concerns are limited, because high-water marks essentially defeat any hedging incentives between fund and wealth. Yet, portfolio separation has a downside – attention separation.

2.4 Attention Separation

As a consequence of portfolio separation, the manager's welfare in (8) is the maximum between the welfare from performance fees, and the welfare from remaining wealth. Thus, while the joint policy in (6) and (7) is optimal in all cases, it is never unique. Indeed, if the manager's welfare in (8) is due to the fund (i.e. performance fees), then the private investment opportunity becomes irrelevant, and can be replaced, for example, with the policy $\pi^F = 0$. Vice versa, if remaining wealth drives welfare, then the fund policy is irrelevant, and utter negligence ($\pi^X = 0$) will deliver the same result.

This rather extreme implication is clearly driven by the assumption of a long-horizon, which focuses on the risk-adjusted long-term growth rate, neglecting all other welfare effects. Still, it makes it clear that a manager's commitment to the fund will easily wane, unless its investments are superior to outside opportunities. The manager's attention inevitably shifts to either the fund, or wealth, whichever is more profitable.

Indeed, equation (8) shows that the manager focuses on the fund if and only if the fund's Sharpe ratio ν^X exceeds the private Sharpe ratio ν^F by a multiple, which depends on the fund's fees and on the manager's risk aversion:

$$\frac{\nu^X}{\nu^F} \geq \left(1 + \frac{\alpha}{(1-\alpha)\gamma}\right)^{\frac{1}{2}}. \tag{11}$$

For example, in the case of a logarithmic manager $\gamma = 1$, and of performance fees of 20%, the manager focuses on the fund, provided that its Sharpe ratio is 11.8% higher than private investments. Such a condition is likely to hold in practice: Getmansky, Lo and Makarov (2004) find high Sharpe ratios in the hedge fund industry, even after controlling for return smoothing and illiquidity.

The right-hand side in (11), which represents the manager's attention threshold, grows as risk aversion declines. The explanation is as follows: as risk aversion declines to zero, the effective risk aversion $\gamma^* = \alpha + (1-\alpha)\gamma$ induced by the high-water mark converges to α, which entails finite leverage in the fund. On the other hand, the private portfolio is driven by the true risk aversion γ, which declines to zero, leading to increasingly high leverage. Because leverage can arbitrarily magnify expected returns, for sufficiently low risk aversion the private portfolio is always more attractive.

Overall, attention separation brings both some bad news, as the manager may grossly neglect a fund if it does not offer sufficiently attractive returns, and some good news, since the conditions for attention to the fund seem mild, and a manager with very low risk aversion is likely to leverage wealth rather than the fund.

2.5 Growth and Fees

A puzzling feature of extant models of performance fees, is that a manager prefers lower performance fees, i.e. welfare is decreasing in α. The explanation of this finding, common to the models of Panageas and Westerfield (2009) with risk-neutrality, and of Guasoni and Obłój (2012) with risk aversion, is that higher fees today reduce the growth rate of the fund, leading to lower fees tomorrow. Both models assume that fees are invested at the safe rate in the manager's account, and raise the question of whether reinvestment can induce preference for higher fees.

Equation (8) offers a qualified negative answer. If private risky investments are available, there will be some threshold α^*, below which the manager prefers lower fees, as in the absence of private investments, and above which the manager is indifferent to changes in fees, because the fund becomes irrelevant, as welfare is entirely driven by wealth. This threshold is in fact the value of alpha for which (11) holds as equality.

This result is essentially a consequence of portfolio separation. Because the manager is unable to compound fund growth with wealth growth, either private investments make fees negligible, or are negligible themselves. Overall, the model shows that the reinvestment value of fees is not sufficient to obtain a manager's preference for higher payout rates, which in turn is likely to involve intertemporal preference for consumption, or fund flows.

3 Heuristic Solution

This section derives a candidate optimal solution with heuristic stochastic control arguments. For brevity, this argument is presented only for the case of logarithmic utility, while the rigorous proof for all cases $0 < \gamma \leq 1$ is in the next section.

To ease notation, in the rest of the paper we drop the superscripts π^X and π^F from X_t and F_t.

Denoting by $Z_t = X_t^* - X_0$, the manager's value function is

$$V(t,x,f,z) = \sup_{\pi^X,\pi^F} \mathbb{E}_t \left[\ln F_T | X_t = x, F_t = f, Z_t = z\right]. \tag{12}$$

By Itô's formula,

$$dV(t,x,f,z) = \left(V_t + x\mu^X \pi_t^X V_x + f\mu^F \pi_t^F V_f + \frac{\left(\sigma^X x \pi_t^X\right)^2}{2} V_{xx} + \frac{\left(\sigma^F f \pi_t^F\right)^2}{2} V_{ff} + \rho\sigma^X \sigma^F x f \pi_t^X \pi_t^F V_{xf}\right) dt$$
$$+ \sigma^X x \pi_t^X V_x dW_t^X + \sigma^F f \pi_t^F V_f dW_t^F + \left(V_z + \frac{\alpha}{1-\alpha}(V_f - V_x)\right) dX_t^*.$$

Thus the Hamilton-Jacobian-Bellman (HJB) equation for $V(t,x,f,z)$ is, for $0 < x < z + x_0$, (where x_0 is the initial fund's value)

$$V_t + \sup_{\pi^X,\pi^F} \left(x\mu^X \pi_t^X V_x + f\mu^F \pi_t^F V_f + \frac{\left(\sigma^X x \pi_t^X\right)^2}{2} V_{xx} + \frac{\left(\sigma^F f \pi_t^F\right)^2}{2} V_{ff} + \rho\sigma^X \sigma^F x f \pi_t^X \pi_t^F V_{xf}\right) = 0,$$

with the boundary condition:

$$V_z + \frac{\alpha}{1-\alpha}(V_f - V_x) = 0 \qquad \text{when} \qquad x = z + x_0.$$

By the usual scaling property of logarithmic utility, and in the long-horizon limit, we can rewrite $V(t,x,f,z) = -\beta t + \ln z + v(\xi,\phi)$, where $\xi = \ln \frac{x}{z}$ and $\phi = \ln \frac{f}{z}$, and the HJB equation becomes

$$-\beta + \sup_{\pi^X,\pi^F} \left(\mu^X \pi_t^X v_\xi + \mu^F \pi_t^F v_\phi + \frac{\left(\sigma^X \pi_t^X\right)^2}{2}(v_{\xi\xi} - v_\xi) + \frac{\left(\sigma^F \pi_t^F\right)^2}{2}(v_{\phi\phi} - v_\phi) + \rho\sigma^X \sigma^F \pi_t^X \pi_t^F v_{\xi\phi}\right) = 0,$$

for $0 < x < z + x_0$, while the boundary condition reduces to

$$(1-\alpha) - v_\xi\left(\alpha \exp(-\xi) + (1-\alpha)\right) + v_\phi\left(\alpha\exp(-\phi) - (1-\alpha)\right) = 0 \qquad \text{when} \qquad x = z + x_0.$$

In the long run, the initial fund's value X_0 should not matter in this optimization problem. Furthermore, since X_t^* becomes large in the long run at optimum, $X_t^* \approx Z_t = X_t^* - X_0$, and we can approximate the HJB equation and the boundary condition with

$$-\beta + \sup_{\pi^X,\pi^F} \left(\mu^X \pi_t^X v_\xi + \mu^F \pi_t^F v_\phi + \frac{\left(\sigma^X \pi_t^X\right)^2}{2}(v_{\xi\xi} - v_\xi) + \frac{\left(\sigma^F \pi_t^F\right)^2}{2}(v_{\phi\phi} - v_\phi) + \rho\sigma^X \sigma^F \pi_t^X \pi_t^F v_{\xi\phi}\right) = 0, \quad -\infty < \xi < 0, \tag{13}$$

$$(1-\alpha) - v_\xi + v_\phi\left(\alpha\exp(-\phi) - (1-\alpha)\right) = 0, \quad \xi = 0, \forall \phi \in \mathbb{R}. \tag{14}$$

Solving (13), the first order conditions are

$$\pi_t^X = -\frac{\sigma^F \mu^X v_\xi (v_{\phi\phi} - v_\phi) - \rho\sigma^X \mu^F v_{\xi\phi} v_\phi}{\left(\sigma^X\right)^2 \sigma^F \left((v_{\xi\xi} - v_\xi)(v_{\phi\phi} - v_\phi) - \rho^2 v_{\xi\phi}^2\right)}, \tag{15}$$

$$\pi_t^F = -\frac{\sigma^X \mu^F v_\phi (v_{\xi\xi} - v_\xi) - \rho\sigma^F \mu^X v_{\xi\phi} v_\xi}{\sigma^X \left(\sigma^F\right)^2 \left((v_{\xi\xi} - v_\xi)(v_{\phi\phi} - v_\phi) - \rho^2 v_{\xi\phi}^2\right)}. \tag{16}$$

Plugging these two maximizers in (13), yields:

$$-\beta - \frac{\left(\mu^X \sigma^F\right)^2 v_\xi^2 (v_{\phi\phi} - v_\phi) - 2\rho\sigma^X \sigma^F \mu^F \mu^X v_\xi v_\phi v_{\xi\phi} + \left(\sigma^X \mu^F\right)^2 v_\phi^2 (v_{\xi\xi} - v_\xi)}{2\left(\sigma^X \sigma^F\right)^2 \left((v_{\xi\xi} - v_\xi)(v_{\phi\phi} - v_\phi) - \rho^2 v_{\xi\phi}^2\right)} = 0, -\infty < \xi < 0. \tag{17}$$

To solve (14), we want a solution such that $v_\xi - v_\phi\left(\alpha\exp(-\phi) - (1-\alpha)\right)$ is a constant for any $\phi \in \mathbb{R}$ when $\xi = 0$, thus we guess the solution to (17) as $v(\xi,\phi) = \delta\xi + b\ln|\alpha - (1-\alpha)\exp(\phi)|$, where β, b and δ are three parameters to be found. Plugging this guess into (14) gives $b = 1-\delta$. Finally, plugging $v(t,\xi,\phi) = \delta\xi + (1-\delta)\ln|\alpha-(1-\alpha)\exp(\phi)|$ into (17),(15) and (16), yields:

$$\beta(\delta) = \frac{\delta}{2}\left(\nu^X\right)^2 + \left(1-\frac{\delta}{1-\alpha}\right)\frac{1}{2}\left(\nu^F\right)^2, \tag{18}$$

$$\hat{\pi}_t^X = \frac{\mu^X}{\left(\sigma^X\right)^2}, \tag{19}$$

$$\hat{\pi}_t^F = \left(1-\frac{\alpha z}{(1-\alpha)f}\right)\frac{\mu^F}{\left(\sigma^F\right)^2}. \tag{20}$$

Notice that $\hat{\pi}_t^X$ and $\hat{\pi}_t^F$ does not depend on δ and taking δ to be 0 when $\frac{1}{2}\left(\nu^F\right)^2 \geq \frac{1-\alpha}{2}\left(\nu^X\right)^2$ and $1-\alpha$ when $\frac{1}{2}\left(\nu^F\right)^2 < \frac{1-\alpha}{2}\left(\nu^X\right)^2$ yields (8). (19) and (20) help us to conjecture the optimal policies for the fund and the private investments: the manager puts the performance fees, which are $\frac{\alpha z}{(1-\alpha)f}$ of wealth in the safe asset, and invests the rest in a Merton portfolio. For the fund, the same strategy as in Guasoni and Obłój (2012) is adopted, as to ensure the maximum ESR from performance fees.

4 Verification

We now show that the policies (6) and (7) are optimal, and lead to the maximum ESR from wealth in (8).

4.1 The Fund Value and The Fees

We start by defining the following processes, which represent cumulative log returns, before fees, on the fund and wealth respectively,

$$\begin{aligned}
R_t^X &= \int_0^t \left[\left(\mu^X\pi_s^X - \frac{1}{2}\left(\sigma^X\pi_s^X\right)^2\right)ds + \sigma^X\pi_s^X dW_s^X\right], \\
R_t^F &= \int_0^t \left[\left(\mu^F\pi_s^F - \frac{1}{2}\left(\sigma^F\pi_s^F\right)^2\right)ds + \sigma^F\pi_s^F dW_s^F\right], \\
R_{t,T}^X &= R_T^X - R_t^X, \\
R_{t,T}^F &= R_T^F - R_t^F.
\end{aligned}$$

Equations (3), (4), and Proposition 7 in Guasoni and Obłój (2012), imply that

$$\begin{aligned}
X_t &= X_0 e^{R_t^X - \alpha\left(R^X\right)_t^*}, \\
X_t^* &= X_0 e^{(1-\alpha)\left(R^X\right)_t^*}, \qquad (21)\\
F_t &= F_0 e^{\int_0^t\left(\mu^F\pi_s^F - \frac{1}{2}\left(\sigma^F\pi_s^F\right)^2\right)ds + \sigma^F\pi_s^F dW_s^F} + \frac{\alpha}{1-\alpha}\int_0^t e^{\int_s^t\left(\mu^F\pi_u^F - \frac{1}{2}\left(\sigma^F\pi_u^F\right)^2\right)du + \sigma^F\pi_u^F dW_u^F} dX_s^* \\
&= F_0 e^{R_t^F} + \frac{\alpha}{1-\alpha}\int_0^t e^{R_{s,t}^F} dX_s^*. \qquad (22)
\end{aligned}$$

4.2 Proof of Theorem 1

The discussion begins with two simple lemmas that are used often in the proof of the main theorem.

Lemma 1. *Let X_t and Y_t be two continuous processes, and define $X_t^* = \max_{0\le s\le t} X_s$ and $Y_{*t} = \min_{0\le s\le t} Y_s$. Then $X_t^* + Y_{*t} \le (X+Y)_t^*$.*

Proof. Since $(X+Y)_t^* \ge X_s + Y_s \ge X_s + Y_{*t}$ for all $0 \le s \le t$, it follows that

$$(X+Y)_t^* \ge X_t^* + Y_{*t}. \tag{23}$$

□

Lemma 2. *Let $(\mathcal{G}_t)_{0\le t\le T}$ be a continuous filtration and sigma algebra $\mathcal{F} \subset \mathcal{G}_0$, and denote by $\mathbb{E}_{\mathcal{F}}$ and $\mathbb{E}_{\mathcal{G}_t}$ the conditional expectation with respect to $\mathcal{F}$ and $\mathcal{G}_t$ respectively. If A_t is an increasing process adapted to $\mathcal{G}_t$ for $0 \le t \le T$, and X_t is a positive, continuous stochastic process such that $\mathbb{E}_{\mathcal{G}_t}[X_{t,T}] \le C$, $0 \le t \le T$, for some constant C, where $X_{t,T} = \frac{X_T}{X_t}$, then*

$$\mathbb{E}_{\mathcal{F}}\left[\int_0^T X_{t,T}\, dA_t\right] \le C\mathbb{E}_{\mathcal{F}}\left[A_T - A_0\right].$$

Proof. Since A_t is an increasing process, for a partition of $[0,T]$: $0 = t_0^n < t_1^n < \cdots < t_n^n = T$, $t_k^n = \frac{k}{n}T$ for $1 \le k \le n$,

$$\int_0^T X_{t,T}\, dA_t = \lim_{n\to\infty} \sum_{k=1}^n X_{t_k^n,T}\left(A_{t_k^n} - A_{t_{k-1}^n}\right). \tag{24}$$

Thus,

$$\mathbb{E}_{\mathcal{F}}\left[\int_0^T X_{t,T}\, dA_t\right] = \mathbb{E}_{\mathcal{F}}\left[\lim_{n\to\infty} \sum_{k=1}^n X_{t_k^n,T}\left(A_{t_k^n} - A_{t_{k-1}^n}\right)\right], \tag{25}$$

and by Fatou's Lemma and the tower property of conditional expectation, the right-hand side is less than or equal to

$$\liminf_{n\to\infty} \mathbb{E}_{\mathcal{F}}\left[\sum_{k=1}^n X_{t_k^n,T}\left(A_{t_k^n} - A_{t_{k-1}^n}\right)\right] = \liminf_{n\to\infty} \mathbb{E}_{\mathcal{F}}\left[\sum_{k=1}^n \mathbb{E}_{\mathcal{G}_{t_k^n}}\left[X_{t_k^n,T}\right]\left(A_{t_k^n} - A_{t_{k-1}^n}\right)\right]. \tag{26}$$

Since $\mathbb{E}_{\mathcal{G}_t}[X_{t,T}] \le C$, $0 \le t \le T$, for any positive n, (26) is less than or equal to

$$C \liminf_{n\to\infty} \mathbb{E}_{\mathcal{F}}\left[\sum_{k=1}^n \left(A_{t_k^n} - A_{t_{k-1}^n}\right)\right] = C \liminf_{n\to\infty} \mathbb{E}_{\mathcal{F}}\left[A_T - A_0\right] = C\mathbb{E}_{\mathcal{F}}\left[A_T - A_0\right]. \tag{27}$$

□

The proof of Theorem 1 is divided into the following two steps. First, any investment policies π^X and π^F satisfy the following:

$$\mathrm{ESR}_\gamma(\pi^X, \pi^F) \le \max\left(\frac{\left(\nu^X\right)^2}{2(\gamma + \frac{\alpha}{1-\alpha})}, \frac{\left(\nu^F\right)^2}{2\gamma}\right), \tag{28}$$

which is proved in Lemma 3. Second, this upper bound is achieved by the candidate optimal policies in (6) and (7), as proved in Lemma 8.

Lemma 3. *For any investment strategies π^X and π^F,*

$$\mathrm{ESR}_\gamma(\pi^X, \pi^F) \le \lambda = \max\left(\frac{\left(\nu^X\right)^2}{2(\gamma + \frac{\alpha}{1-\alpha})}, \frac{\left(\nu^F\right)^2}{2\gamma}\right).$$

We prove this lemma for logarithmic utility and power utility, respectively.

Proof of Lemma 3 for logarithmic utility:

Let F_0 be the manager's initial capital. For convenience of notation, define

$$\tilde{X}_t = \begin{cases} 0 & \text{for } t < 0, \\ \dfrac{1-\alpha}{\alpha} F_0 + X_t^* - X_0 & \text{for } t \geq 0. \end{cases}$$

Then $\tilde{X}_t$ is an increasing process, which has a jump at $t = 0$, and then grows with X_t^*. From (22), rewrite equation (5) as,

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln F_T\right] = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \frac{\alpha}{1-\alpha}\int_0^T e^{R_{t,T}^F} d\tilde{X}_t\right] \tag{29}$$

$$= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{R_{t,T}^F} d\tilde{X}_t\right] = \lambda + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{-\lambda T} e^{R_{t,T}^F} d\tilde{X}_t\right]. \tag{30}$$

Write $W_t^X = \rho W_t^F + \sqrt{1-\rho^2} W^\perp$, where $W_t^\perp$ is a Brownian Motion independent to W_t^F, and let $\mathbb{E}_{W_T^\perp}$ be the expectation conditional on $\left(W_s^\perp\right)_{0\leq s\leq T}$ (the whole trajectory of $W^\perp$ until T). By Lemma 4 below, (30) is less than or equal to

$$\lambda + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{R_{t,T}^F - \int_t^T\left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW_s^F\right)} e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right] \tag{31}$$

$$= \lambda + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\mathbb{E}_{W_T^\perp}\left[\ln \int_0^T e^{R_{t,T}^F - \int_t^T\left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW_s^F\right)} e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right]\right] \tag{32}$$

$$\leq \lambda + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[\int_0^T e^{R_{t,T}^F - \int_t^T\left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW_s^F\right)} e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right]\right], \tag{33}$$

where (32) follows from the tower property of conditional expectation, and (33) from Jensen's inequality.

Next, Lemma 5 below implies that $e^{R_{0,t}^F - \int_0^t\left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW_s^F\right)}$ is a supermartingale with respect the filtration generated by $\left(W_s^F\right)_{0\leq s\leq t}$ and $\left(W_s^\perp\right)_{0\leq s\leq T}$ (the present of W^F and $W^\perp$, plus the future of $W^\perp$). Thus,

$$\mathbb{E}_{W_T^\perp, W_t^F}\left[e^{R_{t,T}^F - \int_t^T\left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW_s^F\right)}\right] \leq 1, \qquad \forall 0 \leq t \leq T. \tag{34}$$

Since $A_t = \int_0^t e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(W^F\right)_t^*} d\tilde{X}_t$ is an increasing process, (34) and Lemma 2 imply that

$$\mathbb{E}_{W_T^\perp}\left[\int_0^T e^{R_{t,T}^F - \int_t^T\left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW_s^F\right)} e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right] \leq \mathbb{E}_{W_T^\perp}\left[\int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right]. \tag{35}$$

Thus, from (30) and (35) it follows that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln F_T\right] \leq \lambda + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[\int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right]\right]. \tag{36}$$

Then, Lemma 6 below proves that $\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[\int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right]\right] \leq 0$, whence

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln F_T\right] \leq \lambda, \tag{37}$$

which concludes the proof for logarithmic utility. □

Lemma 4. *If* $\lambda = \max\left(\frac{1-\alpha}{2}\left(\nu^X\right)^2, \frac{\left(\nu^F\right)^2}{2}\right)$,

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{-\lambda T} e^{R^F_{t,T}} d\tilde{X}_t\right] \leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{R^F_{t,T} - \int_t^T \left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW^F_s\right)} e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X \left(\rho W^F\right)^*_t} d\tilde{X}_t\right].$$

Proof. Define the stochastic process $N^T_s = W^F_T - W^F_{T-s}$, for $0 \leq s \leq T$, and note that N^T_s has the same distribution as W^F_s. It follows that

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[-\nu^F \left(N^T\right)^*_T - (1-\alpha)\nu^X \left(\rho W^F\right)^*_T\right] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[-\nu^F \left(N^T\right)^*_T\right] + \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[-(1-\alpha)\nu^X \left(\rho W^F\right)^*_T\right] \tag{38}$$

$$= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[-\nu^F \left(W^F\right)^*_T\right] + \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[-(1-\alpha)\nu^X |\rho| \left(W^F\right)^*_T\right] = 0, \tag{39}$$

where the last equality uses the fact that, for a, $b \neq 0$,

$$\lim_{T\to\infty} \frac{1}{bT}\mathbb{E}\left[aW^*_T\right] = \lim_{T\to\infty} \frac{1}{bT}\int_0^\infty \frac{ax}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx = 0. \tag{40}$$

Thus,

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{-\lambda T + R^F_{t,T}} d\tilde{X}_t\right] \\
=&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{-\lambda T + R^F_{t,T}} d\tilde{X}_t\right] + \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[-\nu^F \left(N^T\right)^*_T - (1-\alpha)\nu^X \left(\rho W^F\right)^*_T\right] &(41)\\
\leq&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{-\lambda T - \nu^F \left(N^T\right)^*_T - (1-\alpha)\nu^X \left(\rho W^F\right)^*_T + R^F_{t,T}} d\tilde{X}_t\right]. &(42)
\end{aligned}$$

Now, note that $\left(N^T\right)^*_T \geq W^F_T - W^F_t$, $\lambda T \geq \frac{1}{2}\left(\nu^F\right)^2 (T-t) + \frac{1-\alpha}{2}\left(\nu^X\right)^2 t$ and $(1-\alpha)\nu^X \left(\rho W^F\right)^*_T \geq (1-\alpha)\nu^X \left(\rho W^F\right)^*_t$, for any $0 \leq t \leq T$. Thus, it follows that

$$\begin{aligned}
&-\lambda T - \nu^F \left(N^T\right)^*_T - (1-\alpha)\nu^X \left(\rho W^F\right)^*_T + R^F_{t,T} \\
\leq& -\frac{1}{2}\left(\nu^F\right)^2 (T-t) - \frac{1-\alpha}{2}\left(\nu^X\right)^2 t - \nu^F \left(W^F_T - W^F_t\right) - (1-\alpha)\nu^X \left(\rho W^F\right)^*_t + R^F_{t,T} \\
=& R^F_{t,T} - \int_t^T \left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW^F_s\right) - \frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X \left(\rho W^F\right)^*_t.
\end{aligned}$$

Plugging this inequality into (42) yields:

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{-\lambda T} e^{R^F_{t,T}} d\tilde{X}_t\right] \leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \int_0^T e^{R^F_{t,T} - \int_t^T \left(\frac{1}{2}\left(\nu^F\right)^2 ds + \nu^F dW^F_s\right) - \frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X \left(\rho W^F\right)^*_t} d\tilde{X}_t\right]. \tag{43}$$

□

Lemma 5. *Let* π_t *be adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *generated by* $\left(W^X_s\right)_{0\leq s\leq t}$ *and* $\left(W^F_s\right)_{0\leq s\leq t}$*. Define* $\{\mathcal{G}_t\}_{t\geq 0}$ *as the filtration generated by* $\left(W^F_s\right)_{0\leq s\leq t}$ *and* $\left(W^X_s\right)_{0\leq s\leq T}$*. Then* $M_t = e^{\int_0^t \left(\pi_s dW^F_s - \frac{1}{2}\pi^2_s ds\right)}$ *is a supermartingale with respect to* $\{\mathcal{G}_t\}_{t\geq 0}$*.*

Proof. Suppose π_t is a simple process, i.e. $\pi_t = \Sigma^n_{i=1}\pi_i \mathbf{1}_{[t_{i-1},t_i)}$ for a partition of [0,T]: $0 = t_0 < t_1 < t_2 \cdots < t_n = T$ and π_i, $i = 1, \cdots, n$ are n constants. Then for any $0 \leq s < t \leq T$,

$$\mathbb{E}_{W^X_T, W^F_s}\left[e^{\int_0^t \pi_u dW^F_u - \frac{1}{2}\pi^2_u du}\right] = e^{\int_0^s \pi_u dW^F_u - \frac{1}{2}\pi^2_u du}\mathbb{E}_{W^X_T, W^F_s}\left[e^{\int_s^t \pi_u dW^F_u - \frac{1}{2}\pi^2_u du}\right]. \tag{44}$$

Since there exists $1 \le k_s \le k_t \le n$ such that $t_{k_s-1} \le s \le t_{k_s}$ and $t_{k_t-1} \le t \le t_{k_t}$, thus s, t and all the division points in between forms a partition of $[s,t]$, denoted by $s = u_0 < u_1 < u_2 \cdots < u_m = t$. Then, since W_t^X and W_t^F are two independent Brownian Motions,

$$\mathbb{E}_{W_T^X, W_s^F}\left[e^{\int_s^t\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}\right] = \mathbb{E}_{W_T^X, W_s^F}\left[e^{\sum_{i=1}^m \pi_i\left(W_{u_i}^F - W_{u_{i-1}}^F\right) - \frac{1}{2}\pi_i^2(u_i - u_{i-1})}\right] \tag{45}$$

$$= \prod_{i=1}^m \mathbb{E}_{W_T^X, W_t^F}\left[e^{\pi_i\left(W_{u_i}^F - W_{u_{i-1}}^F\right) - \frac{1}{2}\pi_i^2(u_i - u_{i-1})}\right] = 1, \tag{46}$$

and thus,

$$\mathbb{E}_{W_T^X, W_s^F}\left[e^{\int_0^t\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}\right] = e^{\int_0^s\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}. \tag{47}$$

For a general π_t, from the definition of stochastic integral, there exists a sequence of simple processes $\{\pi_t^n\}_{n=1}^{\infty}$, such that $\int_0^t \left(\pi_s^n dW_s^F - \frac{1}{2}\left(\pi_s^n\right)^2 ds\right)$ converges to $\int_0^t \left(\pi_s dW_s^F - \frac{1}{2}\left(\pi_s\right)^2 ds\right)$ a.e., hence

$$\mathbb{E}_{W_T^X, W_s^F}\left[e^{\int_0^t\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}\right] = e^{\int_0^s\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}\mathbb{E}_{W_T^X, W_s^F}\left[e^{\int_s^t\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}\right] \tag{48}$$

$$= e^{\int_0^s\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}\mathbb{E}_{W_T^X, W_s^F}\left[\liminf_{n\to\infty} e^{\int_s^t\left(\pi_u^n dW_u^F - \frac{1}{2}(\pi_u^n)^2 du\right)}\right] \tag{49}$$

$$\le e^{\int_0^s\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)} \liminf_{n\to\infty} \mathbb{E}_{W_T^X, W_s^F}\left[e^{\int_s^t\left(\pi_u^n dW_u^F - \frac{1}{2}(\pi_u^n)^2 du\right)}\right] \tag{50}$$

$$= e^{\int_0^s\left(\pi_u dW_u^F - \frac{1}{2}\pi_u^2 du\right)}, \tag{51}$$

which confirms that M_t is a supermartingale with respect to $\{\mathcal{G}_t\}_{t\ge 0}$. □

Lemma 6.

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^{\perp}}\left[\int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t\right]\right] \le 0.$$

Proof. By integration by parts,

$$\int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} d\tilde{X}_t = F_0 + \int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} dX_t^* \tag{52}$$

$$= F_0 + e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 T - (1-\alpha)\nu^X\left(\rho W^F\right)_T^*} X_T^* - X_0 + \frac{1-\alpha}{2}\left(\nu^X\right)^2 \int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} X_t^* dt$$

$$+ (1-\alpha)\nu^X \int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} X_t^* d\left(\rho W^F\right)_t^*. \tag{53}$$

Since $X_t^* = e^{(1-\alpha)\left(R^X\right)_t^*}$, from Lemma 1, $e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(\rho W^F\right)_t^*} X_t^* \le e^{(1-\alpha)\left(R_\cdot^X - \frac{\left(\nu^X\right)^2}{2}\cdot - \nu^X \rho W_\cdot^F\right)_t^*}$. Thus, from (53),

$$\int_0^T e^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2 t - (1-\alpha)\nu^X\left(W^F\right)_t^*} d\tilde{X}_t$$

$$\le F_0 + e^{(1-\alpha)\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_T^*} + \frac{1-\alpha}{2}\left(\nu^X\right)^2 \int_0^T e^{(1-\alpha)\left(R_\cdot - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_t^*} dt$$

$$+ (1-\alpha)\nu^X \int_0^T e^{(1-\alpha)\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_t^*} d\left(\rho W^F\right)_t^*. \tag{54}$$

Since $e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-\nu^X\rho W^F_\cdot\right)^*_t}\le e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-\nu^X\rho W^F_\cdot\right)^*_T}$ for any $0\le t\le T$, (54) is less than or equal to

$$F_0+e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}(\nu^X)^2\cdot-\nu^X\rho W^F_\cdot\right)^*_T}+\frac{1-\alpha}{2}\left(\nu^X\right)^2e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}(\nu^X)^2\cdot-\nu^X\rho W^F_\cdot\right)^*_T}T+(1-\alpha)\nu^Xe^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}(\nu^X)^2\cdot-\nu^X\rho W^F_\cdot\right)^*_T}\left(\rho W^F\right)^*_T \tag{55}$$

$$=F_0+\left(1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+(1-\alpha)\nu^X|\rho|\left(W^F\right)^*_T\right)e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}(\nu^X)^2\cdot-\nu^X\rho W^F_\cdot\right)^*_T}. \tag{56}$$

Thus, from (54) and (56),

$$\begin{aligned}
&\mathbb{E}_{W^\perp_T}\left[\int_0^Te^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2t-(1-\alpha)\nu^X\left(\rho W^F\right)^*_t}d\tilde{X}_t\right]\\
\le&\mathbb{E}_{W^\perp_T}\left[F_0+\left(1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+(1-\alpha)\nu^X|\rho|\left(W^F\right)^*_T\right)e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-(1-\alpha)\nu^X\rho W^F_\cdot\right)^*_T}\right] && (57)\\
=&F_0+\mathbb{E}_{W^\perp_T}\left[\left(1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+(1-\alpha)\nu^X|\rho|\left(W^F\right)^*_T\right)e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-(1-\alpha)\nu^X\rho W^F_\cdot\right)^*_T}\right] && (58)\\
\le&F_0+\mathbb{E}_{W^\perp_T}\left[e^{(1-\alpha)\delta\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-(1-\alpha)\nu^X\rho W^F_\cdot\right)^*_T}\right]^{\frac{1}{\delta}}\mathbb{E}_{W^\perp_T}\left[\left(1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+(1-\alpha)\nu^X|\rho|\left(W^F\right)^*_T\right)^{\frac{\delta}{\delta-1}}\right]^{\frac{\delta-1}{\delta}}, && (59)
\end{aligned}$$

for any $\delta>1$, by Hölder's inequality.

Since $\delta>1$ and $\frac{\delta}{\delta-1}>1$, by Minkowski inequality ($\mathbb{E}\left[(f+g)^p\right]^{\frac{1}{p}}\le\mathbb{E}\left[f^p\right]^{\frac{1}{p}}+\mathbb{E}\left[f^p\right]^{\frac{1}{p}}$), it follows that

$$\begin{aligned}
&\mathbb{E}_{W^\perp_T}\left[\left(1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+(1-\alpha)\nu^X|\rho|\left(W^F\right)^*_T\right)^{\frac{\delta}{\delta-1}}\right]^{\frac{\delta-1}{\delta}}\\
\le&1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+(1-\alpha)\nu^X|\rho|\mathbb{E}_{W^\perp_T}\left[\left(W^F\right)^{*\frac{\delta}{\delta-1}}_T\right]^{\frac{\delta-1}{\delta}} && (60)\\
=&1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+\sqrt{2}(1-\alpha)\nu^X|\rho|\left(\frac{\Gamma(\frac{1+\frac{\delta}{\delta-1}}{2})}{\sqrt{\pi}}\right)^{\frac{\delta-1}{\delta}}\sqrt{T}. && (61)
\end{aligned}$$

Thus from (59) and (61), setting $C_T=1+\frac{1-\alpha}{2}\left(\nu^X\right)^2T+\sqrt{2}(1-\alpha)\nu^X|\rho|\left(\frac{\Gamma(\frac{1+\frac{\delta}{\delta-1}}{2})}{\sqrt{\pi}}\right)^{\frac{\delta-1}{\delta}}\sqrt{T}$, for any $\delta>1$,

$$\mathbb{E}_{W^\perp_T}\left[\int_0^Te^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2t-(1-\alpha)\nu^X\left(\rho W^F\right)^*_t}d\tilde{X}_t\right]\le F_0+C_T\mathbb{E}_{W^\perp_T}\left[\left(e^{(1-\alpha)\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-(1-\alpha)\nu^X\rho W^F_\cdot\right)^*_T}\right)^\delta\right]^{\frac{1}{\delta}}. \tag{62}$$

Then

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\ln\mathbb{E}_{W^\perp_T}\left[\int_0^Te^{-\frac{1-\alpha}{2}\left(\nu^X\right)^2t-(1-\alpha)\nu^X\left(\rho W^F\right)^*_t}d\tilde{X}_t\right]\right]\\
\le&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\ln\left(F_0+C_T\mathbb{E}_{W^\perp_T}\left[e^{(1-\alpha)\delta\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-(1-\alpha)\nu^X\rho W^F_\cdot\right)^*_T}\right]^{\frac{1}{\delta}}\right)\right] && (63)\\
\le&\limsup_{T\to\infty}\frac{1}{T}\ln C_T && (64)\\
&+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\ln\left(\frac{F_0}{C_T}+\mathbb{E}_{W^\perp_T}\left[e^{(1-\alpha)\delta\left(R^X_\cdot-\frac{1}{2}\left(\nu^X\right)^2\cdot-(1-\alpha)\nu^X\rho W^F_\cdot\right)^*_T}\right]^{\frac{1}{\delta}}\right)\right]. && (65)
\end{aligned}$$

Note that the limit in (64) is 0. Since $\frac{F_0}{C_T} \to 0$ a.s. as $T \uparrow \infty$, for T large enough,

$$\frac{F_0}{C_T} < \mathbb{E}_{W_T^\perp}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - (1-\alpha)\nu^X \rho W_\cdot^F\right)_T^*}\right]^{\frac{1}{\delta}}. \tag{66}$$

Thus, (65) is less than or equal to

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln 2\mathbb{E}_{W_T^\perp}\left[\left(e^{(1-\alpha)\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_T^*}\right)^\delta\right]^{\frac{1}{\delta}}\right] = \limsup_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_T^*}\right]\right] \tag{67}$$

$$= \limsup_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_T^*}\right]\right] + \liminf_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[-\sqrt{1-\rho^2}(1-\alpha)\delta\nu^X \left(W^\perp\right)_T^*\right] \tag{68}$$

$$\leq \limsup_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F\right)_T^* - \sqrt{1-\rho^2}(1-\alpha)\delta\nu^X \left(W^\perp\right)_T^*}\right]\right]. \tag{69}$$

Note that (68) holds because $\liminf_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[-\sqrt{1-\rho^2}(1-\alpha)\delta\nu^X \left(W^\perp\right)_T^*\right] = 0$, which follows from (40). Then, again by Lemma 1, the running maximum and running minimum can be combined, and (69) is less than or equal to

$$\limsup_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X \rho W_\cdot^F - \sqrt{1-\rho^2}\nu^X W_\cdot^\perp\right)_T^*}\right]\right]$$

$$= \limsup_{T\to\infty} \frac{1}{\delta T}\mathbb{E}\left[\ln \mathbb{E}_{W_T^\perp}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X W_\cdot^X\right)_T^*}\right]\right] \leq \limsup_{T\to\infty} \frac{1}{\delta T}\ln \mathbb{E}\left[e^{(1-\alpha)\delta\left(R_\cdot^X - \frac{1}{2}\left(\nu^X\right)^2 \cdot - \nu^X W_\cdot^X\right)_T^*}\right], \tag{70}$$

where (70) follows from Jensen's inequality. $M_t = e^{R_t^X - \frac{1}{2}\left(\nu^X\right)^2 t - \nu^X W_t^X}$ is a local martingale with respect to the filtration generated by $\left(W_s^F\right)_{0\leq s\leq t}$ and $\left(W_s^\perp\right)_{0\leq s\leq t}$. Then, since $M_t^* \leq M_\infty^*$, of which the inverse is uniformly distributed on $[0,1]$ (cf. (54) in Guasoni and Obłój (2012)), for $1 < \delta < \frac{1}{1-\alpha}$, (70) is less than or equal to

$$\limsup_{T\to\infty} \frac{1}{\delta T}\ln \mathbb{E}\left[\left(M_\infty^*\right)^{(1-\alpha)\delta}\right] \leq \limsup_{T\to\infty} \frac{1}{\delta T}\ln\left(\int_0^1 x^{-(1-\alpha)\delta}dx\right) \tag{71}$$

$$= \limsup_{T\to\infty} \frac{1}{\delta T}\ln\left(\frac{1}{1-(1-\alpha)\delta}\right) = 0, \tag{72}$$

which concludes the proof. □

Proof of Lemma 3 for power utility:

For the rest of the paper, let $p = 1-\gamma$. Since $dF_t = F_t\left(\mu^F \pi_t^F dt + \sigma^F \pi_t^F dW_t^F\right) + \frac{\alpha}{1-\alpha}dX_t^*$,

$$F_T = F_0 e^{R_T^{F,\pi^F}} + \frac{\alpha}{1-\alpha}\int_0^T e^{R_{t,T}^{F,\pi^F}} dX_t^*. \tag{73}$$

Suppose now that the fund manager has an additional source of income, such that whenever performance fees are paid, ϵ of fees are matched (like a bonus), with a restriction that this extra income can only be put in the safe asset. Let the manager's wealth under this schedule be $\tilde{F}_t$, with the same strategy π^F for wealth in excess of the extra income, the dynamics of $\tilde{F}_t$ is

$$d\tilde{F}_t = \left(\tilde{F}_t - \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right)\right)\left(\pi_t^F \mu^F dt + \pi_t^F \sigma^F dW_t^F\right) + (1+\epsilon)\frac{\alpha}{1-\alpha}dX_t^*, \tag{74}$$

and

$$\tilde{F}_t = F_0 e^{R_{0,t}^{F,\pi^F}} + \frac{\alpha}{1-\alpha}\int_0^t e^{R_{s,t}^{F,\pi^F}} dX_s^* + \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right) \tag{75}$$

$$= F_t + \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right). \tag{76}$$

Thus $\tilde{F}_t \geq F_t$ and $\tilde{F}_t \geq \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right)$ for all $t \geq 0$, and ESR of F is less than or equal to ESR of $\tilde{F}$. Lemma 7 below shows that this upper bound is also less than or equal to λ. ☐

Lemma 7. $\limsup_{T\to\infty} \frac{1}{pT}\ln\mathbb{E}\left[\tilde{F}_T^p\right] \leq \lambda$ *for all* $0 < p < 1$.

Proof. Let $\tilde{\pi}_t^F = \frac{\left(\tilde{F}_t - \epsilon\frac{\alpha}{1-\alpha}(X_t^* - X_0)\right)}{\tilde{F}_t}\pi_t^F$. Investing π_t^F of $\tilde{F}_t - \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right)$ in S^F is equivalent to investing $\tilde{\pi}_t^F$ of $\tilde{F}_t$. Thus $\tilde{\pi}_t^F$ can be regarded as an investment strategy for $\tilde{F}_t$, and

$$\begin{aligned} d\tilde{F}_t^p &= p\tilde{F}_t^{p-1}\left(\tilde{F}_t - \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right)\right)\left(\pi_t^F\mu^F dt + \pi_t^F\sigma^F dW^F\right) \\ &\quad + \frac{p(p-1)}{2}\tilde{F}_t^{p-2}\left(\tilde{F}_t - \epsilon\frac{\alpha}{1-\alpha}\left(X_t^* - X_0\right)\right)^2\left(\pi_t^F\sigma^F\right)^2 dt + p\tilde{F}_t^{p-1}(1+\epsilon)\frac{\alpha}{1-\alpha}dX_t^* \end{aligned} \tag{77}$$

$$= p\tilde{F}_t^p\left(\left(\tilde{\pi}_t^F\mu^F + \frac{(p-1)}{2}\left(\tilde{\pi}_t^F\sigma^F\right)^2\right)dt + \tilde{\pi}_t^F\sigma^F dW^F\right) + p\tilde{F}_t^{p-1}(1+\epsilon)\frac{\alpha}{1-\alpha}dX_t^*. \tag{78}$$

Solving this differential equation,

$$\tilde{F}_T^p = F_0^p e^{pR_T^{F,\tilde{\pi}^F}} + p(1+\epsilon)\frac{\alpha}{1-\alpha}\int_0^T e^{pR_{t,T}^{F,\tilde{\pi}^F}}\tilde{F}_t^{p-1}dX_t^*. \tag{79}$$

Thus,

$$\limsup_{T\to\infty}\frac{1}{pT}\ln\mathbb{E}\left[\tilde{F}_T^p\right] = \limsup_{T\to\infty}\frac{1}{pT}\ln\mathbb{E}\left[F_0^p e^{pR_T^{F,\tilde{\pi}^F}} + p(1+\epsilon)\frac{\alpha}{1-\alpha}\int_0^T e^{pR_{t,T}^{F,\tilde{\pi}^F}}\tilde{F}_t^{p-1}dX_t^*\right]. \tag{80}$$

Since $0 < p < 1$, from Dembo and Zeitouni (1998), Lemma 1.2.15, for any positive process f_t and g_t,

$$\limsup_{T\to\infty}\frac{1}{pT}\ln\left(f_T + g_T\right) = \max\left(\limsup_{T\to\infty}\frac{1}{pT}\ln f_T, \limsup_{T\to\infty}\frac{1}{pT}\ln g_T\right). \tag{81}$$

It follows that,

$$\begin{aligned} &\limsup_{T\to\infty}\frac{1}{pT}\ln\mathbb{E}\left[F_0^p e^{pR_T^{F,\tilde{\pi}^F}} + p(1+\epsilon)\frac{\alpha}{1-\alpha}\int_0^T e^{pR_{t,T}^{F,\tilde{\pi}^F}}\tilde{F}_t^{p-1}dX_t^*\right] \\ &= \max\left(\limsup_{T\to\infty}\frac{1}{pT}\ln\mathbb{E}\left[F_0^p e^{pR_T^{F,\tilde{\pi}^F}}\right], \limsup_{T\to\infty}\frac{1}{pT}\ln\mathbb{E}\left[p(1+\epsilon)\frac{\alpha}{1-\alpha}\int_0^T e^{pR_{t,T}^{F,\tilde{\pi}^F}}\tilde{F}_t^{p-1}dX_t^*\right]\right). \end{aligned} \tag{82}$$

Note that, since $0 < p < 1$, by Hölder's inequality,

$$\mathbb{E}\left[F_0^p e^{pR_T^{F,\tilde{\pi}^F}}\right]^{\frac{1}{p}}\mathbb{E}\left[e^{q\left(-\nu^F W_T^F - \frac{\left(\nu^F\right)^2}{2}T\right)}\right]^{\frac{1}{q}} \leq \mathbb{E}\left[F_0 e^{\left(R_T^{F,\tilde{\pi}^F} - \nu^F W_T^F - \frac{\left(\nu^F\right)^2}{2}T\right)}\right] \leq F_0, \tag{83}$$

where $q = \frac{p}{p-1}$. Thus,

$$\limsup_{T\to\infty}\frac{1}{pT}\ln\mathbb{E}\left[F_0^p e^{pR_T^{F,\tilde{\pi}^F}}\right] \leq -\limsup_{T\to\infty}\frac{1}{qT}\ln\mathbb{E}\left[e^{q\left(-\nu^F W_T^F - \frac{\left(\nu^F\right)^2}{2}T\right)}\right] = \frac{\left(\nu^F\right)^2}{2(1-p)}. \tag{84}$$

For the second term in (82), since $p<1$, and $\tilde{F}_t \geq \epsilon \frac{\alpha}{1-\alpha}\left(X_t^*-X_0\right)$, $\tilde{F}_t^{p-1} \leq \epsilon^{p-1}\left(\frac{\alpha}{1-\alpha}\right)^{p-1}\left(X_t^*-X_0\right)^{p-1}$, and

$$
\begin{aligned}
& \mathbb{E}\left[p(1+\epsilon) \frac{\alpha}{1-\alpha} \int_0^T e^{p R_{t, T}^{F, \tilde{\pi}^F}} e^{-p \lambda T} \tilde{F}_t^{p-1} d X_t^*\right] \\
& \leq \epsilon^{p-1}(1+\epsilon)\left(\frac{\alpha}{1-\alpha}\right)^p \mathbb{E}\left[\int_0^T e^{p R_{t, T}^{F, \tilde{\pi}^F}-p \lambda(T-t)} e^{-p \lambda t} p\left(X_t^*-X_0\right)^{p-1} d X_t^*\right] \qquad (85)\\
& =\epsilon^{p-1}(1+\epsilon)\left(\frac{\alpha}{1-\alpha}\right)^p \mathbb{E}\left[\int_0^T e^{p R_{t, T}^{F, \tilde{\pi}^F}-p \lambda(T-t)} e^{-p \lambda t} d\left(X_t^*-X_0\right)^p\right] . \qquad (86)
\end{aligned}
$$

Since the process $A_t=\int_0^T e^{-p \lambda t} d\left(X_t^*-X_0\right)^p$ is an increasing process, and $\mathbb{E}_t\left[e^{p R_{t, T}^{F, \tilde{\pi}^F}-p \lambda(T-t)}\right] \leq e^{\frac{p}{2(1-p)}\left(\nu^F\right)^2(T-t)-p \lambda(T-t)} \leq 1$, Lemma 2 implies that (86) is less than or equal to

$$
\epsilon^{p-1}(1+\epsilon)\left(\frac{\alpha}{1-\alpha}\right)^p \mathbb{E}\left[\int_0^T e^{-p \lambda t} d\left(X_t^*-X_0\right)^p\right], \qquad (87)
$$

and hence

$$
\begin{aligned}
& \limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \mathbb{E}\left[p(1+\epsilon) \frac{\alpha}{1-\alpha} \int_0^T e^{p R_{t, T}^{F, \tilde{\pi}}} \tilde{F}_t^{p-1} d X_t^*\right] \\
& =\lambda+\limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \mathbb{E}\left[p(1+\epsilon) \frac{\alpha}{1-\alpha} \int_0^T e^{p R_{t, T}^{F, \tilde{\pi}}} e^{-p \lambda T} \tilde{F}_t^{p-1} d X_t^*\right] \qquad (88)\\
& \leq \lambda+\limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \left(\epsilon^{p-1}(1+\epsilon)\left(\frac{\alpha}{1-\alpha}\right)^p \mathbb{E}\left[\int_0^T e^{-p \lambda t} d\left(X_t^*-X_0\right)^p\right]\right) \qquad (89)\\
& =\lambda+\limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \mathbb{E}\left[\int_0^T e^{-p \lambda t} d\left(X_t^*-X_0\right)^p\right] . \qquad (90)
\end{aligned}
$$

Now, integration by parts implies that

$$
\begin{aligned}
\int_0^T e^{-p \lambda t} d\left(X_t^*-X_0\right)^p & =e^{-p \lambda T}\left(X_T^*-X_0\right)^p+p \lambda \int_0^T e^{-p \lambda t}\left(X_t^*-X_0\right)^p d t \qquad (91)\\
& \leq e^{-p \lambda T}\left(X^*\right)_T^p+p \lambda \int_0^T e^{-p \lambda t}\left(X^*\right)_t^p d t . \qquad (92)
\end{aligned}
$$

By Lemma 1, $e^{-p \lambda t}\left(X^*\right)_t^p \leq e^{(1-\alpha) p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_t^*}$, for all $0 \leq t \leq T$, thus (92) is less than or equal to

$$
\begin{aligned}
& e^{(1-\alpha) p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*}+p \lambda \int_0^T e^{(1-\alpha) p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_t^*} d t \\
\leq & e^{(1-\alpha) p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*}+p \lambda T e^{(1-\alpha) p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*}=(1+p \lambda T) e^{(1-\alpha) p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*} . \qquad (93)
\end{aligned}
$$

Now, Lemma 9 in Guasoni and Obłój (2012) with $\varphi-r=\frac{\lambda}{1-\alpha}$ implies that

$$
\limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \mathbb{E}\left[(1+p \lambda T) e^{p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*}\right]=\limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \mathbb{E}\left[e^{p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*}\right] \leq 0 . \qquad (94)
$$

Thus,

$$
\lambda+\limsup _{T \rightarrow \infty} \frac{1}{p T} \ln \mathbb{E}\left[(1+p \lambda T) e^{p\left(R_{\cdot}^X-\frac{\lambda}{1-\alpha} \cdot\right)_T^*}\right] \leq \lambda . \qquad (95)
$$

Then, (82), (84), and (95) imply:

$$\limsup_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[\tilde{F}_T^p\right] \le \max(\frac{\left(\nu^F\right)^2}{2(1-p)}, \lambda) = \lambda. \tag{96}$$

□

To prove Theorem 1, it now remains to show that the upper bound in Lemma 3 is achieved by the strategies in (6) and (7), and hence they are optimal. Plugging $\hat{\pi}^X$ and $\hat{\pi}^F$ in the dynamics of X_t and F_t, the fund's value and the manager's wealth processes follow

$$d\hat{X}_t = \hat{X}_t \left(\frac{1}{1-(1-\alpha)p}\left(\nu^X\right)^2 dt + \frac{1}{1-(1-\alpha)p}\nu^X dW_t^X\right) - \frac{\alpha}{1-\alpha} d\hat{X}_t^*, \tag{97}$$

$$d\hat{F}_t = \left(\hat{F}_t - \frac{\alpha}{1-\alpha}(\hat{X}_t^* - X_0)\right)\left(\frac{1}{1-p}\left(\nu^F\right)^2 dt + \frac{1}{1-p}\nu^F dW_t^F\right) + \frac{\alpha}{1-\alpha} d\hat{X}_t^*. \tag{98}$$

Denoting by $\hat{R}_T^X = \frac{1}{2}\frac{1-2(1-\alpha)p}{(1-(1-\alpha)p)^2}\left(\nu^X\right)^2 T + \frac{1}{1-(1-\alpha)p}\nu^X W_T^X$, $\hat{R}_T^F = \frac{1}{2}\frac{1-2p}{(1-p)^2}\left(\nu^F\right)^2 T + \frac{1}{1-(1-\alpha)p}\nu^F W_T^F$, Itô's formula shows that:

$$\hat{X}_T = X_0 e^{\hat{R}_T^X - \alpha\left(\hat{R}^X\right)_T^*}, \tag{99}$$

$$\hat{X}_T^* = X_0 e^{(1-\alpha)\left(\hat{R}^X\right)_T^*}, \tag{100}$$

$$\hat{F}_T = F_0 e^{\hat{R}_T^F} + \frac{\alpha}{1-\alpha}(\hat{X}_T^* - X_0). \tag{101}$$

Lemma 8. $\mathrm{ESR}_\gamma\left(\hat{\pi}^X, \hat{\pi}^F\right) = \lambda$.

Proof. Let $G_t = F_0 e^{\hat{R}_T^F}$ and $H_t = \frac{\alpha}{1-\alpha}\left(\hat{X}_t^* - X_0\right)$, then $\hat{F}_T = G_T + H_T$. From Lemma 3, it suffices to prove that $\mathrm{ESR}_\gamma\left(\hat{\pi}^X, \hat{\pi}^F\right) \ge \lambda$.

Case of logarithmic utility.
Since H_t is a positive process,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \hat{F}_T\right] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln(G_T + H_T)\right] \ge \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln G_T\right] = \frac{\left(\nu^F\right)^2}{2}. \tag{102}$$

Likewise, since G_t is a positive process,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \hat{F}_T\right] \ge \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln H_T\right] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \hat{X}_T^*\right] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(1-\alpha)\left(\frac{1}{2}\left(\nu^X\right)^2 \cdot + \nu^X W_\cdot^X\right)_T^*\right]. \tag{103}$$

From Lemma 1,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(1-\alpha)\left(\frac{1}{2}\left(\nu^X\right)^2 \cdot + \nu^X W_\cdot^X\right)_T^*\right] \ge \frac{1-\alpha}{2}\left(\nu^X\right)^2 - \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(1-\alpha)\nu^X\left(W^X\right)_T^*\right] = \frac{1-\alpha}{2}\left(\nu^X\right)^2, \tag{104}$$

where the last equality follows from (40). Thus,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \hat{F}_T\right] \ge \max\left(\frac{\left(\nu^F\right)^2}{2}, \frac{1-\alpha}{2}\left(\nu^X\right)^2\right) = \lambda. \tag{105}$$

Case of $p \in (0,1)$. Since H_t is a positive process,

$$\lim_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[\hat{F}_T^p\right] = \lim_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[(G_T + H_T)^p\right] \geq \lim_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[G_T^p\right] = \frac{\left(\nu^F\right)^2}{2(1-p)}. \tag{106}$$

Likewise, since G_t is a positive process,

$$\lim_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[\hat{F}_T^p\right] \geq \lim_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[H_T^p\right] = \lim_{T\to\infty} \frac{1}{pT} \ln \mathbb{E}\left[\left(\hat{X}^*\right)_T^p\right] = \frac{\left(\nu^X\right)^2}{2\left(1-p+\frac{\alpha}{1-\alpha}\right)}. \tag{107}$$

where the last equality follows Lemma 11 in Guasoni and Obłój (2012). Thus,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\ln \hat{F}_T\right] \geq \max\left(\frac{\left(\nu^F\right)^2}{2(1-p)}, \frac{\left(\nu^X\right)^2}{2\left(1-p+\frac{\alpha}{1-\alpha}\right)}\right) = \lambda. \tag{108}$$

□

5 Conclusion

High-water marks, combined with long horizons, reduce a fund manager's motive to hedge fund exposure with private investments, and vice versa. Indeed, optimal policies for fund and wealth are separate, in that each of them depends only on the respective investment opportunity. The resulting welfare is the maximum welfare between that of fees and of private investment, without any diversification gain. Thus, the manager effectively focuses on either the fund or wealth, whichever is more attractive.

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